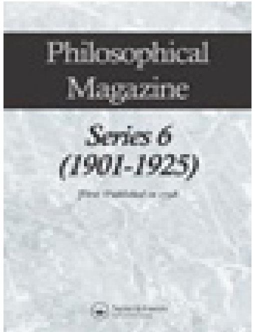

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III. *On Transfinite Cardinal Numbers of the Exponential Form.* *By* PHILIP E. B. JOURDAIN, *B.A.*, *Trinity College, Cambridge* *.

AMONG cardinal numbers of the form

$$\mathfrak{a}^{\mathfrak{b}},$$

where $\mathfrak{b}$, at least, is transfinite, the smallest and most interesting is the cardinal number of the number-continuum:

$$2^{\aleph_0} = \nu^{\aleph_0} = \aleph_0^{\aleph_0} = \aleph_1^{\aleph_0}.$$

Cantor has always been of the conviction that

$$2^{\aleph_0} = \aleph_1,$$

and investigations in the theory of manifolds tend to increase one's belief in the truth of this conviction, although hitherto no proof of it has been given. It is very important to prove that $2^{\aleph_0}$ is equal to some Aleph in order to be certain that the number-continuum is not what I have called an "inconsistent aggregate" †.

A failure to prove the above equality by an attempted arrangement of all the real numbers between 0 and 1 in a well-ordered series ultimately led me to the result of § 1, that the cardinal number of all the real numbers which can be represented by fundamental series of which the general term is known as a rational function of its index is $\aleph_0$, which proves that it is impossible to obtain a series of type ω_1 from such numbers, and consequently the impossibility of actually proving that

$$2^{\aleph_0} = \aleph_1$$

in a large class of cases.

This negative result, which is the only definite result I have as yet been able to obtain on the question of the equality

$$2^{\aleph_\alpha} = \aleph_{\alpha+1},$$

where α is any ordinal number, allows, however, a number of conclusions to be drawn in what I have called the "cardinal theory of functions" (§ 2). The result that only a small

* Communicated by the Author.

† Phil. Mag. January 1904, p. 66. In § 5 (p. 67) of this article I tacitly assumed that the exponential numbers in question belonged to *consistent* aggregates, or manifolds; for, though this is not rigorously proved to be the case, nothing seems more unlikely than that it should not be so. Further information on the subject of inconsistent aggregates is given below, §§ 6–9.

portion of the whole manifold of analytic functions, for example, are analytically representable by no means implies that general theorems cannot be found which apply to *all* analytic functions, and even in particular to all which are not representable (§ 3) ; so that the concept of function taken by Pringsheim, in the recent *Encyclopädie der mathematischen Wissenschaften* *, as the basis of the general theory of functions, appears to be too narrow.

After a digression on the cardinal theory of functions and on the utility of the concept of the "aggregate of definition" (§ 4), I prove (§ 5) a theorem due to Bernstein on exponential numbers, which includes a result of my own † as a special case, and allows us to find the necessary and sufficient conditions that

$$\mathfrak{a}^{\mathfrak{b}}=\mathfrak{a},$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are any cardinal numbers.

In § 6, I make a few remarks on the extended principle of induction used in § 5, which serves to define the series W of ordinal numbers. The series ($\mathfrak{W}$) such that every well-ordered series is ordinally similar either to $\mathfrak{W}$ or to a segment of $\mathfrak{W}$ extends beyond W (§ 7), and this more exact account of W throws a clearer light on my solution of Burali-Forti's contradiction (§ 8).

Finally, in § 9, I revert to the consideration of the concept of "consistency," with especial reference to investigations of Cantor, Hilbert, and Russell.

1.

Every real number is determined by an enumerable sequence of rational numbers, and hence the cardinal number of the aggregate of real numbers is seen without difficulty to be $2^{\aleph_0}$. But, if this enumerable sequence is

$$u_1, u_2, \ldots, u_\nu, \ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

we must, if we are to be able to determine exactly the real number in question, limit the form of u_ν to be a function obtained by performing the elementary operations a finite number of times on ν and a finite number (n) of given rational numbers ; in symbols

$$u_\nu=f(\nu, r_1, r_2, \ldots, r_n).$$

By this limitation, the cardinal number of the aggregate of

* Bd. ii. A. 1, pp. 9–11.

† Phil. Mag. March 1904, p. 302.

the real numbers which can now be represented by the sequence (1) is merely $\aleph_0$. This may be shown as follows.

