XXXII. On a new principle of relativity in electromagnetism

A.H. Bucherer D.Sc.

homœoid by adding to the interior until h is changed from 1 to H,

$$W = -2\pi^2\rho^2a^2b^2c^2\int_{H}^{1} h^{\frac{1}{2}}(1-h)\,dh\int_{0}^{\infty}\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}}$$

$$=2\pi^2\rho^2a^2b^2c^2\{\tfrac{4}{15}-\tfrac{2}{3}H^{\frac{3}{2}}(1-\tfrac{3}{5}H)\}\int_{0}^{\infty}\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (52)$$

If $H=0$, this becomes the result for a solid ellipsoid.

For this case

$$W=\tfrac{8}{15}\pi^2\rho^2a^2b^2c^2\int_{0}^{\infty}\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad . \quad (53)$$

In the particular case of a uniform sphere this becomes

$$W=\tfrac{3}{5}\frac{M^2}{a}$$

if the unit of mass is grvavitational (see § 6, footnote), or

$$W=\tfrac{3}{5}\kappa\frac{M^2}{a}$$

if the ordinary unit of mass is used and κ is the proper value of the gravitational constant. This is the result given by Helmholtz, from which the rate of shrinking of the sun necessary to supply the energy radiated may be calculated.

If the density vary from shell to shell (53) becomes

$$W=2\pi^2a^2b^2c^2\int_{0}^{1}\rho\frac{dh}{h}\int_{h}^{1}\rho dk\int_{0}^{\infty}\frac{du}{\sqrt{(a^2+u)(b^2+u)(c^2+u)}} \quad (54)$$

as the reader may verify by finding the work done in building up a thick homœoid by adding shells of varying density ρ, and then varying the constant h of this homœoid from 0 to 1.

XXXII. *On a New Principle of Relativity in Electromagnetism.* *By* A. H. Bucherer, *D.Sc., Privatdocent in the Bonn University*.*

§ 1. IT is needless to dwell on the serious difficulties which the Maxwellian theory has encountered by the well established experimental fact that terrestrial optics is not influenced by the earth's motion. The endeavours of some distinguished physicists, notably of H. A. Lorentz, to

* Communicated by the Author. A short note on the same subject was published in the *Physik. Zeitschr.* vii. p. 556 (1906).

modify the Maxwellian theory in such a manner as to eliminate the effects of translatory motion have admittedly failed. Now the author, guided by the feeling that the *form* of the Maxwellian equations must correspond somehow to the true laws of electromagnetism, has attempted a new *interpretation* of these equations that should harmonize with facts.

In the following the author will show how this task can be accomplished. We consider two electromagnetic systems A and B in uniform rectilinear motion relatively to each other. Then, whenever we speak of the dynamical interaction of the systems we stipulate that the *system acted upon* —we will call this henceforth the *passive* system, and accordingly the other system the active one—experiences the same force as it would in the Maxwellian theory on the assumption *that it were at rest in the æther* and the other system moving relatively to it. As will be shown later on, the stipulation that the passive system is invariably the system "at rest" implies the principle of relativity.

§ 2. The Maxwellian equations as adapted to the electron theory have this form for empty space :

$$\text{(I.)} \qquad -\frac{d\mathbf{H}}{dt} = \text{curl}\,\mathbf{E},$$

$$\text{(II.)} \qquad \frac{d\mathbf{E}}{dt} = v^2\,\text{curl}\,\mathbf{H} - 4\pi v^2 \rho \mathrm{u},$$

$$\text{(III.)} \qquad \nabla\mathbf{H} = 0,$$

$$\text{(IV.)} \qquad \nabla\mathbf{E} = 0, \quad \text{or} \quad = 4\pi v^2 \rho.$$

As in the Maxwellian theory, **H** and **E** are the field intensities measured in the system "at rest." The forces exerted by the active system are purely electrical or magnetic. There are no electrodynamic forces on a passive system. We will limit our investigation for simplicity to rigid systems. The total force exerted on the passive system takes the form

$$\text{(V.)} \qquad \mathbf{F} = \iint \mathbf{H}\,\sigma_m\,dg + \iiint \mathbf{E}\,\rho\,d\tau.$$

Here σ_m means the surface-density of magnetism ; dg is an element of surface, and $d\tau$ an element of volume. While **H** and **E** are due to the active system, σ_m and ρ belong to the passive system.

We will now proceed to apply our equations to the relative motion of electrons and fictitious magnetic poles. We thus obtain so-called point laws which by suitable integrations can be employed to find the ponderomotive forces for any distribution of electrical and magnetic masses of systems in relative motion. The following point laws are partly known from the Maxwellian theory. They can be easily verified and found in agreement with the differential equations.

