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XI. *On the Partition of Energy between Matter and Æther.* By J. H. JEANS, M.A., Fellow of Trinity College*.

§ 1. **T**HE question discussed in the present paper is one which I stated in an earlier paper on the Theory of Gases (Phil. Trans. cxvii. p. 397, 1901) as follows:—

“If an interaction between matter and æther exists, no matter how small this interaction may be, the complete dynamical system will consist of the molecules of the gas, together with the æther, and must therefore be regarded as a system possessing an infinite number of degrees of freedom. Applying Boltzmann’s Theorem to this system, we are merely led to the conclusion that no steady state is possible until all the energy of the gas has been dissipated by radiation into the æther. This application of the theorem may or may not be legitimate, but it is, I think, certain that no other application is legitimate.”

The same question has recently been discussed by Lord Rayleigh in ‘Nature’*. In the present paper, I have tried to investigate the legitimacy of applying the theorem of equipartition to a system consisting of both matter and æther, assuming for the present that this matter is in the gaseous state.

Let us, to take the simplest case first, consider a gas of which the molecules are rigid spheres or point centres of force, possessing no internal degrees of freedom, and acting on one another partly through “material” forces and partly through their interaction with the æther, this action being very small except when two molecules are near together. If there are N molecules (A, B, C, . . .), we can suppose the configuration of the system to be specified at any instant by

- (i.) $6N$ coordinates $x_a, y_a, z_a, u_a, v_a, w_a, x_b, y_b, \dots$ giving the positions and velocities of the molecules, and
- (ii.) m coordinates $\xi_1, \xi_2, \dots, \xi_m$ (*e. g.* components of electric and magnetic force) giving the state of the æther at every point, these coordinates being independent. We here suppose that any relations which must be satisfied in the free æther (*e. g.*, the vanishing of the divergence of the electric and magnetic forces) have already been taken into account.

* Communicated by the Author. —

† April 13th and May 16th, 1905. The present paper was written in March, and was sent to the Phil. Mag. immediately after the appearance of Lord Rayleigh’s first letter. It is therefore not intended as a reply to the second letter, although the questions discussed happen to be much the same. The postscript was, however, written with special reference to the letter of May 16th.

The system is accordingly specified by $6N + m$ Lagrangian coordinates, and the energy is some function of these $6N + m$ coordinates.

§ 2. If there is *no* interaction between matter and æther, this energy can readily be expressed as a sum of squares of these coordinates. The energy is of the form

$$\frac{1}{2}m \Sigma (u_a^2 + v_a^2 + w_a^2) + f(\xi_1, \xi_2, \dots, \xi_m).$$

The last term represents the energy in free æther which, if we assume the exact linearity of the electromagnetic equations for all electric intensities with which we shall ultimately be concerned, is known to be

$$\frac{1}{8\pi} \iiint (X^2 + Y^2 + Z^2 + \alpha^2 + \beta^2 + \gamma^2) dx dy dz.$$

The energy is not yet expressed as the required sum of squares, for the quantities $X, Y, Z, \alpha, \beta, \gamma$, are not independent Lagrangian coordinates. Let each of the quantities $X, Y, Z, \alpha, \beta, \gamma$, regarded as a function of x, y, z , be expressed in the form

$$f(x, y, z) = \frac{1}{\pi^3} \iiint_0^\infty \iiint_{-\infty}^{+\infty} f(\lambda, \mu, \nu) \cos p(\lambda - x) \cos q(\mu - y) \cos r(\nu - z) dp dq dr d\lambda d\mu d\nu,$$

and let the right-hand integral be further transformed into the sum of an infinite number of terms of the form

$$C \frac{\cos}{\sin} (px + qy + rz),$$

or again

$$C \frac{\cos}{\sin} \kappa (lx + my + nz),$$

where l, m, n are direction cosines, $\kappa^2 = p^2 + q^2 + r^2$, and C is independent of x, y , and z . The value of any one of the six quantities $X, Y, Z, \alpha, \beta, \gamma$, say the first, can now be expressed in the form

$$X = \iiint_{-\infty}^{+\infty} \iiint_0^{2\pi} (X_1 \cos + X_2 \sin)(x \sin \theta \cos \phi + y \sin \theta \sin \phi + z \cos \theta) d\kappa \sin \theta d\theta d\phi \dots \quad (1)^*$$

where X_1 and X_2 are functions of κ, θ , and ϕ only.

