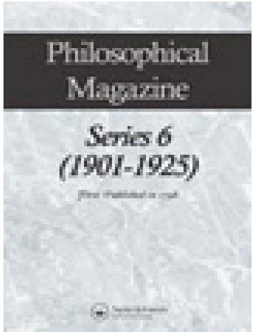


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# I. The relation of mass to energy

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## I. *The Relation of Mass to Energy.* *By* Daniel F. Comstock, *Ph.D., Instructor in Theoretical Physics, Mass. Institute of Technology, Boston**.

1. WHETHER the inertia of matter has or has not a complete electromagnetic explanation is a question that it will perhaps take many years to answer with any degree of certainty. The experiments of Kaufmann seem to prove that in the case of a single electron the mass is entirely of this origin; and it is impossible therefore to avoid the conclusion that at least a fraction of ordinary material inertia is also electromagnetic. Doubtless there is a psychological cause for our reluctance to accept the electromagnetic explanation as complete, constant familiarity with ponderable bodies having blinded us to the possibility of anything being more fundamental; but certain it is, that if we free ourselves from prejudice as much as possible and adopt the well-tried policy of choosing the simplest theory which adequately represents the phenomena,—the theory that is, which involves the least number of variables,—we must decide in favour of the complete electromagnetic explanation, which involves only the æther and its properties.

2. The complexity of the Zeeman effect and the relations

* Communicated by the Author.

between the wave-lengths of the spectral lines, make it seem probable that if matter is to be considered as an electrical system, it must be much more complex than a system composed entirely of electrons separated by distances great in comparison to their size. It becomes therefore of interest to see whether any relations can be found between the mass of an electric system in general, and any of its other properties. It will be found that a general relation does exist, which is not only of considerable interest in itself, but also suggests other relations.

3. The straightforward calculation of the mass of an electric system possessing any distribution of charge and any internal velocities below that of light presents considerable difficulty ; for such calculation involves the use of the scalar and vector potential, and these are not effective instantaneously at all parts of the system. Any expression for the mass of the system calculated in this way will therefore involve terms which vary in an extremely complicated way with the internal velocities when these are not very small, The same is true with respect to the velocity of the system as a whole. In the following discussion the problem is attacked in an entirely different way, which is not open to this objection.

As the constraints of the system are intimately involved, it will be well first to consider them.

4. The position of internal constraints in general electrical theory is a very fundamental one. By "constraints" are meant rigid connexions of any kind. These act merely as reactions to the electrical forces, and do not contribute to the virtual work. If the electrical laws are to hold universally, *i. e.*, for minute distances as well as for greater ones, it is obvious that no electrical system can exist as such unless there are such constraints to balance the electrical forces. Even a single electron would dissipate itself through the mutual repulsion of its elements, were it not for some form of internal constraint. Besides holding the system together, as it were, these constraints also act in another important way. They may become, in common with all geometrical constraints, paths of energy flow. We are accustomed to think of the Poynting vector as representing completely the energy flow in a purely electrical system, but of course this is not in general true.

Take as a simple example the case of a large plane air-condenser moving in a direction perpendicular to the plane of its plates. If the condenser is charged there is obviously a transference of energy at a rate equal to the internal energy multiplied by the velocity of movement. The Poynting vector is, however, zero. The energy transfer is not through space in the ordinary sense, but is along the constraint which holds the condenser-plates apart. The plate in the rear picks up, as we may say, the energy of the field, and after it has been transmitted to the forward plate by means of the constraint it is there set down again. On the other hand, when the condenser is moving parallel to the plane of the plates, there is no energy flow along the constraint and the Poynting vector adequately represents the transfer of energy. So also in the case of a single moving electron, the rate of transfer of energy is not given by the integration of the Poynting vector through all space, but differs from this by an amount corresponding to the energy-flow along the constraints in the body of the electron. This does not mean that there is any energy associated with the constraints, for of course rigid constraints can neither absorb nor give out energy; there is no storing up, but merely a transfer.

5. It is not difficult to find an expression for this rate of transfer. If the constraint is a simple linear one the transfer of energy along its direction is evidently

$$-l\mathrm{T}v',$$

where ($l$) is the length of the constraint, ($v'$) the velocity with which it is moving along its length, and (T) the tension along it. The amount of energy ($\mathrm{T}v'$) per sec. is put at the forward end, and is instantly available at the rear end at a distance ($l$). If the velocity ($v$) makes an angle ($\theta$) with the constraint, $v'=v\cos\theta$, and the transfer in the direction of $\boldsymbol{v}$ is

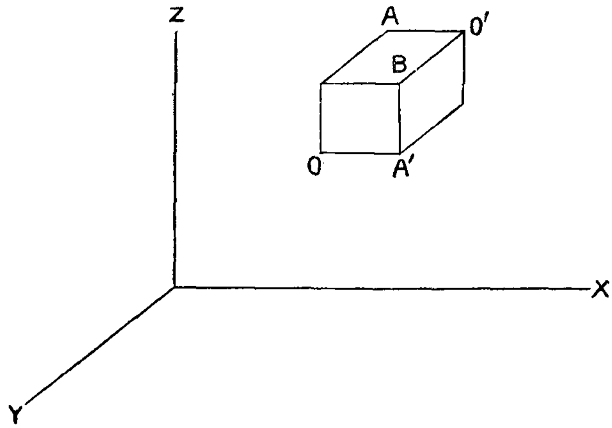
$$-l\mathrm{T}v\cos^2\theta.$$

Another type of transfer enters when there is shearing stress in the constraint, a transfer that is which is in a direction perpendicular to that of velocity. It must be remembered that the constraints are described in no way except geometrically.