When two electrons move relatively to each other the passive one is acted upon by the force

$$\mathbf{F}=\frac{\mathbf{r}_1 q^2 v^2 s}{r^2\left(1-\frac{u^2}{v^2}\sin^2\gamma\right)^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad (1)$$

Here $\mathbf{r}_1$ signifies a unit radius vector drawn from the active to the passive electron; q is the charge of the electron, γ is the angle which the direction of motion makes with $\mathbf{r}$, $\mathbf{u}$ is the velocity of motion, and $s=1-\frac{u^2}{v^2}$.

Of two moving fictitious magnet-poles of unit strength the passive one is acted upon by the force

$$\mathbf{F}=\mathbf{r}_1\frac{s}{r^2\left(1-\frac{u^2}{v^2}\sin^2\gamma\right)^{\frac{3}{2}}}. \quad . \quad . \quad . \quad . \quad (2)$$

Further, when an electron and a unit magnet-pole are moving relatively to each other, the force on the passive electron is

$$\mathbf{F}=\mathrm{V}\,\mathbf{u}\,\mathbf{r}_1\frac{sq}{r^2\left(1-\frac{u^2}{v^2}\sin^2\gamma\right)^{\frac{3}{2}}}, \quad . \quad . \quad . \quad (3)$$

whereas the force on the passive magnet-pole is

$$\mathbf{F}=\mathrm{V}\,\mathbf{r}_1\,\mathbf{u}\frac{sq}{r^2\left(1-\frac{u^2}{v^2}\sin^2\gamma\right)^{\frac{3}{2}}}. \quad . \quad . \quad . \quad (4)$$

Here as elsewhere the radius vector is always drawn to the passive system. *Now an inspection of these four fundamental equations shows that the choice of the passive one of the two systems in relative motion is arbitrary*; i. e., a moving magnetic pole exerts exactly the same force on a "resting" electron as a moving electron on a "resting" pole, and we can at once conclude that with any distribution of magnetic and electric masses in the two systems, the force exerted by A on B is the same as that exerted by B on A. This proves the validity of the third law of Newton and implies the principle of relativity.

Suppose a system A contains only magnets and B only electric charges; then the observer, whom we imagine always to be on the passive system, will interpret the action exerted on his system as electric or magnetic according as he stays on B or on A. Evidently the electric action must equal the magnetic action, and we have the relation

$$\iiint \mathbf{E}\rho_{\mathrm{B}}\, d\tau = -\iint \mathbf{H}\, \sigma_{m_{\mathrm{A}}}\, dg. \quad . \quad . \quad . \quad . \quad (5)$$

Of particular interest are the forces which uniform magnetic and electric field exert on moving electrons. They are different from those of the Maxwellian theory. Suppose an electron moves between the poles of an electromagnet, then, according to equation (3), the force on the passive electron is

$$\mathbf{F} = \iint \frac{\mathrm{V}\, \mathbf{u}\, \mathbf{r}_1\, sq\, \sigma_m\, dg}{r^2 \left(1 - \frac{u^2}{v^2} \sin^2 \gamma\right)^{\frac{3}{2}}}$$

$$= \frac{4\, \pi\, \sigma_m\, q}{1 - \frac{u^2}{v^2} (\mathbf{g}_1 \mathbf{u}_1)^2} \mathrm{V}\, \mathbf{g}_1 \mathbf{u}. \quad . \quad . \quad . \quad . \quad (6)$$

Here $\mathbf{g}$ is the outward normal erected on the positive surface of the pole. For $4\pi\sigma_m$ we can put $\mathbf{H}_0$, the field intensity as measured by an observer at rest with the electromagnet. We then can write for the force exerted on the electron:

$$\mathbf{F} = \mathrm{V}\, \mathbf{H}_0 \mathbf{u} \frac{q}{1 - \frac{u^2}{v^2} (\mathbf{g}_1 \mathbf{u}_1)^2}, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

whereas the Maxwellian theory furnishes

$$q\, \mathrm{V}\, \mathbf{H}_0 \mathbf{u}.$$

Equation (7) can be tested by the deviation of Becquerel rays in a magnetic field.

Equation (4) furnishes the force which an electron experiences in a uniform electric field, for instance in that of a condenser. Let the surface-density of electricity on the condenser-plate be σ, then evidently

$$\mathbf{F} = \iint \frac{\mathbf{r}_1\, q\, v^2 s\sigma\, dg}{r^2 \left[1 - \frac{u^2}{v^2} \sin^2 \gamma\right]^{\frac{3}{2}}}$$

$$= \frac{4\pi\sigma\, v^2\, q}{1 - \frac{u^2}{v^2} (\mathbf{g}_1 \mathbf{u}_1)^2} \left\{ \mathbf{u}_1 \frac{u^2}{v^2} (\mathbf{g}_1 \mathbf{u}_1) - \mathbf{g}_1 \right\}. \quad . \quad . \quad (8)$$

$\mathbf{g}_1$ is the unit normal erected on the positive plate of the condenser; $(\mathbf{g}_1\,\mathbf{u}_1)$ is the cosine of the angle which $\mathbf{u}$ makes with $\mathbf{g}_1$.