The condition $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0$ is now satisfied if

$$X_1 \sin \theta \cos \phi + Y_1 \sin \theta \sin \phi + Z_1 \cos \theta = 0; \quad (2)$$

* Cf. Whittaker's solution of $\nabla^2 V + V = 0$, 'Modern Analysis,' p. 321.

and there is a similar relation for X_2, Y_2, Z_2 , and similarly for $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$.

The quantities $X_1, X_2, Y_1, Y_2, \alpha_1, \alpha_2, \beta_1, \beta_2$ can now be taken to be the independent Lagrangian coordinates $\xi_1 \xi_2 \dots \xi_m$, the values of $Z_1, Z_2, \gamma_1, \gamma_2$ being given by relations of the type (2). Substituting from equations (2) we find that

$$\frac{1}{8\pi} \iiint (X^2 + Y^2 + Z^2 + \alpha^2 + \beta^2 + \gamma^2) dx dy dz$$

becomes a sum of squares of these coordinates.

The total energy is accordingly of the form

$$E = \frac{1}{2} m \sum_N (u_a^2 + v_a^2 + w_a^2) + \sum_m a_1 \xi_1^2.$$

The dynamical equations which regulate the variation of $\xi_1, \xi_2 \dots$ are obtained from the electro-dynamical equations

$$\frac{\partial X}{\partial t} = \frac{1}{V} \left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \right), \text{ etc.}$$

As a typical equation we have

$$\dot{X}_1 = \frac{1}{V \cos \theta} \{ (\sin^2 \theta \sin^2 \phi - \cos^2 \theta) \beta_2 + \sin^2 \theta \sin \phi \cos \phi \cdot \alpha_2 \}. \quad (3)$$

If q_1, p_1 are any pair of coordinates of position and momentum of the material system, we have the variation of p_1, q_1 determined by the usual Hamiltonian equations

$$\dot{p}_1 = - \frac{\partial H}{\partial q_1}, \quad \dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dots \dots \quad (4)$$

and equations (3) and (4) contain between them the dynamics of the system.

Let us represent the motion in a generalised space corresponding to the $6N + m$ dimensions

$$p_1, q_1, \dots \xi_1, \xi_2, \dots \xi_m.$$

If ρ is the density of a fluid moving in this space, the equation of continuity is

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \sum \left(\frac{\partial \dot{p}_1}{\partial p_1} + \frac{\partial \dot{q}_1}{\partial q_1} \right) + \sum \frac{\partial \dot{\xi}_1}{\partial \xi_1} = 0.$$

If the fluid moves in accordance with the dynamical equations, the first sum vanishes as usual by equations (4), and the second sum vanishes by equations (3). Thus $D\rho/Dt = 0$, so that the fluid, if initially homogeneous, remains so through all time.

§ 3. As a result of the assumed absence of interaction between matter and æther, the condition of "continuity of path" is not yet satisfied: the fluid moves as though the space were divided into water-tight compartments. The equations of the divisions between these compartments are simply the integrals of the equations of motion (3) and (4), such, *e. g.*, as the family of surfaces

$$X_1^2 + Y_1^2 + \alpha_2^2 + \beta_2^2 = \text{constant.}$$

If the material system is divided into separate masses of gas, we have as further integrals

$$E_1 = \text{const.} \quad E_2 = \text{const.} \quad \&c.,$$

where $E_1, E_2 \dots$ are the material energies of these systems separately.

§ 4. If we now admit a very small interaction between matter and æther, the conditions are changed. The equations of motion are changed by the introduction of cross-terms which link up the material system with the æther, and we may assume that no integrals are left except the energy equation. Thus the whole system of water-tight compartments is broken down except for the family of surfaces $E = \text{constant}$ where E is the total energy. The new coordinates of position and momentum differ from the old by terms which depend on the interaction just introduced, and so may be regarded as approximately unaltered when this interaction is small. The whole system, regarded now as a single electromagnetic system, will still have equations of motion of the Hamiltonian type, so that the equation $D\rho/Dt = 0$ remains true accurately.

The conditions for the theorem of equipartition are now satisfied, so that the expectation of the energy of each of the degrees of freedom is the same, namely

$$\frac{E}{3N + m}.$$

§ 5. When we have a finite amount of matter in infinite æther, N is finite while m is infinite, so that the ratio of energy of matter to that of æther, namely $3N/m$, is infinitesimal.

