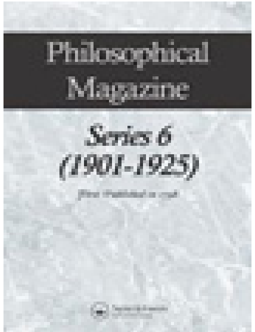

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LXX. Figures of equilibrium of rotating fluid under the restriction that the figure is to be a surface of revolution

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LXX. *Figures of Equilibrium of Rotating Fluid under the restriction that the Figure is to be a Surface of Revolution.*

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THE following paper was begun rather more than a year ago. It was then put aside until a more convenient season, and now, owing to other work which I have undertaken, its completion has been rendered impossible for a long time to come. I believe, however, that there is some interest in the paper as it stands, and I venture to publish it in its unfinished state. There is a great deal of heavy, but straightforward, arithmetic required to complete it. It will be seen that the whole paper presents very striking analogies with that of Mr. J. H. Jeans on the "Equilibrium of Rotating Liquid Cylinders"†.

In the case of a surface of revolution the potential can be very simply written down without a knowledge of the form of the surface. For the potential of a uniform circular disk at a point on its axis is

$$V=2\pi\rho\, d\zeta\,(\sqrt{\varpi^2+z^2}-z),$$

where $d\zeta$ is the thickness, ϖ the radius, ρ the density of the disk, and z is the distance of the point where the potential is measured from the plane of the disk;—if z is negative its sign must be changed in the above expression. Whence it follows that the potential of a surface of revolution at a point on its axis, within the surface, is

$$V=2\pi\rho\left\{\int^{z}[\sqrt{\varpi^2+(z-\zeta)^2}-(z-\zeta)]\,d\zeta+\int_{z}[\sqrt{\varpi^2+(z-\zeta)^2}+(z-\zeta)]\,d\zeta\right\},$$

ϖ being a function of ζ; and the potential at any point within the surface is therefore, by a well-known theorem,

$$V=2\rho\int_0^{\pi}\left\{\int[\varpi^2+(z-\zeta+\iota R\cos\lambda)^2]^{\frac{1}{2}}\,d\zeta-\int^{z}(z+\iota R\cos\lambda-\zeta)\,d\zeta+\int_{z}(z+\iota R\cos\lambda-\zeta)\,d\zeta\right\}d\lambda,$$

z, R being the cylindrical coordinates of the point.

* Communicated by the Author.

† Phil. Trans. A. cc. (1902) pp. 67–104.

Excluding the case where the surface extends to infinity, we may change the order of integration in this result, and obtain, after a slight reduction,

$$V=2\rho\int_{-a}^{a}d\zeta\int_{0}^{\pi}d\lambda\sqrt{\varpi^2+(z-\zeta+\iota R\cos\lambda)^2}-2\pi\rho(a^2+z^2),$$

where it is assumed that the origin is the middle point of the axis, and that $\varpi=0$ when $\zeta=\pm a$.

The equation which has to be satisfied on the surface, namely

$$V+\tfrac{1}{2}\omega^2R^2=\text{constant},$$

ω being the angular velocity, is thus

$$\frac{1}{\pi}\int_{-a}^{a}d\zeta\int_{0}^{\pi}d\lambda\sqrt{\varpi^2+(z-\zeta+\iota R\cos\lambda)^2}=C+z^2-\frac{1}{2}\frac{\omega^2}{2\pi\rho}R^2,$$

or in spherical polar coordinates,

$$\frac{1}{\pi}\int_{-1}^{1}dx\int_{0}^{\pi}d\lambda\,\frac{d}{dx}(rx)\,\sqrt{r^2(1-x^2)+(z+\iota R\cos\lambda-rx)^2}$$
$$=C+\tfrac{1}{2}r^2\left\{\left(2+\frac{\omega^2}{2\pi\rho}\right)x^2-\frac{\omega^2}{2\pi\rho}\right\},$$

where r is the distance from the origin of a point on the surface, and x is the cosine of the angle which r makes with the axis.

If we put $z+\iota R\cos\lambda=\xi$, we have $|\,\xi\,|=\sqrt{z^2+R^2\cos^2\lambda}$, which is less than $\sqrt{z^2+R^2}$, *i. e.* less than r, since z, R is a point on the surface. Hence we may expand the left-hand side in powers of ξ/r. We have then, putting

$$\frac{1}{3}\left(1-\frac{\omega^2}{2\pi\rho}\right)=\nu,$$

$$C+\nu r^2+(1-\nu)r^2P_2(x)=\frac{1}{\pi}\int_{-1}^{1}dx\int_{0}^{\pi}d\lambda\frac{d}{dx}(rx)\sqrt{r^2-2r\xi x+\xi^2}$$

$$=\frac{1}{\pi}\int_{-1}^{1}dx\int_{0}^{\pi}d\lambda\frac{d}{dx}(rx)\left\{1-\frac{\xi}{r}P_1+\sum_{1}^{\infty}\frac{1}{n+1}(P_{n-1}-P_nx)\left(\frac{\xi}{r}\right)^{n+1}\right\}r$$

