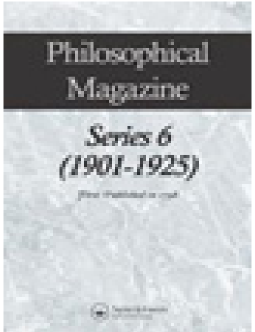

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LXXXII. On the solution of certain problems of two-dimensional physics

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electron or even a positive ion (of smaller radius so as to possess greater mass) could rotate and something like a magneton would be the result, even if the elastic constants were not supposed to be infinite. Actually, however, the electromagnetic "potential" energy will produce effects analogous to those due to a mass density varying from the centre to the circumference. By supposing λ and μ to be infinite, the "semirigid" rotating electron (electron-magneton) could still be used as an hypothesis consistent with the Principle of Relativity.

LXXXII. *On the Solution of Certain Problems of Two-Dimensional Physics.*

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1. A GENERAL method of solution of certain types of physical problem, in which the boundary considered consists of a single analytical curve, may be founded on the obvious remark † that the transformation

$$x+\iota y=\mathrm{X}(\tau)+\iota \mathrm{Y}(\tau),$$

in which $\tau=\eta-\iota\xi$, and X and Y are real when τ is real, makes the real axis in the τ plane correspond to the curve

$$x=\mathrm{X}(\eta), \quad y=\mathrm{Y}(\eta), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in the $x+\iota y$ plane. We may therefore take the equation of any analytical boundary in the form

$$\xi=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

or, if $\theta=\eta+\iota\xi$, we have

$$\theta=\tau.$$

For the sake of brevity, we shall denote $\mathrm{X}(\eta)$ by X, $\mathrm{X}(\tau)$ by X_1, and $\mathrm{X}(\theta)$ by X_2, with a similar notation in the case of Y.

In the simplest type of problem we are required to determine a function ψ from

$$\nabla^2\psi=0,$$

together with the conditions $\psi=f(\eta)$, $\dfrac{\partial\psi}{\partial n}=\mathrm{F}(\eta)$ on the boundary, where dn is an element of the outward drawn normal. The solution is

$$\psi=\tfrac{1}{2}\{f(\theta)+f(\tau)\}+\frac{1}{2\iota}\int_{\tau}^{\theta}(\mathrm{X}'^2+\mathrm{Y}'^2)^{\frac{1}{2}}\,\mathrm{F}(\eta)d\eta.$$

* Communicated by the Author.

† *Cf.* Forsyth, 'Theory of Functions,' § 265, p. 624 (2nd edition); also Jeans, 'Electricity,' p. 264.

But, in general, the boundary conditions are not alone sufficient to determine ψ, and we have to resort to other means in order to obtain the final form of the solution.

In the examples that follow the endeavour has been to give a consistent exposition of the mode of attack on the problems of most frequent occurrence; hence the inclusion of a number of well-known results.

Hydrodynamical Problems.

2. If in hydrodynamical steady motion under forces whose potential is $\Omega(x, y)$ the curve (1) is a free surface, we obtain the stream function (Earnshaw's) ψ by means of the boundary conditions $\psi=0$,

$$\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2=\mathrm{C}+2\Omega(x, y),$$

where C is constant, together with $\nabla^2\psi=0$. The result is

$$\psi=\frac{1}{2\iota}\int_\tau^\theta(\mathrm{X}'^2+\mathrm{Y}'^2)^{\frac{1}{2}}\{\mathrm{C}+2\Omega(\mathrm{X}, \mathrm{Y})\}^{\frac{1}{2}}\,d\eta.$$

The theory of plane progressive waves may be based on this result, but the work is practically identical with that of the well-known method due to Stokes.

3. The motion of a cylinder of any form in perfect fluid at rest at infinity may be obtained with equal ease.

Let the cylinder be moving with a velocity whose components parallel to the axes are U and V, and let it be rotating with angular velocity ω. Then we have $\nabla^2\psi=0$, and on the cylinder

$$\frac{\partial\psi}{\partial s}=\mathrm{U}\frac{\partial y}{\partial s}-\mathrm{V}\frac{\partial x}{\partial s}-\omega\left(x\frac{\partial x}{\partial s}+y\frac{\partial y}{\partial s}\right),$$

i. e. $$\psi=\mathrm{U}y-\mathrm{V}x-\tfrac{1}{2}\omega(x^2+y^2)$$

when $\theta=\tau$. And thus

$$\psi=\frac{1}{2\iota}\{\mathrm{F}(\theta)-\mathrm{F}(\tau)\}+\tfrac{1}{2}\mathrm{U}(\mathrm{Y}_1+\mathrm{Y}_2)-\tfrac{1}{2}\mathrm{V}(\mathrm{X}_1+\mathrm{X}_2)$$
$$-\tfrac{1}{4}\omega(\mathrm{X}_1^2+\mathrm{X}_2^2+\mathrm{Y}_1^2+\mathrm{Y}_2^2),$$

where the function F is to be determined from the fact that ψ is nowhere infinite and vanishes at infinity.

As an illustration we take the familiar case of the elliptic cylinder, for which

$$\mathrm{X}=c\cosh\lambda\cos\eta,\quad \mathrm{Y}=c\sinh\lambda\sin\eta.$$

So that

$$\psi=\frac{1}{2\iota}\{F(\theta)-F(\tau)\}+\tfrac{1}{2}bU(\sin\theta+\sin\tau)-\tfrac{1}{2}aV(\cos\theta+\cos\tau)$$
$$-\tfrac{1}{8}\omega c^2(2\cosh 2\lambda+\cos 2\theta+\cos 2\tau),$$

where $a=c\cosh\lambda,\quad b=c\sinh\lambda.$

Hence, finally, omitting a constant,

$$\psi=e^{-\xi}(bU\sin\eta-aV\cos\eta)-\tfrac{1}{4}\omega c^2 e^{-2\xi}\cos 2\eta.$$