The function obtained by performing the elementary operations a finite number of times on $(n+1)$ arguments is a rational function of arguments, the coefficients of which are integers *. Since, then, in each case we only have a finite number (m) of coefficients to choose, and each coefficient can be chosen out of $\aleph_0$ values (the integers), the cardinal number of those functions of m coefficients is

$$\aleph_0^m.$$

Further, we get *all* such functions by giving m all possible finite values in turn ; consequently the cardinal number of all these functions is

$$\aleph_0^1+\aleph_0^2+\ldots+\aleph_0^m+\ldots,$$

the series being of type ω, and consequently—remembering that each term reduces to $\aleph_0$—the cardinal number in question is

$$\aleph_0 \cdot \aleph_0 = \aleph_0.$$

We may state this result in words as follows: The cardinal number of all the real numbers that we can *actually* determine (that is to say, determine in the sense explained above) is

$$\aleph_0.$$

Accordingly, if, as is the case with some methods that suggest themselves for arranging real numbers in a well-ordered series, we only use such "actually determinable" real numbers, we can never arrange them in a series of type ω_1. For every enumerable manifold can be well-ordered, but the series always breaks off before some number of the second number-class is reached.

Now, this conclusion has applications, which seem to me to be of some importance, in the theory of functions. In the first place, such sequences as (1) enter into Weierstrass' construction of whole transcendental functions with given zeros, Mittag-Leffler's construction of analytic functions whose singularities form an aggregate whose first derivative is enumerable, and the construction of whole transcendental functions which take given values at certain points †. We

* *Cf.* Harnack, "An Introduction to the Elements of the Differential and Integral Calculus," Eng. trans. p. 67 (1891).

† This construction, which forms an extension of Lagrange's interpolation-formula to whole transcendental functions, is given by me in part of an essay "On the General Theory of Functions," which is to appear shortly in Crelle's *Journal für Math.* It is very simple, and is obtained by the multiplication of a whole function constructed by Weierstrass' theorem with a meromorphic function constructed by Mittag-Leffler's theorem.

conclude then that, although the cardinal number of any of the above classes of functions is $2^{\aleph_0}$, the cardinal number of those functions which we can actually represent is $\aleph_0$. Of course, we make the same stipulation as to representability in the case of the extraneous factor in all these constructions. In Weierstrass' construction this factor is

$$e^{g(z)},$$

where $g(z)$ is any whole function. Thus, we cannot, for example, consider

$$c \,.\, P(z)$$

as a constructible function, if $P(z)$ is the product of primary factors and c is *any* real number; for c must be a representable real number.

In the second place, it appears that the postulate of "arithmetical definability," which Pringsheim has introduced as an essential qualification of the functions which can be treated in a general theory of functions, cannot be considered as relevant, for the double reason that it is necessary to take account of functions which cannot be defined by $\aleph_0$ conditions and that even functions which are so definable are not in general "arithmetically representable." The former reason rests on a theorem which constitutes an important part of what I have called "the cardinal theory of functions"; the latter reason rests on a theorem which is easily obtained from what precedes and completes, in a sense, the cardinal theory of functions.

2.

The cardinal theory of functions consists of two parts: The determination of the cardinal numbers of the various aggregates of functions, and the drawing of conclusions, from inequalities between these numbers, as to the non-inclusion of certain aggregates in certain others. Thus, from the results that the cardinal number of all integrable functions is

$$2^{2^{\aleph_0}},$$

while that of all functions representable as limits of sequences of continuous functions is

$$2^{\aleph_0},$$

and

$$2^{2^{\aleph_0}} > 2^{\aleph_0},$$

we conclude that a function, even when it is restricted to be integrable, is not in general, representable as the limit of a sequence of continuous functions *.

This example suffices to substantiate the contention that the requirement of arithmetical definability is unnecessarily narrow for the possibility of a general theory of functions. In other words, there exist propositions in the general theory of functions (on integrable functions, for example) which apply to a much wider class of functions than that of arithmetically definable functions.

Now the class of functions which can be represented as limits of infinite series of continuous functions, or, what is the same thing, of functions to which an "existence-theorem" is applicable, contains, of course, all arithmetically definable functions, but not inversely. For every function of the former class is completely and uniquely determined by the datum of the enumerable sequence of the coefficients of the sequence of polynomials by which it can be replaced, and the cardinal number of the sequences (1) whose general term can be found in the manner indicated, but in which, possibly, a finite number of terms are completely arbitrary, is

$$\aleph_0,$$

and

$$2^{\aleph_0} > \aleph_0.$$

3.