$$=\int_{-1}^{1}r\frac{d}{dx}(rx)\,dx-\frac{1}{\pi}\int_{0}^{\pi}\xi d\lambda\int_{-1}^{1}x\frac{d}{dx}(rx)\,dx$$
$$+\sum_{1}^{\infty}\frac{1}{\pi}\int_{0}^{\pi}\xi^{n+1}\,d\lambda\int_{-1}^{1}\frac{1}{n+1}(P_{n-1}-xP_n)\frac{d}{dx}(rx)\,\frac{dx}{r^n},$$

where P_n denotes the nth zonal harmonic, the argument being x.

Since z, R is on the surface,

$$\xi = r(x + \iota \cos\lambda \sqrt{1-x^2}),$$

and

$$\frac{1}{\pi}\int_0^\pi \xi^n\, d\lambda = r^n \mathrm{P}_n(x).$$

Whence

$$\mathrm{C} + \nu r^2 + (1-\nu) r^2 \mathrm{P}_2 = \int_{-1}^1 r\frac{d}{dx}(rx)dx - r\mathrm{P}_1 \int_{-1}^1 x\frac{d}{dx}(rx)\,dx$$
$$+ \sum_1^\infty r^{n+1}\mathrm{P}_{n+1}\int_{-1}^1 \frac{1}{n+1}(\mathrm{P}_{n-1} - x\mathrm{P}_n)\frac{d}{dx}(rx)\frac{dx}{r^n}.$$

We might, owing to a result of Poincaré's*, assume at once that r is an even function of x, since there must be a plane of symmetry perpendicular to the axis of rotation. It will, however, add but little complexity to the work if we do not make this assumption; we shall thus obtain an independent proof of the theorem in this particular case.

By integration by parts it is readily seen that

$$\int_{-1}^1 \frac{1}{n+1}(\mathrm{P}_{n-1} - x\mathrm{P}_n)\frac{d}{dx}(rx)\frac{dx}{r^n} = -\int_{-1}^1 \frac{1}{n-1}\mathrm{P}_{n+1}\frac{dx}{r^{n-1}},$$

unless $n=1$, in which case

$$\int_{-1}^1 \frac{1}{2}(\mathrm{P}_0 - x\mathrm{P}_1)\frac{d}{dx}(rx)\frac{dx}{r} = \frac{2}{3} + \int_{-1}^1 \mathrm{P}_2 \log r\, dx.$$

Hence the equation which must be satisfied on the surface may be put into the form†

$$\mathrm{C}' + \nu r^2 + (\tfrac{1}{3} - \nu) r^2 \mathrm{P}_2$$
$$+ \sum_0^{\infty}{}' \frac{r^n \mathrm{P}_n}{n-2}\int_{-1}^1 \frac{\mathrm{P}_n dx}{r^{n-2}} - r^2\mathrm{P}_2\int_{-1}^1 \mathrm{P}_2 \log r\, dx = 0, \quad (1)$$

where the accent attached to the Σ means that the term for which $n=2$ is to be omitted. If we suppose that the equation of the surface of equilibrium may be expressed in the form

$$r^2 = \alpha_0 + \alpha_1 r\mathrm{P}_1 + \alpha_2 r^2 \mathrm{P}_2 + \ldots \alpha_n r^n \mathrm{P}_n + \ldots, \quad . \quad (2)$$

* *Acta Mathematica*, vii. p. 331.

† The equation in this form may be obtained more directly and more elegantly by forming the potential of a circular ring, and expanding *before* integration with regard to ϖ. The only advantage of the procedure adopted above is that it avoids the discussion of a nice, though not at all difficult, question which arises in the course of the direct method.

we have, on substituting for νr^2 in equation (1), and equating the coefficients of the various zonal harmonics to zero, the system of equations

$$\left.\begin{aligned} \frac{1}{2}\int_{-1}^{1} r^2\,dx &= C' + \nu\alpha_0, \\ \int_{-1}^{1} r P_1\,dx &= \nu\alpha_1, \\ \int_{-1}^{1} P_2 \log r\,dx = \nu\alpha_2 + (\tfrac{1}{3}-\nu) &= \nu\alpha_2 + \frac{1}{3}\frac{\omega^2}{2\pi\rho}, \\ \text{and, if } n > 2, \qquad & \\ \int_{-1}^{1} P_n \frac{dx}{r^{n-2}} + (n-2)\,\nu\alpha_n &= 0. \end{aligned}\right\} \quad . \quad \text{(A)}$$

The first of these equations merely determines the value of the constant in the equation $V + \frac{1}{2}\omega^2 R^2 = \text{const.}$, and may be discarded. The others determine the form of the surface of equilibrium.