Definite though this result may seem, the partition of energy is not yet completely known. If we attempt to find the law of distribution of energy in the radiation-spectrum, the equipartition theorem directs us to assign equal amounts of energy to each coordinate, and therefore to each value of k , but gives no information as to the "density" of coordinates within the different range of values of k . This circumstance

suggests that the physical problem is not yet fully threshed out.

§ 6. Let us assume, as a preliminary to carrying the investigation a stage further, that the transfer of energy from matter to *æther* does not occur at all on the free-paths of molecules, but that vibrations are set up in the *æther* at collisions. If τ is an average time of duration of a collision, the frequency of the waves "forced" in the *æther* is less than or comparable with τ , waves of frequencies much greater than τ being, as can readily be shown by analysis which I have already given*, of infinitesimal amplitude. We now see that the ultimate distribution of energy in the spectrum would depend on the temperature from which the material system had started, and the radiation at any instant would depend on the temperature at that instant. A mass of gas might at one time emit a spectrum extending into the ultra-violet; by the time the gas had cooled to half its former temperature, the spectrum would be a heat-spectrum only. The actual temperature at which the light-spectrum would disappear is easily found, from the analysis already quoted, to be comparable with 1000° C.

§ 7. Suppose, however, that we now consider a mass of gas shut up in a perfectly reflecting enclosure. The coordinates of the *æther* must no longer correspond to plane waves in unlimited *æther*; they must correspond to the principal vibrations of the *æther* inside the enclosure. The number of these vibrations is still infinite, but the number of which the frequency is below any given limit is finite. Let us suppose that the average time of a collision is τ , and that of the infinite series n of principal vibrations in the *æther*, a finite number s have periods which are less than, or at most comparable with τ , the remaining $n-s$ having periods small compared with τ . Then, of the n degrees of freedom of the *æther*, only s receive any perceptible amount of energy from the molecules at collisions. We may say then that the transfer of energy between the material degrees of freedom and s degrees of *æther* freedom is comparatively rapid, while that to the remaining $n-s$ degrees is very slow. For an enormous time these $n-s$ degrees of freedom will not receive their due share of the energy, while the energy will rapidly equalise itself between the remaining $3N+s$ degrees of freedom. During this time, the ratio of the energy of the *æther* to that of the material system is $s/3N$, and this will generally be very small.

§ 8. For instance, let us enclose gas at atmospheric pressure

* 'The Dynamical Theory of Gases,' Chapter IX., or *Phil. Mag.* August 1903: "On the Vibrations set up in Molecules by Collisions."

and at a temperature of 15° C. in a cubical enclosure of edge equal to l centimetres. For this system $N = 4 \times 10^{19} l^3$ very nearly. The free vibrations of the æther are known (*cf.* Lord Rayleigh's 'Sound,' § 267), each vibration corresponding to values of $X, Y, Z, \alpha, \beta, \gamma$ of the type

$$\frac{\cos}{\sin} \left(\frac{p\pi x}{l} \right) \frac{\cos}{\sin} \left(\frac{q\pi y}{l} \right) \frac{\cos}{\sin} \left(\frac{r\pi z}{l} \right), \dots \dots \dots (5)$$

where p, q, r are integers. These principal vibrations cannot be compounded into plane waves, as was done in § 2 when the space was unlimited.

The frequency κ of this vibration is given by

$$(p^2 + q^2 + r^2) \frac{\pi^2}{l^2} = \frac{\kappa^2}{V^2}, \dots \dots \dots (6)$$

where V is the velocity of light. For air at 15° C. the mean duration of a collision is of the order of 10^{-13} seconds. Let us suppose the s degrees of freedom to consist of all those for which the period is greater than 10^{-14} seconds. The period given by equation (6) is

$$\frac{2\pi}{\kappa} = \frac{2l}{V\sqrt{p^2 + q^2 + r^2}}.$$

If we take $V = 3 \times 10^{10}$, we find that the upper limit of $\sqrt{p^2 + q^2 + r^2}$ in order that the vibration may have a period of not less than 10^{-14} seconds, is $20000l/3$.

The number of sets of positive integral values of p, q, r for which $\sqrt{p^2 + q^2 + r^2} < \theta$, where θ is large, is approximately $\frac{1}{6}\pi\theta^3$, so that in the present case the number of sets of values of p, q, r is approximately

$$\frac{4\pi \times 10^{12}}{81} l^3.$$

Each system of values of p, q, r gives four principal coordinates, so that for our present purpose

$$s = \frac{16\pi \times 10^{12}}{81} l^3.$$

Taking the value $N = 4 \times 10^{19} l^3$, we find

$$\frac{s}{3N} = \frac{4\pi}{243 \times 10^7} = 5 \times 10^{-9} \text{ roughly,}$$

so that the energy of the æther is almost inappreciable, no matter how large the enclosure may be.