If we consider therefore the general case where the stress in the body of the constraint is represented mathematically by the nine stresses commonly used in the theory of elasticity, namely, $\mathrm{X}_x$, $\mathrm{Y}_x$, $\mathrm{Z}_x$, $\mathrm{X}_y$, $\mathrm{Y}_y$, $\mathrm{Z}_y$, $\mathrm{X}_z$, $\mathrm{Y}_z$, $\mathrm{Z}_z$, then there is a rate of transfer of energy in the $x$-direction through unit volume given by

$$f_x' = -(v_x + \mathrm{X}_x v_y \mathrm{Y}_x + v_z \mathrm{Z}_x).$$

This can be readily shown by a consideration of the figure. When the velocity of the element is along $(x)$ there is an




amount of work $(v_x \,.\, X_x)\, dy \,.\, dz$ done per second on the element by the tension $(X_x)$ applied at the surface (O′A′), and this energy is instantly available at the surface (OA), where it is given out. The distance over which the energy is transmitted being $dx$ (the thickness of the element), the rate of energy-flow is

$$-v_x X_x dy\, dz\, dx = -v_x X_x d\tau,$$

where $(d\tau)$ is the element of volume.

In like manner the velocity $(v_y)$ and the shearing stress $(Y_x \,.\, dy \,.\, dz)$ cause energy to be taken up at the surface (O′A′) and given out at the surface (OA), and we have the rate of flow along the $x$-axis

$$-v_y Y_x d\tau\ ;$$

and finally the velocity $(v_z)$ and the shearing stress $(Z_x \,.\, dy \,.\, dz)$ give

$$-v_z Z_x d\tau.$$

Hence adding we have, if we call $(f_x')$ the density of flow along $x$,

$$f_x' d\tau = -(v_x X_x + v_y Y_x + v_z Z_x)\, d\tau.$$

Obtaining the corresponding equations in similar way we have finally for the three components of the density of energy-flow along the constraints in any system

$$\left.\begin{aligned} f_x' &= -(v_x X_x + v_y Y_x + v_z Z_x), \\ f_y' &= -(v_x X_y + v_y Y_y + v_z Z_y), \\ f_z' &= -(v_x X_z + v_y Y_z + v_z Z_z). \end{aligned}\right\} \quad . \quad . \quad . \quad (1)$$

For the total density of energy-flow $(f_x, f_y, f_z)$ we must of course add to the above the components of the Poynting vector. Writing as usual X, Y, Z and $\alpha$, $\beta$, $\gamma$ for the electric and magnetic force intensities and calling (V) the velocity of light, we have

$$\left.\begin{aligned} f_x &= \frac{V}{4\pi}(\gamma Y - \beta Z) - (v_x X_x + v_y Y_x + v_z Z_x), \\ f_y &= \frac{V}{4\pi}(\alpha Z - \gamma X) - (v_x X_y + v_y Y_y + v_z Z_y), \\ f_z &= \frac{V}{4\pi}(\beta X - \alpha Y) - (v_x X_z + v_y Y_z + v_z Z_z), \end{aligned}\right\} \quad . \quad . \quad (2)$$

These equations give the density of the total energy-flow through any purely electrical system, in which the ordinary electrical laws hold universally.

6. Consider an isolated electrical system moving as a whole through space with the constant velocity $(v_1)$. A constant velocity will be possible if the system retains on the average the same internal structure. The total average rate of transfer of energy corresponding to the movement of such a system is evidently $(v_1 \,.\, W)$, where W is the total contained energy. Another expression for the same thing is to be obtained by integrating throughout the system the components along $(v_1)$ of $(f_x, f_y, f_z)$ given in equations (2). In order that the velocity $(v_1)$ may appear explicitly, however, it is necessary that the velocity $(v)$, which was used in equations (2), be written as the sum of $(v_1)$ and another velocity $(v_2)$. Then $(v_2)$ is the velocity with respect to axes moving with the system.

If $l$, $m$, $n$ are the direction cosines of the constant velocity $(v_1)$, we have for the total energy-flow (F) in the direction of $(v_1)$,

$$\begin{aligned} W v_1 = F &= \int (l f_x + m f_y + n f_z) d\tau \\ &= \frac{lV}{4\pi}\int(\gamma Y - \beta Z)d\tau - l v_1 \int (l X_x + m Y_x + n Z_x) d\tau - l \int (v_{2x} X_x + v_{2y} Y_x + v_{2z} Z_x) d\tau \\ &+ \frac{mV}{4\pi}\int(\alpha Z - \gamma X)d\tau - m v_1 \int (l X_y + m Y_y + n Z_y) d\tau - m \int (v_{2x} X_y + v_{2y} Y_y + v_{2z} Z_y) d\tau \\ &+ \frac{nV}{4\pi}\int(\beta X - \alpha Y)d\tau - n v_1 \int (l X_z + m Y_z + n Z_z) d\tau - n \int (v_{2x} X_z + v_{2y} Y_z + v_{2z} Z_z) d\tau \end{aligned} \quad . \quad . \quad . \quad . \quad (3)$$

Since the proof of equations (1) is equally valid for relative motion, the integrals involving $(v_2)$ in the above correspond to flow of energy with respect to axes moving with the system. There is also an *implicitly* involved internal term in each of the Poynting vector integrals. Since the system is isolated, the sum of such "internal" terms must on the average vanish. There remains therefore to represent the actual average rate of transfer of energy through space only the explicit Poynting terms and the terms involving $(v_1)$.

The electromagnetic momentum corresponding to any electrical system is given by the components

$$\left.\begin{aligned} M_x &= \frac{1}{4\pi V}\int(\gamma Y-\beta Z)d\tau \\ M_y &= \frac{1}{4\pi V}\int(\alpha Z-\gamma X)d\tau \\ M_z &= \frac{1}{4\pi V}\int(\beta X-\alpha Y)\,d\tau \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which, except for the factor $V^2$, are the same as the integrals of the components of the Poynting vector throughout the system. Hence equation (3) may be written

$$\begin{aligned} Wv_1 = lM_x + mM_y + nM_z - lv_1\int(lX_x + mY_x + nZ_x)d\tau \\ -mv_1\int(lX_y + mY_y + nZ_y)d\tau - nv_1\int(lX_z + mY_z + nZ_z)d\tau \end{aligned} \quad . \quad . \quad (3\text{ A})$$