4. In the case of a vortex filament bounded by the curve (1), we have, within the vortex $\nabla^2\psi_i=2\zeta$, and without the vortex $\nabla^2\psi_0=0$, where ζ, supposed constant, is the vorticity. And on the boundary we have

$$\frac{\partial\psi_i}{\partial n}=\frac{\partial\psi_0}{\partial n},\quad \psi_0=\psi_i=\tfrac{1}{2}\omega(x^2+y^2),$$

where it is assumed that the vortex rotates with constant angular velocity ω. At infinity ψ_0 must take the same form as the gravitational potential of a cylinder of the same form and of density $-2\pi\zeta$. Further, in order to avoid infinite velocities, $\frac{\partial\psi_i}{\partial\xi}$ and $\frac{\partial\psi_i}{\partial\eta}$ must vanish at the singular points of the transformation

$$z\equiv x+\iota y=X(\tau)+\iota Y(\tau),$$

namely, the points where $\frac{dz}{d\tau}=0$, *i. e.*

$$X'(\tau)=-\iota Y'(\tau),$$

and therefore $X'(\theta)=\iota Y'(\theta).$

We easily find

$$\psi_i=\tfrac{1}{2}\zeta(x^2+y^2)-\frac{\zeta-\omega}{4}(X_1^2+X_2^2+Y_1^2+Y_2^2)+\frac{1}{2\iota}\{F(\tau)-F(\theta)\},$$

$$\psi_0=\tfrac{1}{4}\omega(X_1^2+X_2^2+Y_1^2+Y_2^2)+\frac{\zeta}{2\iota}\int_\tau^\theta(XY'-X'Y)d\eta+\frac{1}{2\iota}\{F(\tau)-F(\theta)\}$$

where F is to be determined by the conditions given above.

The case of variable vorticity may be treated in the same way. It is evident, however, that the problem is precisely the same as that of determining figures of equilibrium of rotating fluid, where ω is put for $-\omega^2$ and ζ for $-2\pi\rho$. (See §§ 10 and 11 *infra.*)

We take, by way of illustration, the hypotrochoid of n oscillations,

$$\left.\begin{aligned} x&=a\cos\eta+b\cos(n-1)\eta,\\ y&=a\sin\eta-b\sin(n-1)\eta.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

This includes the ellipse when $n=2$, and when b is not too large—the greatest possible value for b is $1/(n-1)$—it represents a circle disturbed by an harmonic inequality with n maxima. We have, in fact,

$$r^2=a^2+b^2+2ab\cos n\eta,$$

and if b is so small that squares of b/a may be neglected,

$$r=a+b\cos n\eta,$$

and η differs from θ by a multiple of b/a, where r and θ are polar coordinates.

In general, we have

$$x+\iota y=a\,e^{\xi+\iota\eta}+b\,e^{-(n-1)(\xi+\iota\eta)},$$

$$x^2+y^2=a^2\,e^{2\xi}+b^2\,e^{-2(n-1)\xi}+2ab\,e^{-(n-2)\xi}\cos n\eta,$$

$$\tfrac{1}{2}(X_1^2+X_2^2+Y_1^2+Y_2^2)=a^2+b^2+2ab\cosh n\xi\cos n\eta,$$

$$\frac{1}{2\iota}\int_{T}^{\theta}(XY'-X'Y)\,d\eta=\{a^2-(n-1)b^2\}\xi-(1-2/n)ab\sinh n\xi\cos n\eta.$$

And therefore, provided that the boundary does not cross itself, *i. e.* provided that $b \not> 1/(n-1)$,

$$\psi_i=\tfrac{1}{2}\zeta(x^2+y^2-a^2-b^2-2ab\cosh n\xi\cos n\eta)$$
$$\qquad+\omega ab\,e^{-n\xi}\cos n\eta+(1-2/n)\zeta ab\sinh n\xi\cos n\eta$$
$$\psi_0=\omega ab\,e^{-n\xi}\cos n\eta+\{a^2-(n-1)b^2\}\zeta\xi.$$

The singular points of the transformation are given by

$$n\xi=\log\{(n-1)b/a\},\quad n\eta=2k\pi,$$

where k is an integer or zero. Thus ω is determined by the condition that $\partial\psi_i/\partial\xi=0$ at these points. We have

$$\frac{\omega}{\zeta}=1-\frac{1}{n}-\frac{(n-1)^2}{n}\frac{b^2}{a^2},$$

and

$$\psi_i=\tfrac{1}{2}\zeta(x^2+y^2-a^2-b^2)-\zeta(ab/n)\{e^{n\xi}+(n-1)^2(b^2/a^2)\,e^{-n\xi}\}\cos n\eta,$$

$$\psi_0=\zeta\{a^2-(n-1)b^2\}\{\xi+[(n-1)b/na]\,e^{-n\xi}\cos n\eta\}.$$

In the case of the ellipse, $n=2$, we have the well-known result

$$\omega/\zeta = \tfrac{1}{2}(a^2+b^2)/a^2,$$

where $a+b$, $a-b$ are the semi-axes. And in the case of the n-cusped hypocycloid, for which $b=a/(n-1)$,

$$\omega/\zeta = 1-2/n.$$

As a corollary to the general case we notice that

$$\psi = \tfrac{1}{2}\zeta\{x^2+y^2-2[a^2-(n-1)b^2]\xi - 2ab[e^{-n\xi}+(4/n)\sinh n\xi]\cos n\eta\}$$

satisfies the equation $\nabla^2\psi = 2\zeta$, which is a particular form of $\nabla^2\psi = f(\psi)$, and is such that the velocity vanishes on the cylinder (2). It therefore represents a possible motion of viscous fluid within this cylinder.

5. In the case of viscous fluid motions so slow that the squares of the velocities may be neglected, we have

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u, \quad \frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v.$$

And in the case of steady motions this leads at once to

$$\nabla^4\psi = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

with the conditions $u=v=0$ on the boundary.

The general solution of (3) subject to these conditions is easily seen to be

$$\psi = \frac{1}{2\iota}\left\{ \mathrm{F}(\theta) - \mathrm{F}(\tau) - x\int_\tau^\theta \frac{\mathrm{F}'}{\mathrm{X}}\,d\eta \right\}$$

together with a similar term, with y, Y in place of x, X, which may be written down by symmetry.