In case the electrons move parallel to the plates the force takes the same form as in the Maxwellian theory.

§ 3. We will now proceed to consider the energy relations that follow from our principle of relativity. Suppose we have two systems A and B in relative motion. External forces are required to keep the motion steady. We denote the field intensities due to the active system A by $\mathbf{E}_A$ and $\mathbf{H}_A$, and those due to the active system B by $\mathbf{E}_B$ and $\mathbf{H}_B$, where it is understood that $\mathbf{E}_A$ and $\mathbf{H}_A$ are measured on B and $\mathbf{E}_B$ and $\mathbf{H}_B$ on A. Further, the volume-density of electricity of A is ρ_A, and that of B is ρ_B. As to the electromagnetic nature of the two systems, we will assume for simplicity that the system A contains only electrical masses, while B contains electrical and magnetic masses. Now remembering that the relative motion of electric and magnetic masses produces forces that are perpendicular to the direction of motion, and hence do not perform work, we find for the energy change in the element of time:

$$d\mathrm{W}=dt\iiint \rho_B\,\mathbf{u}\mathbf{E}_A\,d\tau=dt\iiint \rho_A\,\mathbf{u}\mathbf{E}_B\,d\tau.$$

By the aid of the equations I. to IV. and by the general rule

$$\mathbf{E}\operatorname{curl}\mathbf{H} = \operatorname{div}\mathrm{V}\mathbf{H}\mathbf{E} + \mathbf{H}\operatorname{curl}\mathbf{E},$$

we can easily transform this expression and obtain:

$$\frac{d\mathrm{W}}{dt}=-\frac{1}{8\pi}\frac{d}{dt}\iint\{\mathrm{V}\mathbf{E}_A\,\mathbf{H}_B+\mathrm{V}\mathbf{E}_B\,\mathbf{H}_A\}d\mathbf{g}$$
$$-\frac{1}{8\pi}\frac{d}{dt}\iiint\mathbf{H}_A\,\mathbf{H}_B\,d\tau-\frac{1}{8\pi v^2}\frac{d}{dt}\iiint\mathbf{E}_A\,\mathbf{E}_B\,d\tau. \quad . \quad (9)$$

In this equation $d\mathbf{g}$ denotes a vectorial element of surface which has the direction of the outward normal. The surface integral vanishes by removing the surface to infinity since radiation is excluded. Thus:

$$\frac{d\mathrm{W}}{dt}=-\frac{1}{8\pi}\frac{d}{dt}\iiint\mathbf{H}_A\mathbf{H}_B\,d\tau-\frac{1}{8\pi v^2}\frac{d}{dt}\iiint\mathbf{E}_A\,\mathbf{E}_B\,d\tau. \quad (10)$$

We examine now what happens when the external forces which kept the motion steady are removed. Acceleration will set in, and if the inertia of the system B is infinitely larger than than that of A, the latter system alone will be accelerated. The form of equation (10) suggests putting for the increase of the electromagnetic energy of A in virtue of its acceleration :

$$\frac{d\mathrm{W}_{\mathrm{A}}}{dt}=\frac{1}{8\pi}\frac{d}{dt}\iiint\overline{\mathbf{H}_{\mathrm{A}}{}^{2}}\,d\tau+\frac{1}{8\pi v^{2}}\frac{d}{dt}\iiint\mathbf{E}_{\mathrm{A}}{}^{2}\,d\tau. \qquad (11)$$

This equation, which holds for quasistationary motions, is compatible with our principle of relativity but does not follow from it.

§ 4. We consider an electron in motion, and we inquire after its electromagnetic masses. Let μ_l denote its longitudinal and μ_t its transversal mass. External forces acting in the direction of motion will increase its electromagnetic energy W. Proceeding exactly as in mechanics we can put

$$\mu_l=\frac{1}{u}\frac{\partial \mathrm{W}}{\partial u}=\frac{1}{\pi u}\frac{\partial}{\partial u}\iiint\mathrm{H}^2\,d\tau+\frac{1}{8\pi u v^2}\frac{\partial}{\partial u}\iiint\mathrm{E}^2\,d\tau. \quad (12)$$

Now the right member of this equation has been evaluated by G. F. C. Searle for the case of a charged sphere. Supposing an electron to be a rigid charged sphere, we find by an easy calculation the same expression as Abraham derived for the longitudinal mass.