Also if the electrical system here dealt with is to represent a material body, we may assume that the resultant momentum (M) is in the direction of the velocity, and hence

$$M = lM_x + mM_y + nM_z.$$

This may be considered as due to the fact that the lack of symmetry necessarily involved in the intimate structure of any electromagnetic system has become a symmetrical average in particles large enough to be dealt with. This symmetrical point may of course have been reached in the case of single atoms. We may now write (3 A) in the form

$$\begin{aligned} Wv_1 = V^2M - v_1\int\{(l^2X_x + m^2Y_y + n^2Z_z) \\ + lm(X_y + Y_x) + ln(X_z + Z_x) + mn(Y_z + Z_y)\}d\tau \end{aligned} \quad . \quad . \quad (5)$$

7. To reduce this expression further requires some relation to be established between the stresses and the electric and magnetic force intensities. This process is closely analogous to the derivation of the Maxwell stress in the free æther

except that we here have to deal with, besides the forces in the constraint, only the electromagnetic force on electricity embedded in the constraint, and we have nothing to do with hypothetical stresses in the free æther.

If ($\rho$) represents the electric density and ($\mathscr{F}_x$) the $x$-component of the total electromagnetic force on unit charge embedded in the constraint, we have

$$\left.\begin{aligned}\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}&=-\rho\mathscr{F}_x\\ &=-\rho X-\frac{\rho}{V}(v_y\gamma-v_z\beta)\\ &=-\rho X-(k_y\gamma-k_z\beta)\end{aligned}\right\}, \quad . \quad . \quad . \quad (6)$$

where ($k$) is the density of convection current caused by the movement with velocity ($v$) of the electricity of density ($\rho$).

Making use temporarily of the vector terminology for the sake of brevity and calling the electric force (E), the magnetic (H), and the sign [ ] denoting the vector product, we have

$$-\rho\mathscr{F}_x=-\rho E_x-[kH].$$

Since $\operatorname{div} E=4\pi\rho$ and $\operatorname{curl} H-\frac{1}{V}\frac{\partial E}{\partial t}=4\pi k$,

$$\begin{aligned}-\rho\mathscr{F}_x&=-\frac{1}{4\pi}\left\{E_x\operatorname{div}E+\left[\operatorname{curl}H-\frac{1}{V}\frac{\partial E}{\partial t},\ H\right]_x\right\}\\ &=-\frac{1}{4\pi}\left\{E_x\operatorname{div}E+[\operatorname{curl}H,\ H]_x-\left[\frac{1}{V}\frac{\partial E}{\partial t},\ H\right]_x\right\}. \quad (7)\end{aligned}$$

Now it is an easily verifiable identity that

$$\begin{aligned}&E_x\operatorname{div}E-[E,\ \operatorname{curl}E]_x+H_x\operatorname{div}H+[\operatorname{curl}H,\ H]_x\\ &=\frac{\partial}{\partial x}\{\tfrac{1}{2}(E_x^2-E_y^2-E_z^2)+\tfrac{1}{2}(H_x^2-H_y^2-H_z^2)\}\\ &+\frac{\partial}{\partial y}(E_xE_y+H_xH_y)+\frac{\partial}{\partial z}(E_xE_z+H_xH_z); \quad . \quad . \quad (8)\end{aligned}$$

and hence, remembering that $\operatorname{div} H=0$, equation (7) becomes

$$\begin{aligned}-\mathscr{F}_x=-\frac{1}{4\pi}\Big\{&\frac{\partial}{\partial x}\{\tfrac{1}{2}(E_x^2-E_y^2-E_z^2)+\tfrac{1}{2}(H_x^2-H_y^2+H_z^2)\\ &+\frac{\partial}{\partial y}(E_xE_y+H_xH_y)+\frac{\partial}{\partial z}(E_xE_z+H_xH_z)\\ &+[E,\ \operatorname{curl}E]_x-\left[\frac{1}{V}\frac{\partial E}{\partial t},\ H\right]_x\Big\}. \quad . \quad . \quad . \quad . \quad . \quad (9)\end{aligned}$$

Since

$$[\mathrm{E},\ \mathrm{curl}\,\mathrm{E}]_x = -\left[\mathrm{E},\ \frac{1}{\mathrm{V}}\frac{\partial \mathrm{H}}{\partial t},\ \mathrm{H}\right]_x,$$

the last two terms in the bracket of (9) become together

$$-\frac{1}{\mathrm{V}^2}\frac{\partial}{\partial t}[\mathrm{EH}]_x,$$

which is minus the time rate of the density of momentum at the point. The time rate, however, refers to a point fixed in space, and to change to a point moving with the system we make use of the usual expression and write

$$-\frac{\partial}{\partial t}m_x = -\frac{\partial'}{\partial t}m_x + v_1\left(l\frac{\partial m_x}{\partial x} + m\frac{\partial m_x}{\partial y} + n\frac{\partial m_x}{\partial z}\right), \quad (10)$$

where $m_x$ is the $x$-component of the density of momentum and $(l, m, n)$ are, as formerly, the direction cosines of the constant velocity $(v_1)$. The operator $\frac{\partial'}{\partial t}$ now refers to the rate of change at a point moving with the system.

Substituting (10) for the two last terms in (9) and noticing that

$$\frac{\partial' m_x}{\partial t} \text{ may be written } \frac{\partial}{\partial x}\frac{\partial'}{\partial t}\int_{\mathrm{R}}^{\mathrm{P}} m_x dx,$$

where the integration is to be taken from R (meaning merely from a point outside the system where $m_x$ is zero) up to the point P in question, account being taken of any discontinuity at the bounding surface, we have in place of (9)