In the case of the ellipse, for which

$$x+\iota y = c\cosh(\lambda+\xi+\iota\eta),$$

we have

$$\psi = \frac{1}{2\iota}\left\{ \mathrm{F}(\eta+\iota\xi) - \mathrm{F}(\eta-\iota\xi) \right.$$
$$\left. -\operatorname{sech}\lambda\cos\eta\cosh(\lambda+\xi)\int_{\eta-\iota\xi}^{\eta+\iota\xi} \mathrm{F}'(\eta)\sec\eta\,d\eta \right\} + \&c.$$

In accordance with a result first given by Stokes, it appears to be impossible to determine a solution corresponding to the case of the elliptic cylinder in a steady stream. Consider, however, the stream function

$$\psi=(\mathrm{A}+\mathrm{B})\xi-\frac{1}{2\iota}\int_{\eta-\iota\xi}^{\eta+\iota\xi}\left(\frac{\mathrm{A}x}{a}\sec\eta+\frac{\mathrm{B}y}{b}\operatorname{cosec}\eta\right)d\eta$$

$$=(\mathrm{A}+\mathrm{B})\xi-\frac{\mathrm{A}x}{a}\tan^{-1}\left(\frac{\sinh\xi}{\cos\eta}\right)-\frac{\mathrm{B}y}{b}\tan^{-1}\left(\frac{\sinh\xi}{\sin\eta}\right).$$

We obtain the velocity at any point from the formulæ

$$u=\frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial y}+\frac{\partial\psi}{\partial\eta}\frac{\partial\xi}{\partial x},\quad v=-\frac{\partial\psi}{\partial\xi}\frac{\partial\xi}{\partial x}+\frac{\partial\psi}{\partial\eta}\frac{\partial\xi}{\partial y}.$$

And we have immediately

$$\frac{\partial\psi}{\partial\xi}=\mathrm{A}+\mathrm{B}-\frac{\mathrm{A}c}{a}\left\{\sinh(\lambda+\xi)\cos\eta\tan^{-1}\left(\frac{\sinh\xi}{\cos\eta}\right)+\frac{\cosh\xi\cosh(\lambda+\xi)}{1+(\sinh^2\xi/\cos^2\eta)}\right\}$$
$$-\frac{\mathrm{B}c}{b}\left\{\cosh(\lambda+\xi)\sin\eta\tan^{-1}\left(\frac{\sinh\xi}{\sin\eta}\right)+\frac{\cosh\xi\sinh(\lambda+\xi)}{1+(\sinh^2\xi/\sin^2\eta)}\right\},$$

$$\frac{\partial\psi}{\partial\eta}=\frac{\mathrm{A}c}{a}\cosh(\lambda+\xi)\sin\eta\left\{\tan^{-1}\left(\frac{\sinh\xi}{\cos\eta}\right)-\frac{\sinh\xi\cos\eta}{\sinh^2\xi+\cos^2\eta}\right\}$$
$$-\frac{\mathrm{B}c}{b}\sinh(\lambda+\xi)\cos\eta\left\{\tan^{-1}\left(\frac{\sinh\xi}{\sin\eta}\right)-\frac{\sinh\xi\sin\eta}{\sinh^2\xi+\sin^2\eta}\right\}.$$

At infinity we have

$$\frac{\partial\psi}{\partial\xi}=-\frac{\pi c}{4}e^{\lambda+\xi}\left(\frac{\mathrm{A}}{a}\cos\eta+\frac{\mathrm{B}}{b}\sin\eta\right),$$
$$\frac{\partial\psi}{\partial\eta}=\frac{\pi c}{4}e^{\lambda+\xi}\left(\frac{\mathrm{A}}{a}\sin\eta-\frac{\mathrm{B}}{b}\cos\eta\right).$$

And therefore

$$u=-\frac{\pi}{2}\frac{\mathrm{B}}{b},\quad v=\frac{\pi}{2}\frac{\mathrm{A}}{a}.$$

Further, on the ellipse $\xi=0$ we have $u=v=0$, provided that neither $\cos\eta$ nor $\sin\eta$ vanishes.

In the neighbourhood of the extremities of the minor axis put

$$\sinh\xi=\tan\tfrac{1}{2}\mu\cos\eta.$$

Then, at these points,

$$u=\frac{\mathrm{A}}{2b}(\mu-\sin\mu),\quad v=\frac{\mathrm{A}}{2b}(1+\cos\mu)\mp\frac{\mathrm{B}}{b},$$

where $-\pi\leqq\mu<\pi$, but μ is otherwise arbitrary.

Similarly, in the neighbourhood of the extremities of the major axis, put

$$\sinh \xi = \tan \tfrac{1}{2}\nu \sin \eta,$$

Then at these points,

$$u = \pm \frac{B}{a} - \frac{B}{2a}(1 + \cos \nu), \quad v = -\frac{B}{2a}(\nu - \sin \nu),$$

where, again, $-\pi \leqq \nu \leqq \pi$, but ν is otherwise arbitrary. And in each case the upper sign corresponds to the positive end of the axis, the lower to the negative.

The velocity is thus indeterminate, and it is easy to verify that the vorticity is infinite, at the extremities of the axes. In other words, eddies are formed at these points. The eddies do not disappear when the ellipse becomes a circle. And this appears to be the true explanation of the result obtained by Stokes in attempting to find a slow steady motion of an infinite cylinder in viscous fluid—namely, that such a motion is impossible.

If the cylinder is moving, with a velocity whose components parallel to the axes are U and V, in fluid at rest at infinity, we have

$$\psi = Uy - Vx + \frac{2}{\pi}(bU - aV)\xi$$
$$+ \frac{2V}{\pi} x \tan^{-1}\left(\frac{\sinh \xi}{\cos \eta}\right) - \frac{2U}{\pi} y \tan^{-1}\left(\frac{\sinh \xi}{\sin \eta}\right).$$

In the case of a circular cylinder moving with velocity V parallel to the axis of y this becomes

$$\psi = -Vx\left(1 + \frac{2}{\pi}\tan^{-1}\frac{x^2 + y^2 - a^2}{2ax}\right) - \frac{aV}{\pi}\log\frac{x^2 + y^2}{a^2}.$$

6. For a cylinder slowly rotating with angular velocity ω in viscous fluid at rest at infinity, we have

$$\psi = \frac{1}{2\iota}\left\{F(\theta) - F(\tau) - x\int_\tau^\theta \frac{F'}{X}\,d\theta\right\} - \tfrac{1}{2}\omega(x^2 + y^2).$$

In the case of the ellipse it is easy to verify that the stream function is

$$\psi = -\tfrac{1}{4}\omega c^2\{2\xi \cosh 2\lambda + e^{-2(\xi + \lambda)} + \cos 2\eta\},$$

which, in the case of the circle, reduces to the obvious result,

$$\psi = -\tfrac{1}{4}\omega a^2 \log r.$$

6*a*. In the case of a problem somewhat similar to that of motion in a viscous fluid, which it may sometimes be taken to replace, namely, motion (not irrotational) in a perfect fluid, with the condition that the velocity should vanish on the boundary, the solution may be written down.