2. We know beforehand that a particular solution of equations (A) will be the series of Maclaurin spheroids, for which $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = \ldots = 0$, and it is easy to verify that these values do, in fact, satisfy the equations. The value of α_2 is not, however, arbitrary, but is connected with that of $\omega^2/2\pi\rho$ by the equation

$$\int_{-1}^{1} P_2 \log r\,dx = \nu\alpha_2 + \frac{1}{3}\frac{\omega^2}{2\pi\rho} \quad . \quad . \quad . \quad . \quad (3)$$

It will be convenient to put

$$\alpha_2 = (-2k^2 + 3a_2)/(3+k^2), \quad \text{and}$$
$$\alpha_n = 3a_n/(3+k^2), \qquad (n \neq 2)$$

so that equation (2) takes the form

$$r^2(1+k^2x^2) = a_0 + a_1 r P_1 + a_2 r^2 P_2 + \ldots a_n r^n P_n + \ldots, \quad (2')$$

where in the case of Maclaurin's spheroid $a_n = 0$, if $n \neq 0$.

Equation (3) then becomes (since for the spheroid $a_2 = 0$)

$$\int_{-1}^{1} P_2 \log r\,dx = \frac{1}{3}\frac{\omega^2}{2\pi\rho} - \frac{2k^2\nu}{3+k^2},$$

or, if $K = \nu/(1+\frac{1}{3}k^2)$,

$$\frac{1}{3}\frac{\omega^2}{2\pi\rho} - \frac{2}{3}k^2 K = -\int_0^1 P_2 \log(1+k^2x^2)\,dx$$
$$= \frac{1}{3} - \frac{1+k^2}{k^3}(k - \tan^{-1} k),$$

whence
$$K=\frac{k-\tan^{-1}k}{k^3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$
or
$$\frac{\omega^2}{2\pi\rho}=\frac{3+k^2}{k^3}\tan^{-1}k-\frac{3}{k^2},$$
which is the familiar result for the spheroids*.

Incidentally we see that we may re-write equations (A) in the, for subsequent purposes, rather more convenient form

$$\left.\begin{aligned}&\int_{-1}^{1} rP_1\,dx=Ka_1,\\ &\frac{1}{2}\int_{-1}^{1} P_2\log\left[r^2(1+k^2x^2)\right]dx=Ka_2+(1+k^2)(K_0-K),\\ &\text{and, } n>2,\\ &\int_{-1}^{1} P_n\frac{dx}{r^{n-2}}+(n-2)\,Ka_n=0,\end{aligned}\right\}\quad (B)$$

where K_0 is the value of K for the spheroid.

3. To determine points of bifurcation on the series of spheroids we need retain only the first powers of a_1, a_2, &c. in equations (B), which we shall suppose, for the moment, to be denoted by

$$Ka_n=f_n(a_1, a_2, a_3 \ldots\ldots) \qquad [n=1, 2, \ldots\ldots$$

Points of bifurcation will then be determined by the equation

$$\begin{vmatrix}\frac{\partial f_1}{\partial a_1}-K & \frac{\partial f_1}{\partial a_2} & \frac{\partial f_1}{\partial a_3} & \cdot\;\cdot\;\cdot\;\cdot \\ \frac{\partial f_2}{\partial a_1} & \frac{\partial f_2}{\partial a_2}-K & \frac{\partial f_2}{\partial a_3} & \cdot\;\cdot\;\cdot\;\cdot \\ \frac{\partial f_3}{\partial a_1} & \frac{\partial f_3}{\partial a_2} & \frac{\partial f_3}{\partial a_3}-K & \cdot\;\cdot\;\cdot \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{vmatrix}=0. \quad (5)$$

Putting $a_0=1$, as we may with no loss of generality, retaining only the first powers of $a_1, a_2 \ldots$, and denoting

$$P_n/(1+k^2x^2)^{\frac{1}{2}n}$$

* If $k=\tan\alpha$, the eccentricity of the spheroid is $\sin\alpha$.

by the single letter R_n, equations (B) become

$$Ka_1=\int_{-1}^{1} R_1[1+\tfrac{1}{2}(a_1R_1+a_2R_2+\dots a_nR_n+\dots)]\,dx,$$

$$Ka_2+(1+k^2)(K_0-K)=\frac{1}{2}\int_{-1}^{1} P_2(a_1R_1+a_2R_2+\dots a_nR_n+\dots)\,dx,$$

and, $n>2$,

$$Ka_n=\frac{1}{2}\int_{-1}^{1} P_n(1+k^2x^2)^{\frac{1}{2}n-1}\Big(-\frac{2}{n-2}+a_1R_1+\dots a_nR_n+\dots\Big)dx.$$

In these equations every integral containing an odd power of $(1+k^2x^2)^{\frac{1}{2}}$ vanishes, and they plainly reduce to

$$Ka_1=\int_0^1 R_1(a_1R_1+a_3R_3+a_5R_5+\dots)dx,$$

$$Ka_2+(1+k^2)(K_0-K)=\int_0^1 P_2(a_2R_2+a_4R_4+a_6R_6+\dots)\,dx,$$

and, $n>2$,

$$Ka_n=\int_0^1 P_n(1+k^2x^2)^{\frac{1}{2}n-1}(a_nR_n+a_{n+2}R_{n+2}+\dots)\,dx.$$

It follows that the Hessian (equation 5) reduces to the product of the terms of the leading diagonal, and the points of bifurcation are given by the system of equations

$$(n\neq 2)\qquad K_0=\int_0^1 \frac{P_n^2\,dx}{1+k^2x^2}, \quad \dots\dots \quad (6)$$

in which K_0 has been written for K, because the points of bifurcation belong to the series of spheroids.