$$\begin{aligned}
&\frac{\partial \mathrm{X}_x}{\partial x} + \frac{\partial \mathrm{X}_y}{\partial y} + \frac{\partial \mathrm{X}_z}{\partial z} = -\rho\mathcal{G}_x \\
&= \frac{\partial}{\partial x}\left\{-\frac{1}{8\pi}(\mathrm{E}_x^2 - \mathrm{E}_y^2 - \mathrm{E}_z^2) - \frac{1}{8\pi}(\mathrm{H}_x^2 - \mathrm{H}_y^2 - \mathrm{H}_z^2) - v_1 l m_x + \frac{\partial'}{\partial t}\int_{\mathrm{R}}^{\mathrm{P}} m_x dx\right\} \\
&+ \frac{\partial}{\partial y}\left\{-\frac{1}{4\pi}(\mathrm{E}_x\mathrm{E}_y + \mathrm{H}_x\mathrm{H}_y) - v_1 m m_x + \frac{\partial'}{\partial t}\int_{\mathrm{R}}^{\mathrm{P}} m_x dy\right\} \\
&+ \frac{\partial}{\partial z}\left\{-\frac{1}{4\pi}(\mathrm{E}_x\mathrm{E}_z + \mathrm{H}_x\mathrm{H}_z) - v_1 n m_x + \frac{\partial'}{\partial t}\int_{\mathrm{R}}^{\mathrm{P}} m_x dz\right\}. \quad \ldots \ldots \quad (11)
\end{aligned}$$

This equation gives us what we were seeking, namely, the values of $\mathrm{X}_x$, $\mathrm{X}_y$, and $\mathrm{X}_z$ in terms of the electric and magnetic forces and the density of the momentum.

Thus we may take

$$X_x = -\frac{1}{8\pi}(E_x^2 - E_y^2 - E_z^2) - \frac{1}{8\pi}(H_x^2 - H_y^2 - H_z^2) - v_1 l m_x + \frac{\partial'}{\partial t}\int_R^P m_x dx$$

$$X_y = -\frac{1}{4\pi}(E_x E_y + H_x H_y) - v_1 m m_x + \frac{\partial'}{\partial t}\int_R^P m_x dy$$

$$X_z = -\frac{1}{4\pi}(E_x E_z + H_x H_z) - v_1 n m_x + \frac{\partial'}{\partial t}\int_R^P m_x dz. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Similar values are of course to be found for the other six components of stress in the constraint $Y_x$, $Y_y$, $Y_z$, $Z_x$, $Z_y$, $Z_z$.

8. The values for these nine stress-components are now to be substituted in equation (5). In doing this it is to be noticed that the last term in each of equations (12) will, after substitution, furnish a term of the type

$$\int d\tau \frac{\partial'}{\partial t}\int_R^P v_1 l^2 m_x dx = \frac{\partial'}{\partial t}\int d\tau \int_R^P v_1 l^2 m_x dx, \quad . \quad . \quad (13)$$

and this, being a time derivative, gives an average value of zero when the time is allowed to increase indefinitely, since all quantities in the system remain finite. Also

$$l^2 + m^2 + n^2 = 1.$$

Making the substitution in (5) and simplifying, we have as the value for $(v_1 W)$

$$v_1 W = V^2 M - v_1 \int \left\{ \frac{1}{8\pi}(E_x^2 + E_y^2 + E_z^2) + \frac{1}{8\pi}(H_x^2 + H_y^2 + H_z^2) \right\} d\tau$$
$$+ v_1 \frac{1}{4\pi}\int \left\{ (lE_x + mE_y + nE_z)^2 + (lH_x + mH_y + nH_z)^2 \right\} d\tau$$
$$+ v_1^2 \int (lm_x + mm_y + nm_z) d\tau. \quad . \quad . \quad (14)$$

Now the first integral represents the total included energy, the two parts of the second integral represent the squares of the components of the electric and magnetic forces in the direction of the motion of the system, and the last integral represents the momentum in the direction of motion, which in this case is the whole momentum M, since we have assumed that M and $v_1$ are in the same direction.

Calling

$$\frac{1}{8\pi}\int \left\{ (lE_x + mE_y + nE_z)^2 + (lH_x + mH_y + nH_z)^2 \right\} d\tau = W_L$$

the longitudinal energy of the system, and

$$W - W_L = W_T$$

the transverse energy of the system, we may rewrite equation (14) as

$$v_1 W = V^2 M - v_1 W + 2v_1 W_L + v_1^2 M, \quad . \quad . \quad (15)$$

and hence

$$M = \frac{2W_T v_1}{V^2\left(1 + \left\{\frac{v_1}{V}\right\}^2\right)}. \quad . \quad . \quad . \quad . \quad (16)$$

*This gives the total momentum of any isolated, moving, purely electrical system, which has on the average the same internal structure, in terms of its transverse energy, i. e.*, the energy represented by the components of the electric and magnetic forces which are perpendicular to the velocity of the system. The mass of the system is then

$$\text{Mass} = \frac{dM}{dv_1} = \frac{2}{V^2}\frac{W_T}{\left\{1+\left(\frac{v_1}{V}\right)^2\right\}}\left\{1-2\left(\frac{v_1}{V}\right)^2\frac{1}{\left\{1-\left(\frac{v_1}{V}\right)^2\right\}}\right\} + \frac{4\left(\frac{v_1}{V}\right)\frac{dW_T}{dv_1^2}}{\left\{1+\left(\frac{v_1}{V}\right)^2\right\}} \quad . \quad . \quad . \quad (17)$$

If, as we have assumed in deriving this expression, the system possesses the same momentum for uniform translation in any direction, this formula for the mass can contain terms of even powers only in the ratio of the velocity of the system to the velocity of light. If we neglect terms of the second and higher orders $W_T$ has the same value as for $v_1 = 0$, which from symmetry of the system must be two-thirds the total energy W. Therefore

$$\text{Mass} = \frac{4}{3}\frac{W}{V^2}, \quad . \quad . \quad . \quad . \quad . \quad (18)$$

if second order terms be neglected. This formula would apply with extreme accuracy for the electromagnetic mass of ponderable bodies, for no such bodies have in nature a velocity large enough to make $\left(\frac{v_1}{V}\right)^2$ appreciable.