§ 9. We can now trace the course of events when one or more masses of gas are left to themselves in undisturbed æther. At first we may suppose that the total energy is entirely that of the principal degrees of freedom. The transfer of energy between the different degrees of freedom of the gas at any point is, as we know, extremely rapid. The first phenomenon, then, is that the energy of these degrees of freedom arranges itself according to Maxwell's Law. The time required is a small fraction of a second. The next phenomenon, at any rate if the masses of gas are small, is an equalization of temperature by conduction through each mass. Simultaneously with this, however, a transfer of energy is taking place between the principal degrees of freedom of the molecules, and the vibrations of low frequency in the æther. This equalises the temperatures of different masses of gas, and endows the æther with a small amount of energy, equal to that of a finite number of molecules of the gas, but small compared with the total material energy. The time required for these phenomena must be measured in minutes, days, or centuries, according to the linear scale of the system. After this, a third transfer of energy begins to show itself, but the time required for this must be measured in millions or billions of years unless the gas is very hot. A transfer takes place between the energy of the principal degrees of freedom of the gas and that of degrees of freedom which may either be in the æther or in the atoms of the gas, but which have the common characteristic that they represent vibrations of high frequency. If the gas is in vacant space, the energy set free streams away into space, but if the whole system is enclosed by an ideal perfectly reflecting boundary, the energy accumulates in the æther.

POSTSCRIPT, added June 7th.

§ 10. As in § 8, the number of degrees of freedom of the æther, of which the frequency is less than k , inside a cube of edge l , is $\frac{2}{3}l^3k^3/\pi^2V^2$. Hence the number of vibrations of frequency between k and $k+dk$ is $\frac{2l^3}{\pi^2V^2}k^2dk$. At absolute temperature T each degree of freedom possesses energy $\frac{1}{2}RT$, where, if the units are those of the C.G.S. system and of the centigrade thermometer, the value of R is 9.3×10^{-17} (*cf.* "The Dynamical Theory of Gases," § 130). Thus the
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energy per cubic cm. of vibrations of frequencies between k and $k + dk$, is

$$\frac{RT}{\pi^2 V^3} k^2 dk,$$

or, in terms of the wave-length λ in free æther, the energy of wave-lengths between λ and $\lambda + d\lambda$ is

$$8\pi RT\lambda^{-4}d\lambda (7)$$

This is one-eighth of the amount found by Lord Rayleigh ('Nature,' May 16th), but agrees exactly with that given by Planck (Drude's *Annalen*, iv. p. 553) for large values of λ . It seems to me that Lord Rayleigh has introduced an unnecessary factor 8 by counting negative as well as positive values of his integers ξ, η, ζ . From formula (7), it follows that the total energy of radiation at temperature T is

$$8\pi RT \int_{\lambda_0}^{\infty} \lambda^{-4} d\lambda + \int_0^{\lambda_0} f(\lambda, T, t) d\lambda, . . . (8)$$

in which λ_0 is the shortest wave-length for which the vibrations may be supposed to possess their full energy, and the second integral represents the energy of waves of wave-length less than λ_0 , the energy of radiation of these waves being a function not only of T and λ , but also of t , the time which has elapsed since the closing in of the æther. Formula (8) does not, of course, claim to express the partition of energy in the radiation emitted by a hot solid: it is the radiation when a mass of gas has been shut up for time t in a perfectly reflecting enclosure. And the formula applies only to the continuous spectrum of the gas produced by molecular motions; no account is taken of the line spectrum, produced, so far as we know, by atomic vibrations.

XII. *On the Radioactive Matter present in the Atmosphere.*
By A. S. EVE, M.A., McGill University, Montreal*.

THE presence of radium in the earth, and of the emanation of radium in the atmosphere, has been well established. Exact measurements of the amount of the radioactive substance in the air are, however, needed. Professor Rutherford was kind enough to propose to the writer some methods and experiments by which to estimate the quantities present, and to throw further light on their effects.

The main objects of these investigations were:—

1. To estimate the amount of radioactive matter present in

* Communicated by Professor E. Rutherford, F.R.S.