We proceed to show that equation (6), considered as an equation to determine k, has one and only one solution if n is even, and no solution if n is odd, unless $n=1$, in which case it is an identity.

Putting $z=1/k^2$, and remembering equation (4), we see that equation (6) may be put into the form

$$1-\sqrt{z}\cot^{-1}\sqrt{z}=\int_0^1 \frac{P_n^2\,dx}{x^2+z}. \quad \dots \quad (7)$$

To solve this equation in z we consider the two curves

$$y=1-\sqrt{z}\cot^{-1}\sqrt{z}, \quad \dots\dots \quad (8)$$

$$y=\int_0^1 \frac{P_n^2\,dx}{x^2+z}, \quad \dots\dots \quad (9)$$

in which y is the ordinate, z the abscissa.

Both curves plainly asymptote the z axis, and when z is large the curve (9) is below the curve (8) unless $n=1$, for when z is large the equations are

$$y=1-\sqrt{z}\left(\frac{1}{\sqrt{z}}-\frac{1}{3z\sqrt{z}}+\dots\right)=\frac{1}{3z},$$

and
$$y=\frac{1}{z}\int_0^1 P_n{}^2\,dx=\frac{1}{(2n+1)z}.$$

Moreover, both curves continually descend from $z=0$ to ∞, and both are convex to the axis of z, for in both $\frac{dy}{dz}$ is negative, and $\frac{d^2y}{dz^2}$ positive, for all positive values of z.

Since, further, both $\frac{dy}{dz}$ and $\frac{d^2y}{dz^2}$ are continuous functions of z from $z=0$ to ∞, it follows that there cannot be more than one real solution of equation (7) between 0 and ∞. Again, when $z=0$, $y=1$ on curve (8), and on curve (9) y is infinite if n is even, but is unity if n is odd, for

$$\begin{aligned}
\int_0^1 P^2_{2n-1}\frac{dx}{x^2} &= -\frac{1}{2}\int_{-1}^1 P^2_{2n-1}\,d\left(\frac{1}{x}\right)\\
&= -\frac{1}{2}\left[\frac{P^2_{2n-1}}{x}\right]_{-1}^1+\int_{-1}^1 P_{2n-1}P'_{2n-1}\frac{dx}{x}\\
&= -1+\left[\frac{P_{2n-1}}{x}\right]_{-1}^1-\int_{-1}^1 P_{2n-1}\frac{d}{dx}\left(\frac{P_{2n-1}}{x}\right)dx\\
&= 1-\int_{-1}^1 P_{2n-1}\times(\text{a rational integral expression of degree } 2n-3)\,dx\\
&= 1.
\end{aligned}$$

Hence there is one and only one root of equation (6) if n is even, and no root if n is odd, unless $n=1$, in which case it will be found that (7) is an identity.

The proof further shows that two of the curves (9), for different values of n, cannot intersect. For when z is small the difference of the ordinates is $\int_0^1\left(P^2_{2m}-P^2_{2n}\right)\frac{dx}{x^2}$, which is positive (infinite) if n is greater than m; and when z is large the difference is $\left(\frac{1}{2m+1}-\frac{1}{2n+1}\right)\frac{1}{z}$, which also is positive.

We have thus proved that that member of the system of equations (6) which has the smallest root* is found by taking $n=4$, and therefore that the first point of bifurcation on the series of spheroids is given by the positive root of the equation

$$\mathrm{K}_0=\int_0^1 \frac{\mathrm{P}_4{}^2 dx}{1+k^2x^2},$$

or, putting $k=\tan\alpha$, $c=\cot\alpha$,

$$1-\alpha c=\int_0^1 \frac{\mathrm{P}_4{}^2 dx}{x^2+c^2},$$

which on simplification becomes

$$c\,(119+655\,c^2+\frac{5075}{3}c^4+1225\,c^6)$$
$$-\alpha(1+c^2)(9+235\,c^2+875\,c^4+1225\,c^6)=0. \quad (10)$$

I find that the solution of this equation is

$$\alpha=80^\circ\ 8'\ 19''\cdot 7$$
$$=1\cdot 3986858\,.\,.\text{, in circular measure.}$$

Whence we obtain for the bifurcating spheroid,

$$\frac{ae}{c}=\tan\alpha=5\cdot 7528,$$

$$\frac{a}{c}=\sec\alpha=5\cdot 8391,$$

$$\frac{c}{a}=\cos\alpha=\cdot 17126,$$

$$e=\sin\alpha=\cdot 985226,$$

$$\frac{\omega^2}{2\pi\rho}=\cdot 17452,$$

where c is the length of the semi-axis of rotation, and a is the radius of the circular section through the centre of the spheroid.