It should be noticed that in equation (17) second order terms may enter in either $(W_T)$ or its derivative with respect to $v_1^2$. In fact such terms do enter for two reasons. In the first place, the setting of the body in motion requires work and hence adds new energy, through a second order term; and secondly there is an effect due to the change which motion causes in the velocity of propagation *through the system* of electrical disturbances. This is seen in the simple case of a moving electron where the crowding of the lines towards the equator with increase of velocity is only partly due to the added energy. It is evident, therefore, that for velocities so great that the second order terms cannot be neglected, the mass depends on complicated terms which vary with the internal structure and motions of the system, and it does not appear as if a general expression for the mass of a system for such high velocities could be found.

The second order terms may in the future make themselves experimentally manifest through an increase of mass of rapidly moving $\alpha$-particles.

9. Expression (18) may readily be verified for simple symmetrical systems. For a single charged conducting sphere of radius $(a)$ the mass for slow velocities is well-known to be

$$\frac{2}{3}\frac{1}{V^2}\frac{e^2}{a}=\frac{2}{3}\frac{e}{V^2}\frac{e}{a}=\frac{4}{3V^2}\frac{1}{2}(e\,.\,\text{Potential})=\frac{4}{3V^2}W.$$

An interesting verification of equation (16) for the special case of a general, rigid, electrostatic system in translatory motion has been furnished me privately by Mr. G. F. C. Searle. He obtains for such a system (Phil. Mag. Jan. 1907, p. 129) the expression

$$M_x=\frac{2T}{v_1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where $M_x$ is the momentum of the entire system along the direction $(x)$ of motion, (T) is the total magnetic energy due to this motion, and $(v_1)$ is the common translatory velocity possessed by all parts of the system.

Now it is well known that where the Faraday tubes move through space uniformly, as in the present case, the magnetic force (H) is given in terms of the electric force (E) by the expression

$$H=\frac{v_1}{V}E\sin\theta,$$

$(\theta)$ representing the angle between (E) and the velocity of motion $(v_1)$, and (H) being in a direction perpendicular both to (E) and $(v_1)$. In the present notation

$$H=\frac{v_1}{V}\sqrt{E_y^2+E_z^2},$$

and hence we have

$$T=\frac{H^2}{8\pi}=\frac{1}{8\pi}\frac{v_1^2}{V^2}(E_y^2+E_z^2). \quad . \quad . \quad . \quad (20)$$

Combining (19) and (20) we can obtain

$$M_x = \frac{v_1 \cdot 2\left\{\frac{1}{8\pi}(E_y^2 + E_z^2) + T\right\}}{V^2\left(1 + \frac{v_1^2}{V^2}\right)}, \quad . \quad . \quad . \quad (21)$$

and remembering that

$$T = \frac{1}{8\pi}(H_y^2 + H_z^2),$$

we have finally

$$M_x = \frac{2W_T v_1}{V^2\left(1 + \frac{v_1^2}{V_2}\right)}, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

which, since $(v_1)$ is along $(x)$, is identical with (16). Thus (16) is verified for the case where the moving system possesses no internal motion.

Perhaps the simplest symmetrical system containing magnetic as well as electric energy is that formed by a great number of charged spheres moving in straight lines out from a common centre, with velocities small enough so that the fourth and higher powers may be neglected. They are to be at distances from each other great in comparison with their size, and at equal distances from the common centre. If the system be now given the slow velocity $(v_1)$ as a whole the total momentum accompanying this motion may be determined. Because of limited space the calculation will not be here given, but if it be carried out along established lines it will be found that the mass is $\frac{4}{3}\frac{1}{V^2}$ times the sum of the electric and magnetic energies, thus verifying equation (18).

10. We conclude that, if ordinary material mass has an electromagnetic basis, such mass for slow velocities is proportional to the total electromagnetic energy-content of the body, and the laws of conservation of mass and energy become closely related if not identical. In any case the expression given represents the electromagnetic part of the total mass whatever that may be.

### *Considerations suggested by the Foregoing. The Atomic Weights.*

11. If the conclusion of the last article is correct a diminution in mass should follow a loss of energy in material transformations. Calculation shows, however, that in the

case of the powerful reaction between hydrogen and oxygen forming water, the change of mass would only be of the order $10^{-10}$ gram. In the case of radioactivity, however, the energy change is very much greater and an appreciable effect is to be expected. Thus if a radium atom gives off an $\alpha$-particle of mass $(m)$ with velocity $(\mu)$, then there should be a diminution in the sum of the masses of the $\alpha$-particle and the remaining atom equal to

$$\frac{4}{3}\frac{1}{V^2}(\tfrac{1}{2}m\mu^2),$$

since $\frac{1}{2}m\mu^2$ represents the energy lost, and this, calling $m=4$ (using gram-atomic weight) and $\mu=2{\cdot}5 \,.\, 10^9$, gives

$$\Delta\,(\text{Mass}) = -1{\cdot}7 \,.\, 10^{-2} \text{ gram};$$

an amount large enough to cause discrepancies in calculating the atomic weights of radioactive substances from the number of $\alpha$-particles lost. Since $\Delta$ (Mass) is proportional to the square of the velocity of the $\alpha$-particle, its value would be greatly increased by a slight error in the determination of $(\mu)$ and the effect could easily be much larger.

12. A consideration of some interest is the following. If we adopt the disintegration theory, we are obliged to think of the various atoms as combinations or groups, more or less modified, of the lighter atoms. If there were perfect conservation of mass this would introduce a certain uniformity in the relations between the atomic weights, a uniformity which apparently does not exist. On the other hand, if we take into consideration the inevitable change of mass when the electromagnetic energy of the system is modified, the atomic weights will involve a correction term depending upon the change in this energy, and hence they will no longer bear simple, exact relations to each other. In a highly important paper (*Zeitschrift für Anorg. Chemie*, xiv. p. 66, 1897) Rydberg has shown that the atomic weights of the first twenty-seven elements of the periodic system approximate to whole numbers very much more closely than chance could bring about. He has also shown that the atomic weights of these elements are best considered as the sum of two parts $(N+D)$ where N is an integer and D is a fraction, in general positive and smaller than unity. If M is the number of the element in the system (called by Rydberg the "Ordnungszahl"), then N is equal to 2M for the elements of even valence and $2M+1$ for the elements of odd valence. Below is given a table showing the various quantities. I have used,

however, the International Atomic Weight values for 1907 instead of those Rydberg used.