Let $$\nabla^2\psi = 2\zeta = \nabla^2\chi, \text{ say.}$$

And let the cylinder bounded by the curve (1) be moving with velocity components U and V, and rotating with angular velocity ω. Then, when $\theta = \tau$, we have

$$\psi = Uy - Vx - \tfrac{1}{2}\omega(x^2+y^2),$$
$$\partial\psi/\partial x = -V - \omega x, \quad \partial\psi/\partial y = U - \omega y.$$

Thus, after reduction, we find

$$= \chi(x,y) - \tfrac{1}{2}\left\{\chi(X_1, Y_1) + \chi(X_2, Y_2)\right\} - \frac{1}{2\iota}\int_\tau^\theta \left(Y'\frac{\partial\chi}{\partial X} - X'\frac{\partial\chi}{\partial Y}\right)d\eta$$
$$+ Uy - Vx - \tfrac{1}{4}\omega(X_1^2 + X_2^2 + Y_1^2 + Y_2^2) - \frac{\omega}{2\iota}\int_\tau^\theta (XY' - X'Y)\,d\eta.$$

Remembering that

$$\frac{\partial\psi}{\partial\tau} = \tfrac{1}{2}\left(\frac{\partial\psi}{\partial x} - \iota\frac{\partial\psi}{\partial y}\right)(X_1' + \iota Y_1') = \frac{1}{2\iota}(u - \iota v)(X_1' + \iota Y_1'),$$

we may write down the velocity in the form

$$u - \iota v = \frac{\partial\chi}{\partial y} + \iota\frac{\partial\chi}{\partial x} - \frac{\partial\chi_1}{\partial Y_1} - \iota\frac{\partial\chi_1}{\partial X_1} + U - \iota V - \omega(Y_1 + \iota X_1),$$

where $\chi_1 = \chi(X_1, Y_1)$.

If it is possible to choose χ so that this expression vanishes at infinity, we obtain a solution for the case in which the cylinder moves through the fluid at rest at infinity. (*Cf.* §§ 5 and 6.)

If in addition to making ψ vanish at infinity χ satisfies the equation $\nabla^4\chi = 0$, we have $\nabla^4\psi = 0$, and the analytical form of the solution is precisely the same as in the case of slow motion of viscous fluid with the same boundary conditions. In the problem under consideration in the present paragraph we are not limited to the case of slow motions, but it must not therefore be supposed that the solution may be applied to problems of viscous fluid motion in which the velocities contemplated are not small. It is only in exceptional cases that this is true.

Electrical Problems.

7. The potential of a freely electrified cylinder is evidently

$$\phi = V + \frac{1}{2\iota}\left\{F(\theta) - F(\tau)\right\},$$

where F is so chosen that, at infinity

$$\phi = -2E\log r,$$

E being the charge per unit length.

Take, for instance, the case of the cylinder

$$X = a\cos^n\eta, \quad Y = b\sin^n\eta, \quad . \quad . \quad . \quad . \quad (4)$$

in which n must be a positive integer.

We have
$$x + \iota y = a\cos^n(\eta - \iota\xi) + \iota b\sin^n(\eta - \iota\xi);$$
and at $\xi = +\infty$,
$$x + \iota y = \frac{a+b}{2^n}e^{n(\xi+\iota\eta)},$$
so that
$$\log r = n\xi,$$
approximately.

Thus the potential of the cylinder bounded by the curve (4) is

$$\phi = V - 2nE\xi,$$

provided that the singular points of the transformation do not fall within the field of variation of ϕ. If $n > 1$ there are singular points at $\xi = 0, \eta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$; *i. e.* the electric density is infinite at these points.

8. The potential of a cylinder magnetized transversely may be determined in the same way.

For simplicity we shall assume that the components of the intensity of magnetization are derivatives of a single function J, so that

$$A = -\frac{\partial J}{\partial x}, \quad B = -\frac{\partial J}{\partial y}.$$

We then have

$$\nabla^2\Omega_0 = 0, \quad \nabla^2(\Omega_i + 4\pi J) = 0,$$

and on the boundary

$$\Omega_0 = \Omega_i, \quad \frac{\partial\Omega_0}{\partial\xi} = \frac{\partial}{\partial\xi}(\Omega_i + 4\pi J),$$

where Ω_0 is the external, Ω_i the internal magnetic potential.

In addition we have $\Omega_0=0$ at infinity, and

$$\partial\Omega_i/\partial\xi=\partial\Omega_i/\partial\eta=0$$

at the singular points of the transformation.

We have immediately

$$\Omega_0=\tfrac{1}{2}\left\{f(\tau)-4\pi\mathrm{J}(\mathrm{X}_1,\mathrm{Y}_1)+f(\theta)-4\pi\mathrm{J}(\mathrm{X}_2,\mathrm{Y}_2)\right\}$$
$$+\frac{1}{2\iota}\left\{\mathrm{F}(\theta)-\mathrm{F}(\tau)\right\},$$

$$\Omega_i=\tfrac{1}{2}\left\{f(\tau)+f(\theta)\right\}-4\pi\mathrm{J}(x,y)+\frac{1}{2\iota}\left\{\mathrm{F}(\theta)-\mathrm{F}(\tau)\right\},$$

where F and f are to be determined by the conditions stated above.

As a very simple illustration take the case of an elliptic cylinder uniformly magnetized. We have, in this case,

$$\mathrm{J}=-\mathrm{A}x-\mathrm{B}y.$$

And it immediately follows that

$$\Omega_0=e^{-\xi}\{(\mathrm{E}+4\pi a\mathrm{A})\cos\eta+(\mathrm{F}+4\pi b\mathrm{B})\sin\eta\},$$
$$\Omega_i=\Omega_0-4\pi\sinh\xi(b\mathrm{A}\cos\eta+a\mathrm{B}\sin\eta).$$

In this result E and F are constants to be determined from the values of $\partial\Omega/\partial\xi$ and $\partial\Omega_i/\partial\eta$ at the singular points. We have

$$\frac{\partial\Omega_i}{\partial\xi}=-e^{-\xi}\left\{(\mathrm{E}+4\pi a\mathrm{A})\cos\eta+(\mathrm{F}+4\pi b\mathrm{B})\sin\eta\right\}$$
$$-4\pi\cosh\xi(b\mathrm{A}\cos\eta+a\mathrm{B}\sin\eta),$$

$$\frac{\partial\Omega_i}{\partial\eta}=e^{-\xi}\left\{-(\mathrm{E}+4\pi a\mathrm{A})\sin\eta+(\mathrm{F}+4\pi b\mathrm{B})\cos\eta\right\}$$
$$+4\pi\sinh\xi(b\mathrm{A}\sin\eta-a\mathrm{B}\cos\eta).$$