We can connect the electromagnetic masses of the electron with the quantity of motion $\mathbf{M}$

$$\frac{d\mathbf{M}}{dt}=\mathbf{M}_1\frac{d\mathrm{M}}{dt}+\mathrm{M}\frac{d\mathbf{M}_1}{dt}. \quad . \quad . \quad . \quad . \quad (13)$$

The first term of the right member represents the longitudinal force, while the second member represents the transverse force.

From mechanics we know

$$\mathrm{M}=\int_0^u\frac{1}{u}\frac{\partial \mathrm{W}}{\partial u}\,du=\int_0^u\mu_l\,du$$

But the general expression for the transverse force is

$$\mathrm{M}\frac{d\mathbf{M}_1}{dt}=-\frac{u^2\mathbf{R}_1}{\mathrm{R}}\mu_t,$$

where $\mathbf{R}$ denotes the radius of curvature.

And

$$\frac{d\mathbf{M}_1}{dt}=-\mathbf{R}_1\frac{u}{\mathrm{R}}.$$

Therefore

$$\mu_t=\frac{1}{u}\int_0\mu_l\,du. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Substituting the value of μ_l we find the same value for μ_t as was obtained by Abraham.

It is worthy of note that whereas in general the forces acting on the electron in a uniform magnetic or electric field are different from those of the Maxwellian theory, yet the masses acted upon are identical. Becquerel rays moving obliquely to the field should therefore exhibit a different law of deviation from that expected on the basis of the Maxwellian theory. Here there is the possibility for carrying out an *experimentum crucis* to decide for or against the principle of relativity as here introduced.

§ 5. It is of especial interest to investigate the bearing of our principle on the phenomena of radiation. It is evident that relativity excludes any influence of the translatory motion of the earth on terrestrial optics; and it is likewise apparent that for relative motions of a source of light and an observer the principle of Doppler holds and that the phenomena of aberration are easily accounted for.

The phenomena hitherto considered were governed by the Maxwellian equations for *vacuous space*; and the question imposes itself whether the adopted principle of the relativity of motion is capable of accounting for phenomena in which the structure of ponderable matter plays an essential rôle.

The electron theory, as is well known, interprets the properties of matter by assuming that it is made up of complex assemblages of negative and positive charges. Thus, while our equations still hold in the interstices of matter for the single electrons, the observable properties are due to the combined effects of numerous assemblages of electrons. We obtain these effects by a process of averaging, and as Lorentz has shown the result is a set of equations for ponderable matter which is identical in form with that of Maxwell,—leaving out of regard phenomena of dispersion.

Now it is evident that with our principle of relativity we arrive at the same results as Lorentz as long as we restrict our investigation to bodies at rest. But there are two well-known experiments which have always been adduced as proving the existence of an æther at rest. These experiments are that of Röntgen on the magnetic effect of the rotation of a dielectric between charged condenser-plates, and that of Fizeau with the propagation of radiation in streaming water. It will suffice to show that the former experiment is in perfect accord with our hypothesis. For let a liquid dielectric flow between charged condenser-plates. At the boundary of the dielectric free charges will form and will take part in the motion of the liquid. This motion constitutes a convection current, and as such will be proportional to the free charge; and the magnetic field outside the condenser will therefore likewise be proportionate to the free charge. The latter for a given condenser charge, as follows from Maxwell's theory and from Lorentz's method, is proportionate to $\frac{K-1}{K}$, where K denotes the dielectric constant. And this is in perfect accord with the experimental result that the magnetic force is proportionate to the fraction $\frac{K-1}{K}$.

Now any theory which yields this coefficient for the Röntgen current, leads to the same coefficient called Fresnel's coefficient for the Fizeau experiment. Evidently according to Maxwell $\frac{K-1}{K}$ equals $1-\frac{1}{n^2}$ where n is the refractive index, if we disregard dispersion effects. I intend working out a complete theory of the Fizeau experiment on the basis of the principle of relativity.

I hope to have shown by this short survey that this principle can account for all the known facts of electromagnetism.

Bonn University,
January 1907.

XXXIII. *The Magnetic Field and Inductance Coefficients of Circular, Cylindrical, and Helical Currents.* By ALEXANDER RUSSELL, *M.A., M.I.E.E.**

TABLE OF CONTENTS.

1. *Introduction.*

SERIES formulæ † are usually given for the components of the magnetic force at a point in the neighbourhood of a current flowing in a circular filament. They are generally

* Communicated by the Physical Society: read February 8, 1907.

† E. Mascart and J. Joubert, *Leçons sur l'Electricité et le Magnetisme*, vol. ii. § 566 (1897).