| Sign. | M. | N. 2M. | N. 2M+1 | Atomic Weight. | D. | Sign. | M. | N. 2M. | N. 2M+1 | Atomic Weight. | D. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| He ... | 2 | 4 | | 4 | — | P...... | 15 | | 31 | 31·0 | ·0 |
| Li ... | 3 | | 7 | 7·03 | ·03 | S...... | 16 | 32 | | 32·06 | ·06 |
| Be ... | 4 | 8 | | 9·1 | 1·1 | Cl ... | 17 | | 35 | 35·45 | ·45 |
| B...... | 5 | | 11 | 11·0 | ·0 | A ..... | 18 | 36 | | 39·9 | 3·9 |
| C...... | 6 | 12 | | 12·00 | ·0 | K ... | 19 | | 39 | 39·15 | ·15 |
| N ... | 7 | | 15 | 14·01 | −·99 | Ca ... | 20 | 40 | | 40·1 | ·1 |
| O...... | 8 | 16 | | 16·00 | ·0 | Sc ... | 21 | | 43 | 44·1 | 1·1 |
| Fl ... | 9 | | 19 | 19·0 | ·0 | — | 22 | 44 | | — | — |
| Ne ... | 10 | 20 | | 20·0 | ·0 | — | 23 | | 47 | — | — |
| Na ... | 11 | | 23 | 23·05 | ·05 | Ti ... | 24 | 48 | | 48·1 | ·1 |
| Mg ... | 12 | 24 | | 24·36 | ·36 | V ... | 25 | | 51 | 51·2 | ·2 |
| Al ... | 13 | | 27 | 27·1 | ·1 | Cr ... | 26 | 52 | | 52·1 | ·1 |
| Si ... | 14 | 28 | | 28·4 | ·4 | Mn ... | 27 | | 55 | 55·0 | ·0 |
| | | | | | | Fe ... | 28 | 56 | | 55·9 | ·1 |

The orderly arrangement of the series is striking. It will be noticed that in three cases only are the D's greater than unity and only in two cases are they negative.

Rydberg points out that although the heavier elements do not conform well to this scheme, *i. e.*, do not in general give the small fractional values of (D) noticed above, yet this is in reality no valid objection, for the numerical values of the weights of heavier elements depend much more on the value of the arbitrary unit chosen than do those of the lighter weight elements, and hence they can have little influence one way or the other in estimating the validity of the curious relations he sets forth.

The whole question is of course whether these differences represent real physical deviations from something or whether they are merely mathematical remainders. Rydberg certainly believes them to represent physical realities, and considering the before-mentioned overwhelming improbability that the approximation of the atomic weights to whole numbers is due to chance, we can hardly doubt that he is right.

13. Now it is to be noticed that these deviations find a ready explanation when the conclusions of the present paper are combined with the theory, so much favoured recently, that one element breaks down into two or more others with an accompanying expulsion of energy. The deviations are then to be explained as resulting from loss of mass accompanying the dissipation of energy. On the other hand, if no such loss of mass takes place, the existence of these deviations in the

table of atomic weights becomes a well-nigh insuperable difficulty in the path of the evolutionary theory of the elements.

If we follow the present suggestion, we must search for the components of an element, not by comparing atomic weights, but by comparing the corresponding values of N, for the atomic weights deviate because of the lost mass accompanying the dissipation of internal energy.

Very recently Sir W. Ramsay has announced several striking discoveries which seem to add much weight to the disintegration theory, and, indirectly, to the views here set forth. He found helium, neon, or argon appearing as a product of radium emanation according to the exterior conditions imposed, and he found lithium appearing when a copper-sulphate solution was left in the presence of the emanation. Prof. Ramsay states, I believe, that every source of error was eliminated and that the results were obtained many times.

14. It should be noticed that this theory of loss of mass and its consequences does not require that the *whole* material mass should be of electromagnetic nature. It only requires that the energy lost in the transformations, explosive or otherwise, should be at the expense of internal electromagnetic energy, *i. e.*, that the forces which expel the $\alpha$-particles should be electric or magnetic.

## *Respecting Gravitation.*

15. The experiments of many investigators have shown that up to a high degree of accuracy the ratio of mass to weight for different substances is the same. Now if the mass is proportional to the internal energy as here suggested, instead of being proportional to the number of electric nuclei as might be supposed, the conclusion is apparently forced upon us that gravitational attraction is between quantities of confined energy, and not between quantities of "matter" in any other sense.

On this basis, the weight of a calorie at the earth's surface would be of order $10^{-11}$ dyne. This is apparently too small to explain the temperature gradient in the earth although the calculation, depending as it does on the mechanical force on confined energy due to a temperature gradient, would certainly depend to a large degree on the medium.

If we assume this gravitational effect, it is interesting to ask whether free energy would also show an attraction for itself. If so, the energy radiated from a gravitational centre like the sun would leave some of itself behind along its path as it moved through space, and it might be possible to account in this way for some of the energy which is ordinarily thought of as totally dissipated.

Another conclusion which is suggested by the foregoing is that, assuming the loss of mass accompanying dissipation of energy, the sun's mass must have decreased steadily through millions of years. If too, our conclusion respecting the gravitating quality of confined energy be correct, the gravitation constant of the sun has also decreased and the distances of the planets must have increased accordingly. This last increase of planetary distance can be calculated by making the angular momentum of the planet about the sun a constant, and allowing the mass of the planet, together with the gravities of both sun and planet, to grow less with time.

So little is known as to the former radiating power of the sun that no even approximate calculation can be made, but it is not difficult to show that the order of magnitude is such as might make the increase in the planetary distances not altogether negligible during great lapses of time.