3*a*. It is of interest to compare the bifurcation equations with those given by Poincaré (*Acta Mathematica*, vii. pp. 319 and 329, 1885). Poincaré gives no numerical results, but

* It is, at first sight, not quite evident why we should not take $n=2$. But if, with this value of K_0, we assume an expression of the form (D) for r, we shall find that all the coefficients vanish, except that of $r^2\mathrm{P}_2$; *i. e.* the surface is still a spheroid. It is in fact the spheroid for which ω is a maximum.

his equations for the bifurcation of the spheroids into surfaces of revolution may be stated in the following terms :—

Let R_n be a polynomial in c of degree n, satisfying the equation

$$(1+c^2)\frac{d^2R_n}{dc^2}+2c\frac{dR_n}{dc}-n(n+1)R_n=0,$$

and such that $R_n(\iota)=\iota^n$. Let S_n be a second solution of the equation, connected with R_n by the relation

$$S_n=R_n\int_c^\infty \frac{dc}{R_n^2(1+c^2)}.$$

Then the equations of bifurcation * are

$$R_1S_1=R_{2n}S_{2n},$$

and of these the one with the smallest root is that for which $n=2$. When written at full length this equation is

$$(1-\alpha c)\,c=\alpha R_4^2-cR_4\left(\frac{55}{24}+\frac{35}{8}c^2\right),$$

where $R_4=\frac{1}{8}\,[3+30\,c^2+35\,c^4]$,

i.e. $$\alpha(R_4^2+c^2)=\frac{c}{64}\left(119+655\,c^2+\frac{5075}{3}\,c^4+1225\,c^6\right),$$

which reduces to equation (10) above.

4. For the sake of conciseness we shall speak of the series of figures of equilibrium into which the spheroids pass as "pear-shaped figures," or simply "pears," although the original significance of the name is here entirely lost.

To determine the form of the pear-shaped figure we re-write equation (2′) in the form

$$r^2(1+k^2x^2)=a+\sum_{s=1}^{\infty}\theta^s\sum_{n=0}^{\infty}a_{sn}\,r^n\,P_n \quad . \quad . \quad \text{(D)}$$

and we also put

$$K=K_0+K_1\theta+K_2\theta^2+\ldots, \quad . \quad . \quad . \quad (11)$$

where θ is a parameter which vanishes for the particular pear which is also the bifurcating spheroid. Before substituting this value of r in equations (B) put

$$R_{\frac{1}{2}n}=P_n/(1+k^2x^2)^{\frac{1}{2}n},$$

* Comparison of the two forms of the bifurcation equations shows that

$$\int_0^1\frac{P^2_{2n}\,dx}{x^2+c^2}=R_{2n}S_{2n}/c,$$

a result which it does not seem easy to obtain directly, although it is immediately reducible to the simpler form,

$$\int_0^1\frac{P_{2n}\,dx}{x^2+c^2}=(-)^nS_{2n}/c.$$

so that the above equation becomes

$$r^2(1+k^2x^2)=a+\sum_{s=1}^{\infty}\theta^s\sum_{n=0}^{\infty}a_{sn}\left[r(1+k^2x^2)^{\frac{1}{2}}\right]^n\mathrm{R}_{\frac{1}{2}n}$$

$$=a+\sum_{n=0}^{\infty}\left[r^2(1+k^2x^2)\right]^{\frac{1}{2}n}\mathrm{R}_{\frac{1}{2}n}\sum_{s=1}^{\infty}a_{sn}\theta^s.$$

Whence, by Lagrange's expansion, we obtain

$$\mathrm{F}[r^2(1+k^2x^2)]=\mathrm{F}(a)+\sum_{m=1}^{\infty}\frac{1}{m!}\frac{d^{m-1}}{da^{m-1}}\left\{\left[\sum_{n=0}^{\infty}a^{\frac{1}{2}n}\mathrm{R}_{\frac{1}{2}n}\sum_{s=1}^{\infty}a_{sn}\theta^s\right]^m\mathrm{F}'(a)\right\},$$

so that equations (B) become

$$\mathrm{K}\sum_{s=1}^{\infty}a_{sn}\theta^s=\frac{1}{2}\int_{-1}^{1}\mathrm{P}_n(1+k^2x^2)^{\frac{1}{2}n-1}\sum_{m=1}^{\infty}\frac{1}{m!}\frac{d^{m-1}}{da^{m-1}}\left\{\left[\sum_{p=0}^{\infty}a^{\frac{1}{2}p}\mathrm{R}_{\frac{1}{2}p}\sum_{s=1}^{\infty}a_{sp}\theta^s\right]^m a^{-\frac{1}{2}n}\right\}dx$$

unless $n=2$, in which case the term $(1+k^2)(\mathrm{K}_0-\mathrm{K})$ must be added to the left-hand side of the equation.