And these must both vanish when $\xi=-\lambda$, $\sin\eta=0$. Thus we have

$$\mathrm{E}=-4\pi\mathrm{A}(a+b\,e^{-\lambda}\cosh\lambda)=-4\pi a\mathrm{A}(a+2b)/(a+b),$$
$$\mathrm{F}=-4\pi\mathrm{B}(b+a\,e^{-\lambda}\sinh\lambda)=-4\pi b\mathrm{B}(2a+b)/(a+b).$$

And, finally,

$$\Omega_0=-4\pi\{ab/(a+b)\}e^{-\xi}(\mathrm{A}\cos\eta+\mathrm{B}\sin\eta),$$
$$\Omega_i=-4\pi\sinh\xi(b\mathrm{A}\cos\eta+a\mathrm{B}\sin\eta)+\Omega_0.$$

9. In the case of a cylinder of dielectric material, we have directly

$$V_0=\tfrac{1}{2}\left\{f(\theta)+f(\tau)\right\}+\frac{K_i}{2\iota}\left\{F(\theta)-F(\tau)\right\},$$

$$V_i=\tfrac{1}{2}\left\{f(\theta)+f(\tau)\right\}+\frac{K_0}{2\iota}\left\{F(\theta)-F(\tau)\right\},$$

where V_0, K_0 are the external, V_i, K_i the internal potential and specific inductive capacity, and f and F are to be determined from the usual conditions of finiteness, &c. For the boundary conditions are

$$V_0=V_i,\quad K_0\frac{\partial V_0}{\partial \xi}=K_i\frac{\partial V}{\partial \xi}$$

when $\theta=\tau$.

If, however, the boundary be charged to surface-density σ, we must add to V_i the term

$$\frac{2\pi}{2\iota K_i}\int_\tau^\theta \sigma(X'^2+Y'^2)^{\frac{1}{2}}\,d\eta.$$

Let us consider the simple case of an uncharged elliptic cylinder of radius a and specific inductive capacity K, surrounded by air, and in the presence of a line charge, E per unit length, cutting the xy plane at the point whose coordinates are x_0, y_0. Let

$$r_1=\left\{(x-x_0)^2+(y-y_0)^2\right\}^{\frac{1}{2}},$$

be the distance of the point x, y from the line charge. Then in the neighbourhood of $r_1=0$ we have

$$V_0=-2E\log r_1.$$

But, putting

$$x_0+\iota y_0=c\cosh(\lambda+\xi_0+\iota\eta_0),$$

we have

$$\begin{aligned}r_1^2&=(x-x_0)^2+(y-y_0)^2\\&=c^2\{\cosh(\lambda+\xi+\iota\eta)-\cosh(\lambda+\xi_0+\iota\eta_0)\}^2\\&=c^2\{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)\}\{\cosh(2\lambda+\xi+\xi_0)-\cos(\eta+\eta_0)\}.\end{aligned}$$

And therefore, in the neighbourhood of the point ξ_0, η_0,

$$\begin{aligned}\log r_1&=\tfrac{1}{2}\log\{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)\}\\&=\tfrac{1}{2}\log\{2\sin\tfrac{1}{2}(\theta-\eta_0-\iota\xi_0)\sin\tfrac{1}{2}(\tau-\eta_0+\iota\xi_0)\}\\&=\tfrac{1}{4}\log\{[\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)][\cosh(\xi+\xi_0)-\cos(\eta-\eta_0)]\}\\&\quad+\tfrac{1}{4}\log\frac{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)}{\cosh(\xi+\xi_0)-\cos(\eta-\eta_0)};\end{aligned}$$

and to get the corresponding term of V_i we have to divide the coefficient of the second logarithm in the expression last printed by K. Hence a term of V_0 is

$$-E \log \{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)\},$$

and the corresponding term of V_i is

$$-\tfrac{1}{2}E(1+1/K) \log \{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)\}$$
$$-\tfrac{1}{2}E(1-1/K) \log \{\cosh(\xi+\xi_0)-\cos(\eta-\eta_0)\}.$$

But we must remove the second logarithm, since (if $\xi_0 < \lambda$) it becomes infinite at $-\xi_0$, η_0. By precisely the same analysis as that just used we thus find a term of V_i in the form

$$-\tfrac{1}{2}E(1+1/K) \log \{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)\},$$

and the corresponding term of V_0 is

$$-\{(K+1)^2/4K\}E \log \{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)\}$$
$$+\{(K^2-1)/4K\}E \log \{\cosh(\xi+\xi_0)-\cos(\eta-\eta_0)\}.$$

In order that this last term should take the correct form at ξ_0, η_0 we must divide through by $(K+1)^2/4K$; and we thus obtain a term of V_0 equal to

$$-E \log \{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)\}$$
$$+\{(K-1)/(K+1)\}E \log \{\cosh(\xi+\xi_0)-\cos(\eta-\eta_0)\},$$

and the corresponding term of V_i

$$-\{2E/(K+1)\} \log \{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)\}.$$

At the singular points $\xi=-\lambda$, $\eta=0$ or π, the part of $\partial V_i/\partial\xi$ arising from the logarithm in this term is

$$-\sinh(\lambda+\xi_0)/\{\cosh(\lambda+\xi_0)\mp\cos\eta_0\},$$

and the part of $\partial V_i/\partial\eta$ is

$$\pm\sin\eta_0/\{\cosh(\lambda+\xi_0)\mp\cos\eta_0)\}.$$

Exactly the same terms arise from the expression

$$-\log\{\cosh(2\lambda+\xi_0+\xi)-\cos(\eta+\eta_0)\}. \quad . \quad . \quad (6)$$

Hence a possible term for the internal potential is

$$-\{4E/(K+1)\} \log r_1;$$

and we have to determine the effect on V_0 of the addition of the proper multiple of the term (6) to V_i. This term is equal to

$$-\{2E/(K+1)\}\log\{2\sin\tfrac{1}{2}[\theta+\eta_0+\iota(2\lambda_0+\xi)]\sin\tfrac{1}{2}[\tau+\eta_0-\iota(2\lambda+\xi_0)]\}$$

$$=-\{E/(K+1)\}\log\{4\sin(\phi+\iota\alpha)\sin(\phi-\iota\alpha)\sin(\psi-\iota\alpha)\sin(\psi+\iota\alpha)\}$$

$$-\frac{E}{K+1}\log\frac{2\sin(\phi+\iota\alpha)\sin(\psi-\iota\alpha)}{2\sin(\phi-\iota\alpha)\sin(\psi+\iota\alpha)},$$

where $\phi=\tfrac{1}{2}(\theta+\eta_0),\quad \psi=\tfrac{1}{2}(\tau+\eta_0),\quad \alpha=\lambda+\tfrac{1}{2}\xi_0.$