### *A Proof from a different Point of View.*

16. The proof of expression (17) which has been given has the advantage of entering intimately into the structure of the general system and showing the part that non-electrical forces in the form of constraints must play if the fundamental laws of electrical action are to hold for *every infinitesimal element* of the finite volume occupied by any electrical system. Although this is assumed in every mathematical derivation of the mass of an electron, and in fact in all problems of a similar nature, many will doubtless object to this assumption on the ground that probably the ordinary electrical laws do not apply when the distance between "elements of charge," so called, is comparable with the diameter of an electron.

Although it is difficult to see how a coherent mathematical theory of electricity can at present be formed without this assumption, yet it was thought best to add a more general proof of (17). The following is therefore given as avoiding the explicit use of constraints.

17. The statement of the law of the conservation of energy for an element of volume in any electrical system possessing electrical charges in motion, is the well-known expression

$$\frac{\partial w}{\partial t} = -\left(\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial z}\right) - (v_x\rho\mathscr{F}_x + v_y\rho\mathscr{F}_y + v_z\rho\mathscr{F}_z) \quad . \quad (23)$$

Here $(w)$ is the density of the total electromagnetic energy, $S_x$, $S_y$, $S_z$ are the components of the Poynting vector, $\mathcal{F}_x$, $\mathcal{F}_y$, $\mathcal{F}_z$ are the components of the total electromagnetic force on unit charge, $(\rho)$ is the density of electrification at the given point, and $v_x$, $v_y$, $v_z$ represent the velocity through space of this electrification. Thus

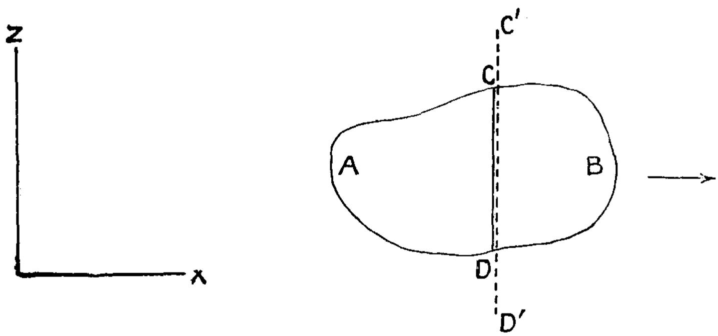
$$w = \frac{1}{8\pi}\{(X^2+Y^2+Z^2)+(\alpha^2+\beta^2+\gamma^2)\},$$

where X, Y, Z, and $\alpha$, $\beta$, $\gamma$, are the electric and magnetic force intensities respectively, and

$$\mathcal{F}_x = X + (v_y\gamma - v_z\beta),$$
$$\mathcal{F}_y = Y + (v_z\alpha - v_x\gamma),$$
$$\mathcal{F}_z = Z + (v_x\beta - v_y\alpha).$$

Equation (23) states merely that the rate of increase of energy in an elementary volume is equal to the activity of any foreign (*i.e.*, non-electrical) forces which may act therein minus the outward flow of energy.

Now suppose we consider an electromagnetic system bounded by a rigid surface (AB), which moves uniformly




through space with the velocity $(v_1)$ along the axis of $(x)$; and further suppose that the volume inside this closed surface is divided into two parts by the plane partition (CD) which is perpendicular to the $x$-axis and which, although fixed in the moving system, coincides at a given instant with the plane (C′D′) fixed in space. If this system be considered as isolated, then no disturbance passes through the bounding surface (AB).

In equation (23) the time derivative of the energy density refers to a point fixed in space, and if we wish it to refer to a point moving with the system we must write as usual

$$\frac{\partial w}{\partial t} = \frac{\partial' w}{\partial t} - v_1 \frac{\partial w}{\partial x}, \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where $\frac{\partial' w}{\partial t}$ now means the rate of change measured from the moving point. Likewise, if we wish the velocities which enter into (23) to be expressed in terms of velocities relative to axes moving with the system, we must write

$$\left.\begin{aligned} v_x &= v_1 + v_{2x} \\ v_y &= v_{2y} \\ v_z &= v_{2z} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where $v_{2x}$, $v_{2y}$, and $v_{2z}$ are the components of these relative velocities.

Substituting (24) and (25) in (23), and remembering the simple proportionality between $S_x$, $S_y$, and $S_z$ and the density of momentum $m_x$, $m_y$, and $m_z$, we easily obtain

$$V^2 \frac{\partial m_x}{\partial x} - v_1 \frac{\partial w}{\partial x} + v_1 \rho \mathcal{F}_x = -\frac{\partial' w}{\partial t} - \left(\frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial z}\right)$$
$$- (v_{2x}\rho\mathcal{F}_x + v_{2y}\rho\mathcal{F}_y + v_{2z}\rho\mathcal{F}_z). \quad . \quad . \quad (26)$$

Now $(\rho\mathcal{F}_x)$ may be expressed in terms of the electric and magnetic force intensities, together with the density of the momentum. This involves only the fundamental equations of electromagnetic theory and has been done in paragraph 7, reference to which will show that with the present notation

$$\rho\mathcal{F}_x = \frac{\partial}{\partial x}\left\{\frac{1}{8\pi}(X^2 - Y^2 - Z^2) + \frac{1}{8\pi}(\alpha^2 - \beta^2 - \gamma^2)\right\}$$
$$+ \frac{\partial}{\partial y}\left\{\frac{1}{4\pi}(XY + \alpha\beta)\right\} + \frac{\partial}{\partial z}\left\{\frac{1}{4\pi}(XZ + \alpha\gamma)\right\}$$
$$- \frac{\partial' m_x}{\partial t} + v_1 \frac{\partial m_x}{\partial x}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Substituting this for the $(\rho\mathcal{F}_x)$ which occurs on the left-hand side of (26), rearranging the latter, and putting