These equations must be satisfied for all values of θ, that is to say the coefficients of the various powers of θ on the two sides of an equation are to be equated; but it must be remembered that K is a function of θ which is determined by the equation for which $n=2$. Further, since the middle point of the axis has been taken as origin, the value of r when $x=1$ is equal to its value when $x=-1$; and it will be found on substitution that the coefficients of all the odd harmonics must be zero. We have thus proved the symmetry of the pear about a plane perpendicular to the axis of revolution.

We may assume that the equation of the pear is, using a slightly different and rather more convenient notation,

$$\begin{aligned}r^2(1+k^2x^2)=a&+\theta\,(a_{10}+a_{11}r^2\mathrm{P}_2+a_{12}r^4\mathrm{P}_4)\\&+\theta^2(a_{20}+a_{22}r^4\mathrm{P}_4+a_{23}r^6\mathrm{P}_6)\\&+\theta^3(a_{30}+a_{32}r^4\mathrm{P}_4+a_{33}r^6\mathrm{P}_6+a_{34}r^8\mathrm{P}_8)\\&+\text{\&c.},\qquad\ldots\ldots\ldots\qquad(12)\end{aligned}$$

where use has been made of the fact that harmonics of order higher than $2s+2$ cannot occur in the coefficient of θ^s. This fact may readily be proved; its truth will appear in the course of the work of determining the coefficients. We have also determined θ (except its scale of measurement) by taking it proportional to the coefficient of $r^2\mathrm{P}_2$.

Equations (B) may now be written

$$(1+k^2)(\mathrm{K}_0-\mathrm{K})+\mathrm{K}a_{11}\theta=\int_0^1\mathrm{P}_2\log\left[r^2(1+k^2x^2)\right]dx,$$

and, $n > 1$,

$$(n-1)K \sum_{n-1}^{\infty} a_{sn}\theta^s + \int_0^1 P_{2n}\, dx/r^{2n-2} = 0,$$

which, on making use of Lagrange's expansion, become

$$K \sum_{s=n-1}^{\infty} a_{sn}\theta^s = \int_0^1 P_{2n}(1+k^2x^2)^{n-1} \sum_{m=1}^{\infty} \frac{1}{m!}\frac{d^{m-1}}{da^{m-1}}\left\{\left[\sum_{s=1} \theta^s \sum_{p=0}^{s+1} a_{sp}a^p R_p\right]^m a^{-n}\right\} dx, \quad \ldots \quad (C)$$

for all positive integral values of n, except that when $n=1$ the left-hand side is $(1+k^2)(K_0-K)+Ka_{11}\theta$.

To equations (C) must be added the condition of constant mass, *i. e.* of constant volume, which may be expressed by saying that $\int_0^1 r^3\, dx$ must be independent of θ, *i. e.*

$$\int_0^1 \frac{dx}{(1+k^2x^2)^{\frac{3}{2}}} \sum_{m=1}^{\infty} \frac{1}{m!}\frac{d^{m-1}}{da^{m-1}}\left\{\left[\sum_{s=1}^{\infty} \theta^s \sum_{p=0}^{s+1} a_{sp} R_p a^p\right]^m a^{\frac{3}{2}}\right\} = 0,$$

for all values of θ. $\ldots \quad (C')$

5. To determine the stability of the pear we must examine the expression for the angular momentum, which must be stationary, and for stability a minimum, when $\theta=0$. That is to say, we must retain squares of θ, but need not retain higher powers in the equations (C) and (C'). They then become

$$(1+k^2)(K_0-K)+Ka_{11}\theta = \theta\int_0^1 P_2\left\{a_{11}R_1+a_{12}R_2+\theta\Big(a_{22}R_2+a_{23}R_3 + a_{10}a_{12}R_2+\frac{1}{2}a_{11}{}^2R_1{}^2+2a_{11}a_{12}R_1R_2+\frac{3}{2}a_{12}{}^2R_2{}^2\Big)\right\}dx, \quad (13)$$

$$K(a_{12}\theta+a_{22}\theta^2) = \theta\int_0^1 P_4(1+k^2x^2)\left\{a_{12}R_2+\theta(a_{22}R_2+a_{23}R_3 + a_{11}a_{12}R_1R_2+a_{12}{}^2R_2{}^2)\right\}dx, \quad \ldots \quad (14)$$

$$Ka_{23}\theta^2 = \theta^2\int_0^1 P_6(1+k^2x^2)^2\left\{a_{23}R_3+\frac{1}{2}a_{12}{}^2R_2{}^2\right\}dx, \quad \ldots \quad (15)$$