The term to be added to V_0 is found by multiplying the coefficient of the second logarithm by K. Hence

$$\begin{aligned}V_0=&-E\log\{\cosh(\xi-\xi_0)-\cos(\eta-\eta_0)\}\\&+\{(K-1)/(K+1)\}E\log\{\cosh(\xi+\xi_0)-\cos(\eta-\eta_0)\}\\&-E\log\{\cosh(2\lambda+\xi_0+\xi)-\cos(\eta+\eta_0)\}\\&+\{(K-1)/(K+1)\}E\log\{\cosh(2\lambda+\xi_0-\xi)-\cos(\eta+\eta_0)\}.\end{aligned}$$

And we have $V_i=-\{4E/(K+1)\}\log r_1.$

This, however, is not the solution of the problem we set out to solve, for there is a line charge of strength

$$-\{(K-1)/(K+1)\}E \text{ at } \xi=2\lambda+\xi_0,\quad \eta=2\pi-\eta_0.$$

In the case of the circle λ is infinite ($c\to 0$ in such a way that $\tfrac{1}{2}ce^{\lambda}=a$, the radius of the circle) and the terms given are sufficient for the complete solution. In the more general case we superimpose the solution for a charge

$$\{(K-1)/(K+1)\}E \quad\text{at}\quad 2\lambda+\xi_0,\quad 2\pi-\eta_0,$$

thus obtaining the solution of a problem involving a charge

$$-\{(K-1)/(K+1)\}^2E \quad\text{at}\quad 4\lambda+\xi_0,\ \eta_0.$$

And proceeding in this way we have, finally,

$$\begin{aligned}V_i=&-\{4E/(K+1)\}\sum_{n=0}^{\infty}\{(K-1)/(K+1)\}^n\\&\times\Big\{\log[\cosh(2n\lambda+\xi_0-\xi)-\cos\{\eta-(-)^n\eta]\\&+\log[\cosh\{(2n+2)\lambda+\xi_0+\xi\}-\cos\{\eta+(-)^n\eta_0\}]\Big\},\end{aligned}$$

$$\begin{aligned}V_0=&-E\log\{\cosh(\xi_0-\xi)-\cos(\eta_0-\eta)\}\\&+\{(K-1)/(K+1)\}E\log\{\cosh(\xi_0+\xi)-\cos(\eta_0-\eta)\}\\&-\{4KE\ (K+1)^2\}\\&\sum_{n=1}^{\infty}\{(K-1)/(K+1)\}^{n-1}\log[\cosh(2n\lambda+\xi_0+\xi)\\&-\cos\{\eta-(-)^n\eta_0\}],\end{aligned}$$

provided that these series converge,—as they evidently do, for all finite values of ξ, except that at $\xi=\xi_0$, $\eta=\eta_0$, V_0 becomes infinite like $-2E\log r_1$. And when ξ is infinite we have

$$V_0=-\{2E/(K+1)\}\xi-\{4KE/(K+1)^2\}\sum_{n=1}^{\infty}\{(K-1)/(K+1)\}^{n-1}\xi$$
$$=-2E\log r_1.$$

So that all the necessary conditions are satisfied.

There is little interest in the case of the circular cylinder, as the solution can immediately be derived by the method of images. But there is some interest in the case of the thin plate of dielectric, $\lambda=0$, for the series for V_0 and V_i can then be summed. For all points outside the plate (which cuts the xy plane in the line $y=0$, $-c \leqq x \leqq c$) we have

$$V=-E\log\{\cosh(\xi_0-\xi)-\cos(\eta_0-\eta)\}$$
$$-E\log\{\cosh(\xi_0+\xi)-\cos(\eta_0+\eta)\},$$

where $$x+\iota y=c\cosh(\xi+\iota\eta).$$

Gravitational Potential.

10. If V_0 and V_i be respectively the external and internal potentials of a gravitating cylinder bounded by the curve (1), we have

$$\nabla^2 V_i=-4\pi\rho, \quad \nabla^2 V_0=0,$$

and when $\theta=\tau$,

$$V_i=V_0, \quad \partial V_i/\partial\xi=\partial V_0/\partial\xi.$$

Also $$V_0 \to -A\log r \quad \text{as } r\to\infty,$$

where A is a constant, which in the case of constant density is $2\rho\times$ the area of cross-section of the cylinder.

Let ψ be determined so that

$$-4\pi\rho=\nabla^2\psi.$$

Then

$$V_i=\psi(x,y)+\tfrac{1}{2}\{f(\theta)+f(\tau)\}+\frac{1}{2\iota}\{F(\theta)-F(\tau)\},$$

$$V_0=\tfrac{1}{2}\{g(\theta)+g(\tau)\}+\frac{1}{2\iota}\{G(\theta)-G(\tau)\}.$$

And we have immediately

$$g=f+\psi(X,Y),$$

$$G'=F'-\left(X'\frac{\partial}{\partial Y}-Y'\frac{\partial}{\partial X}\right)\psi(X,Y).$$

Thus with the previous value of V_i we have

$$V_0=\tfrac{1}{2}\{f(\theta)+f(\tau)\}+\frac{1}{2\iota}\{F(\theta)-F(\tau)\}+\tfrac{1}{2}\{\psi(X_1,Y_1)+\psi(X_2,Y_2)\}$$
$$+\frac{1}{2\iota}\int_\tau^\theta\left(Y'\frac{\partial\psi}{\partial X}-X'\frac{\partial\psi}{\partial Y}\right)d\eta,$$

where f and F are functions to be determined by the form of V_0 at infinity, and by the conditions that $\partial V_i/\partial\xi=\partial V_i/\partial\eta=0$ at the singular points of the transformation, where

$$Y_1'=\iota X_1', \quad Y_2'=-\iota X_2'.$$

But

$$\frac{\partial}{\partial\theta}\left\{\psi(X_1,Y_1)+\psi(X_2,Y_2)-\iota\int_\tau^\theta\left(Y'\frac{\partial\psi}{\partial X}-X'\frac{\partial\psi}{\partial Y}\right)\right\}$$
$$=\frac{\partial\psi}{\partial X_2}X_2'+\frac{\partial\psi}{\partial Y_2}Y_2'-\iota\frac{\partial\psi}{\partial X_2}Y_2'+\iota\frac{\partial\psi}{\partial Y_2}X_2'$$
$$=0,$$

at the singular points. In the same way the differential coefficient of the same function with regard to τ vanishes at these points. And it is obvious that the differential coefficients of $\psi(x,y)$ with regard to ξ and η also vanish at the singular points. Hence $\partial V_0/\partial\xi$ and $\partial V_0/\partial\eta$ vanish at these points.