$$w = \frac{1}{8\pi}\{X^2 + Y^2 + Z^2 + \alpha^2 + \beta^2 + \gamma^2\}$$

and

$$w_t = \frac{1}{8\pi}\{Y^2 + Z^2 + \beta^2 + \gamma^2\},$$

we get

$$\frac{\partial}{\partial x}\left\{(V^2+v_1^2)m_x-2v_1w_t\right\}$$
$$=-\frac{\partial' w}{\partial t}+v_1\frac{\partial' m_x}{\partial t}-\left(\frac{\partial S_y}{\partial y}+\frac{\partial S_z}{\partial z}\right)$$
$$-\frac{\partial}{\partial y}\left\{\frac{1}{4\pi}(XY+\alpha\beta)\right\}-\frac{\partial}{\partial z}\left\{\frac{1}{4\pi}(XZ+\alpha\gamma)\right\}$$
$$-(v_{2x}\rho\mathscr{F}_x+v_{2y}\rho\mathscr{F}_y+v_{2z}\rho\mathscr{F}_z). \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Now if this expression is integrated through the part of the volume (AB) which lies on the side (A) of the partition (CD), the terms on the left of (28) become equal to

$$\int\{(V^2+v_1^2)m_x-2v_1w_t\}dS,$$

where the integral is taken over the part of the plane (C′D′) which is included in the surface. This follows from the fact that the rest of the surface of part A belongs to the surface (AB) and outside of (AB) there is no disturbance whatever. The terms on the right (of 28) give an average value of zero. This last will be evident if they are considered separately. The first two give directly a time average of zero, after great elapse of time, since neither $(w)$ nor $(v_1m_x)$ ever becomes infinite. The terms involving the $(y)$ and $(z)$ derivatives when integrated may be written as surface integrals over the bounding surface (of part A), and they then represent the flux of energy through this surface *in a direction perpendicular* to $(v_1)$, *i. e.*, in a direction perpendicular to the $x$-axis. This flux being everywhere zero over the surface, these terms vanish.

The terms involving $\mathscr{F}_x$, $\mathscr{F}_y$, $\mathscr{F}_z$ also give a time average of zero because they represent what might be called the "internal activity" of the forces which act on the charges in the A-part of the volume (AB), and since this part A is isolated on all sides except on the side (CD), the activity of these forces really means the rate at which the part (B) by means of them is doing work on A. In the long run B's work on A must be equal to A's work on B, if the system is to be conservative and the internal motions are to be stationary.

Thus we learn finally from equation (28) that the average value of

$$\int\{(V^2+v_1^2)m_x-2v_1w_t\}dS=0,$$

where the integral is taken over the enclosed part of the plane (CD). Since this is true for any position of the plane (CD) so long as it is perpendicular to $(v_1)$ we evidently have on the average, integrating along ($x$) throughout the entire system,

$$\int\{(V^2+v_1^2)m_x-2v_1w_t\}d\tau=0. \quad . \quad . \quad . \qquad (29)$$

This gives, using the former notation, and remembering that on the average the internal structure is assumed to remain the same,

$$M_x=\frac{2v_1W_T}{V^2+v_1^2}=\frac{2v_1W_T}{V^2\left(1+\frac{v_1^2}{V^2}\right)} \quad . \quad . \quad . \qquad (30)$$

which, since ($v_1$) is here along ($x$), is precisely the result of equation (16), and becomes (17) on differentiation.

*Conclusion.*

It has been shown in the foregoing that the electromagnetic mass of an isolated, symmetrical, purely electric system possessing any structure which on the average remains the same, and any internal motions or constraints, is expressible in terms of its velocity as a whole through space together with its "transverse energy" and the derivative of the latter with respect to the velocity. If second-order terms in the velocity be neglected, the mass is a simple constant multiplied by the total included electromagnetic energy.

If the mass of ponderable bodies has an electromagnetic origin, then the inertia of matter is to be considered merely as a manifestation of confined energy. From this point of view, matter and energy are thus very closely related and the laws of the conservation of mass and energy become practically identical.

It has been pointed out that the loss of mass, inevitable on this view, which takes place when energy is lost to the system, is large enough to be detected in the case of radioactive changes. If we assume the disintegration theory of the elements, this loss of mass affords a ready explanation of the general, small irregularities to be found in the list of atomic weights, and thus removes a serious difficulty from the path of the disintegration theory. For this loss of mass to take place however, it is not necessary that the *whole* of the mass be electromagnetic.

It has been shown that if material mass be electromagnetic and if lighter elements are formed from heavier ones through violent energy changes, it follows that gravity acts between quantities of confined energy and not between masses in any other sense. Several speculations are indulged in as to the results of assuming gravitation between quantities of energy.

Finally, the fundamental proposition is dealt with mathematically from an entirely different point of view and the same result obtained.

In conclusion I wish to express my thanks to Prof. J. J. Thomson and to Mr. G. F. C. Searle for several valuable criticisms and suggestions.

Cambridge, England,
August 14th, 1907.

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## II. *The Evolution and Devolution of the Elements.* *By* A. C. *and* A. E. Jessup *.

[Plate VII.]

THE hypothesis that the elements are different forms of one original substance was first formulated in modern times by Prout, and though his idea that hydrogen was that substance has since been shown to be incorrect, yet modern theories have given us, in the corpuscle, a body which may well be the root basis of all matter.

The recent researches of M. and Mme. Curie, Sir William Ramsay and Mr. Soddy, Professor Rutherford and others, have brought to light the fact that some of the elements are undoubtedly degrading into simpler forms of matter. But when we look for a reversal of this process on the earth, it is not apparent. In other words, we have as yet found no indications that elements with low atomic weight are changing into other elements with a higher atomic weight, that is, we have no proof of inorganic evolution. But when we turn our attention to the heavens, the case is altered, and it is entirely upon astrophysical observations that the ideas of evolution we are about to bring forward are based.

It was originally our intention to give these observations in full, but it has appeared advisable to give in the present paper only such of them as are essential for an understanding of what follows.

Spectroscopic evidence shows us that the nebulæ contain but few elements, all of which are in a highly attenuated form. The only two which have been recognized on the earth are hydrogen and helium, and the atomic weights of these are less than those of any other elements with which we are acquainted.

As the nebula becomes more compact, and assumes the form of a star, more and more complex elements appear, such

* Communicated by Sir William Ramsay, K.C.B., F.R.S.