$$\int_0^1 \frac{dx}{(1+k^2x^2)^{\frac{3}{2}}}(a_{10}+a_{11}R_1+a_{12}R_2) = 0, \quad \ldots \quad (16)$$

$$\int_0^1 \frac{dx}{(1+k^2x^2)^{\frac{3}{2}}}\Big(a_{20}+a_{22}R_2+a_{23}R_3+\frac{1}{4}a_{10}{}^2+\frac{3}{2}a_{10}a_{11}R_1+\frac{5}{2}a_{10}a_{12}R_2 + \frac{5}{4}a_{11}{}^2R_1{}^2+\frac{7}{2}a_{11}a_{12}R_1R_2+\frac{9}{4}a_{12}{}^2R_2{}^2\Big) = 0. \quad \ldots \quad (17)$$

In these equations we have put $a=1$, as may be done without loss of generality.

The moment of inertia is proportional to

$$\int_0^1 r^5(1-x^2)\,dx,$$

i. e., as far as θ^2, to

$$\int_0^1 \frac{1-x^2}{(1+k^2x^2)^{\frac{5}{2}}}dx\left\{1+\frac{5}{2}\theta(a_{10}+a_{11}R_1+a_{12}R_2)+\frac{5}{2}\theta^2\Big[a_{20}+a_{22}R_2\right.$$
$$+a_{23}R_3+\frac{1}{4}(3a_{10}{}^2+10a_{10}a_{11}R_1+14a_{10}a_{12}R_2+7a_{11}{}^2R_1{}^2$$
$$\left.+18a_{11}a_{12}R_1R_2+11a_{12}{}^2R_2{}^2)\Big]\right\}.$$

Also ω, the angular velocity, is proportional to

$$1+\frac{3+k^2}{2\dfrac{\omega_0{}^2}{2\pi\rho}}(K_0-K)-\frac{(3+k^2)^2}{8\left(\dfrac{\omega_0{}^2}{2\pi\rho}\right)^2}(K_0-K)^2+\ldots.$$

The angular velocity will be stationary for $\theta=0$, so that (from equation (13)),

$$a_{11}\left(K_0-\int_0^1 P_2R_1dx\right)=a_{12}\int_0^1 P_2R_2dx.$$

This value of a_{11} makes the moment of inertia, and therefore also the angular momentum, stationary for $\theta=0$; for equation (16) may be written

$$a_{10}-\frac{1}{3}\sin^2\alpha \quad a_{11}+\frac{1}{5}\sin^4\alpha \quad a_{12}=0;$$

and the coefficient of θ in the expression for the moment of inertia is proportional to

$$\int_0^1\frac{dx}{(1+k^2x^2)^{\frac{5}{2}}}(a_{10}+a_{11}R_1+a_{12}R_2),$$

i. e. to

$$\cos\alpha\left(1-\frac{1}{3}\sin^2\alpha\right)a_{10}-\frac{1}{15}\cos\alpha\sin^2\alpha\,(4+3\cos^2\alpha)\,a_{11}$$
$$+\frac{1}{35}\cos\alpha\sin^4\alpha\,(6+5\cos^2\alpha)\,a_{12},$$

i. e. to

$$a_{11}-\frac{6}{7}\sin^2\alpha\,a_{12},$$

where we have divided out by $1-\frac{2}{3}\sin^2\alpha$ and by $\cos\alpha\sin^2\alpha$.

The coefficient of θ in the angular momentum is thus proportional to

$$7\int_0^1 P_2R_2dx(1+c^2)+6\left(\int_0^1 P_2R_1dx-K_0\right).$$

But $$K_0=(1-\alpha c)c^2;$$

$$\int_0^1 P_2R_1dx=\frac{1}{4}\alpha c(1+3c^2)^2-\frac{3}{4}c^2(1+3c^2);$$

$$\int_0^1 P_2R_2dx=\frac{1}{32}\Big[\alpha c(-3+39c^2+375c^4+525c^6)$$

$$-\cos^2\alpha(3+39c^2+125c^4+105c^6)-180c^4-420c^6\Big].$$

Therefore the coefficient of θ is proportional to

$$\alpha c(1+c^2)(9+235c^2+875c^4+1225c^6)$$
$$-119c^2-655c^4-\frac{5075}{3}c^6-1225c^8$$
$$=0,$$

by equation (10); whence it follows that the angular momentum is stationary when $\theta=0$.

Added September 18th, 1914.

It is but right to point out that while correcting the proofs of this paper I have met with a difficulty which I am at present unable to solve.

We know that ω, and therefore K, must be stationary when $\theta=0$, so that K_1 in equation (11) must be zero. In fact it is by varying the total energy, while ω remains constant, that Poincaré derives the bifurcation equations. Or, since the angular momentum and therefore the moment of inertia are stationary for $\theta=0$, we might have begun by making the coefficient of θ in the moment of inertia vanish. This, since the volume is constant, would lead us to the equation

$$7a_{11}=6a_{12}\sin^2\alpha,$$

and on substituting in (13), remembering that $K_1=0$, we should again find the bifurcation equation, in the form given in § 3*a*. But the vanishing of K_1 requires a certain relation between the coefficients of a_{11}, a_{12}, &c. in equations (13), (14), and (15), and this relation (which I had before assumed to hold) I have been unable to obtain.