This result greatly simplifies the work of obtaining the potential of a cylinder of any given form. For example, let ρ be constant and let

$$x+\iota y=ae^{\xi+\iota\eta}+b_1e^{-(\xi+\iota\eta)}+b_2e^{-2(\xi+\iota\eta)}+\ldots+b_ne^{-n(\xi+\iota\eta)}.$$

Then

$$\tfrac{1}{2}(X_1^2+X_2^2+Y_1^2+Y_2^2) \quad\text{and}\quad \frac{1}{2\iota}\int_\tau^\theta(XY'-X'Y)d\eta$$

contain η only through multiples of $\cos\eta$, $\cos 2\eta$, $\ldots, \cos(n+1)\eta$. And therefore V_0 is of the form

$$-2\pi\rho(a^2-b_1^2-2b_2^2\ldots-nb_n^2)\{\xi+A_1e^{-\xi}\cos\eta+\ldots$$
$$+A_{n+1}e^{-(n+1)\xi}\cos(n+1)\eta\},$$

and both $\partial V_0/\partial\xi$ and $\partial V_0/\partial\eta$ must vanish when

$$ae^{\xi+\iota\eta}-b_1e^{-(\xi+\iota\eta)}-2b_2e^{-2(\xi+\iota\eta)}\ldots-nb_ne^{-n(\xi+\iota\eta)}=0.$$

Thus

$$V_0 = -2\pi\rho(a^2 - b_1^2 - 2b_2^2 \dots - nb_n^2)\left\{\xi + \frac{1}{2}\frac{b_1}{a}e^{-2\xi}\cos 2\eta\right.$$

$$\left. + \frac{2}{3}\frac{b_2}{a}e^{-3\xi}\cos 3\eta + \dots + \frac{n}{n+1}\frac{b_n}{a}e^{-(n+1)\xi}\cos(n+1)\eta\right\},$$

and V_i may be written down by means of the general formula.

The form taken by V_0 at infinity assumes that the curve does not cross itself. This, of course, restricts the variability of the coefficients $b_1, b_2, \dots b_n$.

The value of V_0 may be similarly written down in the rather more general case of the transformation

$$x + \iota y = a_n e^{n(\xi+\iota\eta)} + a_{n-1}e^{(n-1)(\xi+\iota\eta)} + \dots + a_1 e^{\xi+\iota\eta} + a_0$$

$$+ b_1 e^{-(\xi+\iota\eta)} + b_2 e^{-2(\xi+\iota\eta)} + \dots + b_m e^{-m(\xi+\iota\eta)}.$$

But in this case we have, at infinity,

$$\log r \rightarrow n\xi,$$

and therefore the coefficient of ξ in V_0 is $-2\pi\rho n \times$ the area of cross-section, so that a term in ξ alone occurs in V_i. For example, for the evolute of the ellipse, we have

$$x + \iota y = a\cos^3(\eta - \iota\xi) + \iota b \sin^3(\eta - \iota\xi)$$

$$= c\{3\cosh(\lambda + \xi + \iota\eta) + \cosh(\lambda - 3\xi - 3\iota\eta)\},$$

on putting

$$a = 4c\cosh\lambda, \quad b = 4c\sinh\lambda.$$

At the singular points

$$1 + e^{2\lambda - 2\xi - 2\iota\eta} - e^{-4\xi - 4\iota\eta} - e^{2\lambda - 6\xi - 6\iota\eta} = 0.$$

Hence we obtain, by integration,

$$V_0 = -\tfrac{9}{4}\pi\rho ab\{\xi - \tfrac{1}{2}e^{2(\lambda-\xi)}\cos 2\eta + \tfrac{1}{4}e^{-4\xi}\cos 4\eta + \tfrac{1}{6}e^{2(\lambda - 3\xi)}\cos 6\eta\},$$

and therefore

$$V_i = -\pi\rho(x^2 + y^2) + \tfrac{3}{2}\pi\rho ab\xi + V_0 - \tfrac{3}{16}\pi\rho ab\sinh 4\xi\cos 4\eta$$

$$+ \tfrac{1}{2}\pi\rho c^2(15\cosh 2\xi\cos 2\eta + \cosh 6\xi\cos 6\eta)$$

$$+ 3\pi\rho c^2\cosh 2\lambda\cosh 4\xi\cos 4\eta.$$

It is not possible to pass to the case of the four-cusped hypocloid by making $\lambda \rightarrow \infty$, $2ce^{\lambda} \rightarrow a$, for the transformations take different forms at infinity.

Figures of Equilibrium of Rotating Fluid.

11. If the gravitational potential of a cylinder take the form* $C-\frac{1}{2}\omega^2\{(x-k)^2+y^2\}$ on the surface $\xi=0$, *i. e.* if it is possible (in the case of constant density) to have

$$V_i=-\pi\rho\{(x-k)^2+y^2\}+\tfrac{1}{4}(2\pi\rho-\omega^2)(X_1{}^2+X_2{}^2+Y_1{}^2+Y_2{}^2)+\frac{1}{2\iota}\{F(\theta)-F(\tau)\},$$

$$V_0=-\tfrac{1}{4}\omega^2(X_1{}^2+X_2{}^2+Y_1{}^2+Y_2{}^2)+\tfrac{1}{2}\omega^2k(X_1+X_2)+\pi\rho\iota\int_\tau^\theta(XY'-X'Y)d\eta+\frac{1}{2\iota}\{F(\theta)-F(\tau)\},$$

then the cylinder is a possible form of equilibrium of liquid rotating, under the influence of its own attraction, with angular velocity ω.