It is not difficult to prove that

$$\int_0^1 \frac{P_4P_6\,dx}{(1+k^2x^2)^2} = -\frac{15}{11}\sin^2\alpha\left(K_0 - \int_0^1 \frac{P_6^2\,dx}{1+k^2x^2}\right).$$

From the coefficient of θ in (13) we have

$$a_{11} = \frac{6}{7}a_{12}\sin^2\alpha,$$

and substituting in (14) this value of a_{11} and the value of a_{23} derived from (15), we find, on account of the relation above, that the coefficient of θ^2 leads to the equation

$$a_{12}^2\left\{\int_0^1 \frac{P_4^3\,dx}{(1+k^2x^2)^3} + \sin^2\alpha\left(\frac{6}{7}\int_0^1 \frac{P_2P_4^2\,dx}{(1+k^2x^2)^2} - \frac{15}{22}\int_0^1 \frac{P_6P_4^2 dx}{(1+k^2x^2)^2}\right)\right\}$$
$$=0.$$

I find that the coefficient of a_{12}^2 is

$$\cdot 0043482 + \sin^2\alpha(-\cdot 0060458 + \cdot 0025975) = \cdot 00100.$$

Each of the three constants has been evaluated by three distinct processes, and there is substantial agreement between the different results, so that it is not easy to believe that it is merely an arithmetical error which causes this coefficient not to vanish. We thus apparently arrive at the conclusion * that $K_1 \neq 0$. I am unable to see where there can be any mistake in equations (C), and must for the present content myself with merely noting the difficulty. It is important that it should be cleared up, because further progress is impossible until it can be shown either that $K_1=0$ is in reality a consequence of the equations to determine the coefficients in equation (12), or that the reverse is true. The direct disagreement with the work of Poincaré and Darwin makes one hesitate to say definitely that $K_1 \neq 0$, but it may be remarked that the whole difficulty arises from the second order terms in equation (14), and that these terms are not considered at all in Poincaré's paper; they do not in fact affect the form of the bifurcation equations, and so long as we restrict ourselves to the consideration of first order terms we are not led to any inconsistency in supposing that $K_1=0$. It is, however, to be remembered that Darwin, in his papers on The Stability of the Pear-shaped Figure of

* It is easy to assure oneself that it will not do to suppose that $a_{10}=a_{11}=a_{12}=0$, and so to avoid this conclusion. For, if all the first order terms disappear, the bifurcation equations become illusory.

It is, further, to be observed that if $K_1 \neq 0$ the angular momentum is not stationary.

Equilibrium *, and Mr. Jeans, in the paper cited in § 1, consider second (and higher) order terms, and that they are led to no inconsistency in supposing that ω (*i. e.* K) is stationary. If it is true that the case of surfaces of revolution is exceptional and that for them $K_1 \neq 0$, then there will be considerable difficulty in the examination of the stability of these figures of equilibrium.

In conclusion I desire to express my indebtedness to the kindness of my colleague, Mr. G. H. Livens, for a careful reading of the proofs of this paper.

LXXI. *On Concentration Cells in Ionized Gases.*

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THE cause of the characteristic potential difference which exists between a metal and air with which it is in contact is not known with any certainty. It has been supposed to be connected with incipient oxidation of the metal, with the occlusion of gas in its surface, and also with the corpuscular pressure in the interior of the metal. Apart from any of these theories we may, however, draw conclusions with regard to the potential step if we may assume that the passage of electricity from metal to air is a phenomenon which can be treated as reversible in a thermodynamic scheme. Let us suppose that if a net transfer of Q units of + electricity is made from a metal to air by sending an infinitely small current for an infinitely long time through the surface, then when it has been effected, xQ units of + electricity have passed from metal to air and $(1-x)$ Q units of − electricity have passed from air to metal; and that the effect is reversible in the sense that it will be exactly annulled if now Q units of − electricity are passed. We may also make the further supposition that + and − electricity, both in the air and in the metal, exist as entities exerting pressures which are theoretically separately measurable. Then the thermodynamical scheme which follows is exactly the same as that originally worked out by Nernst in his theory of electrolytic solution pressure for the P.D. between a metal and an electrolyte, and it will follow in the same way in the metal-air case that the potential step from metal up to air must be given by the formula

$$V = \frac{RT}{\epsilon}(2x-1)\log\frac{P}{p};$$

* Phil. Trans. A. cc. (1902), pp. 251-314; and A. ccviii. (1908), pp. 1-19. Collected Works, vol. iii. Paper 12.

† Communicated by Dr. S. R. Milner.