In particular, the hypotrochoids of equation (2), § 4, are possible figures of equilibrium if $k=0$ and

$$\frac{\omega^2}{2\pi\rho}=\frac{n-1}{n}\left\{1-(n-1)\frac{b^2}{a^2}\right\}.$$

Thus for any given positive integral value of n, and for values of ϵ varying from 0 to 1, the hypotrochoids

$$x\sqrt{n(n-\epsilon^2)}=c(n\cos\eta+\epsilon\cos n\eta),$$
$$y\sqrt{n(n-\epsilon^2)}=c(n\sin\eta-\epsilon\sin n\eta),$$
$$[\text{with}\quad \omega^2/2\pi\rho=(n-\epsilon^2)/(n+1)]$$

form a linear series of figures of equilibrium (unstable if $n>1$) passing out of the circle of radius c. The case of $n=1$ is that of the elliptic cylinders, which are stable if $\epsilon<\frac{1}{2}$, the bifurcating ellipse being that for which $\epsilon=\frac{1}{2}$. The case of $n>1$ is that of the hypotrochoid which passes, as ϵ increases to unity, into the $n+1$-cusped hypocycloid, after which fluid escapes at the cusps, as is easily verified in any particular case.

For example, take

$$x+\iota y=a\,e^{\xi+\iota\eta}+b\,e^{-(\xi+\iota\eta)}+c\,e^{-2(\xi+\iota\eta)},$$

and therefore

$$\tfrac{1}{2}(X_1{}^2+X_2{}^2+Y_1{}^2+Y_2{}^2)=a^2+b^2+c^2+2bc\cosh\xi\cos\eta+2ab\cosh 2\xi\cos 2\eta+2ca\cosh 3\xi\cos 3\eta,$$

$$\frac{1}{2\iota}\int_\tau^\theta(XY'-X'Y)d\eta=(a^2-b^2-2c^2)\xi-3bc\sinh\xi\cos\eta-\tfrac{1}{3}ca\sinh 3\xi\cos 3\eta.$$

* By taking the surface-condition in this form we are really making use of Poincaré's theorem that there is a plane of symmetry ($y=0$) through the axis.

Thus

$$V_0 = -2\pi\rho(a^2-b^2-2c^2)\xi - \omega^2(bc\,e^{-\xi}\cos\eta + ab\,e^{-2\xi}\cos 2\eta + ca\,e^{-3\xi}\cos 3\eta) + k\omega^2\{(a+b)e^{-\xi}\cos\eta + c\,e^{-2\xi}\cos 2\eta\}.$$

Hence

$$\frac{\partial V_0}{\partial\xi} - \iota\frac{\partial V_0}{\partial\eta} = -2\pi\rho(a^2-b^2-2c^2) - \omega^2\{[bc-k(a+b)]e^{-(\xi+\iota\eta)} + 2(ab-kc)\,e^{-2(\xi+\iota\eta)} + 3ca\,e^{-3(\xi+\iota\eta)}\},$$

and this must vanish at the singular points, where

$$a - b\,e^{-2(\xi+\iota\eta)} - 2c\,e^{-3(\xi+\iota\eta)} = 0.$$

Consequently we have

$$k = bc/(a+b), \quad 4c^2 = a(a+b),$$
$$\omega^2/2\pi\rho = (a+b)(a-2b)/3a^2. \quad . \quad . \quad . \quad . \quad (7)$$

Thus the curve

$$x = (a+b)\cos\eta + \tfrac{1}{2}a\sqrt{(1+b/a)}\cos 2\eta,$$
$$y = (a-b)\sin\eta - \tfrac{1}{2}a\sqrt{(1+b/a)}\sin 2\eta,$$

which is the three-cusped hypocycloid when $b=0$, is a possible form of rotating figure of equilibrium, provided that ω is given by (7). But if $b>0$ the curve possesses loops, and it is therefore not a proper solution, but must be regarded as indicating that as the angular velocity diminishes the fluid escapes at the cusps of the hypocycloid.

More general cases of figures of equilibrium of this type may be found without difficulty, but there is no great interest in carrying on the investigation as all the figures so obtainable are, with the exception of the ellipse, unstable.

Problems of Elastic Equilibrium.

12. It is also possible to obtain solutions of certain problems of elastic equilibrium, namely, the torsion problem *, the flexure problem †, the problem of plane strain for a cylinder bent by its own weight ‡, and the approximate theory of the equilibrium of a plane plate clamped or supported at the edge §. But, except in those cases in which the solution is well known, the analysis is tedious, and the results do not appear to be of sufficient interest to repay the labour of investigation.

* Love, 'Elasticity' (second edition) p. 301, §§ 217–8. This is merely the hydrodynamical problem of fluid in a rotating cylinder (with $\omega=-1$).

† *Loc. cit.* p. 317, § 229.

‡ *Loc. cit.* p. 347, § 244.

§ *Loc. cit.* p. 465, § 313.

Using the notation of the section last quoted, consider as an example the problem of the bending of a plate by its own weight, the edge being clamped in a horizontal plane. We have, in this case,

$$\nabla^4 w = Z'/D = W/AD = 64\Omega, \text{ say,}$$

where W is the weight and A is the area of the plate, with the conditions

$$w = \partial w/\partial \xi = 0$$

at the edge $\xi = 0$. The general solution is

$$w = \Omega\left\{(x^2+y^2)^2 - \tfrac{1}{2}(X_1^2+Y_1^2)^2 - \tfrac{1}{2}(X_2^2+Y_2^2)^2 + 2\iota\int_\tau^\theta (X^2+Y^2)(XY-X'Y)d\eta\right\}$$
$$-\frac{1}{2\iota}\left\{\int_\tau^\theta [XF'(\eta)+YG'(\eta)]d\eta + x[F(\theta)-F(\tau)] + y[G(\theta)-G(\tau)]\right\},$$

where F and G are functions to be determined from the conditions of finiteness, &c.

A sufficient illustration will be furnished by the consideration of the case of the elliptic boundary, for which we may evidently take

$$F(\theta) = \Omega(A\sin\theta + B\sin 3\theta),\quad G(\theta) = -\Omega(C\cos\theta + E\cos 3\theta),$$

where A, B, C, & E are constants which may be determined by equating to zero the coefficient of ξ—since ξ becomes infinite at the centre when the ellipse is a circle—and the value of $\partial w/\partial\xi$ when $\xi = -\lambda$. The equations thus obtained lead to

$$A = 2b(a^2+b^2),\quad C = 2a(a^2+b^2),$$

$$B = \frac{2b(3a^2+b^2)(a^4-b^4)}{3a^4+2a^2b^2+3b^4},\quad E = \frac{2a(a^2+3b^2)(a^4-b^4)}{3a^4+2a^2b^2+3b^4}.$$

On substituting these values in the expression for w, we obtain after reduction the—otherwise obvious—result,

$$w = (W/8\pi D)a^3b^3\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2 \Big/ (3a^4+2a^2b^2+3b^4).$$

The solution of the problem here given is, of course, to be regarded merely as an illustrative example of the general process.