Although there is thus no possibility of actually constructing a greater cardinal number of functions than $\aleph_0$, it by no means follows that definite theorems cannot be found which hold for a greater number. The fact of the existence of a general theory of analytic functions is alone sufficient to disprove this, and, consequently, also for this reason the requirement of the arithmetical definability of functions is too narrow.

Further, it is interesting to see that there is a theorem which holds of actually non-representable analytic functions, due to Borel and Fabry †. The series

$$a_0 + a_1 z + a_2 z^2 + \ldots + a_\nu z^\nu + \ldots, \quad . \quad . \quad . \quad . \quad (2)$$

where z is a complex variable, represents either the whole of an analytic function or part of one within a circle on whose circumference is at least one singularity. The theorem of

* Messenger of Math. Sept. 1903.

† *Cf.* Hadamard, 'La série de Taylor et son prolongement analytique,' Paris, 1901, pp. 33–36.

Borel and Fabry now says that, when the sequence

$$a_0, a_1, \ldots, a_\nu, \ldots,$$

is a "série écrite au hasard,"—that is to say, a sequence whose general term cannot be given in the manner explained above,—this circle of convergence is a line entirely composed of essential singularities, so that (2) cannot be continued beyond this line, and represents the whole function.

4.

I have regarded the conception of an "aggregate of definition" as an essential part of the cardinal and ordinal theories of functions. By an "aggregate of definition" I understand any aggregate of values among those of the independent variable such that, when the values of the (one-valued) function are given for the points of merely *this* aggregate, the values for all other points in the domain of existence are determined. When the domain of the variable is the continuum of real numbers, the cardinal number of this aggregate, when the function is continuous or analytic, is

$$\aleph_0,$$

and the ordinal types are respectively

$$\eta \text{ and } \omega \text{ or } {}^*\omega.$$

Since, however, a knowledge of the means whereby the value of the function at one point is calculated from its values at other points (which varies for different classes of functions) appears indispensable in addition to a knowledge of the aggregate of definition, and the latter knowledge then follows from the former, it might appear that the "aggregate of definition" is always a superfluous conception. The following example will show that this is not the case.

It has long been known that a one-valued analytic function $f(z)$ reduces to a constant if it has a period smaller in absolute amount than any assignable positive number. This proposition, without the necessary restriction to one-, or at least finite-valuedness, was treated, without complete justification, as obvious by Jacobi; so that Weber in the reprint*, edited by him, of Jacobi's memoir, has given a proof of the proposition in question together with an analogous theorem on one-valued continuous functions of a real variable.

* "Ueber die vierfach periodischen Functionen zweier Variabeln, auf die sich die Theorie der Abel'schen Transcendenten stützt," von C. G. J. Jacobi (Crelle's *Journal*, Bd. xiii. (1834); Ostwald's *Klassiker d. exakten Wiss.* No. 64, hrsgb. von H. Weber, Leipzig, 1895, pp. 36–39).

Now, these theorems, together with an extension, follow at once from a consideration of the character of the aggregates of definition.

Suppose that a one-valued analytic function, $f(z)$, with a period smaller in absolute amount than any real positive number without being a constant exists, and let $f(z)$ have a finite value for every point z such that $|z|<a$, where a is some positive constant. Then there must be a sequence of points $\{z_\nu\}$ condensing at some point z within the circle and such that $|z_\nu|<a$, such that

$$f(z_\nu)=f(z)=\mathrm{A}.$$

Now $\{z_\nu\}$ forms an aggregate of definition, and consequently

$$f(z)=\mathrm{A}.$$

Further, if x and $\mathrm{F}(x)$ are real, and $\mathrm{F}(x)$ is one-valued and continuous, and $\mathrm{F}(x)$ has a period of the above nature, it is easy to see that the points where $\mathrm{F}(x)$ is equal to some number A lie everywhere dense, and thus form an aggregate of definition of the continuous function. But it is evident that this argument applies also to the case where x and $\mathrm{F}(x)$ are complex and the periodicity of $\mathrm{F}(x)$ is double. Hence, a real or complex function of a real or complex variable cannot have a (respectively single or double) period smaller in absolute amount than any positive non-zero number *provided only that the function is continuous.*

5.

I now return to the consideration of exponential numbers in general, and prove the theorem of Bernstein * that, if $\aleph_\alpha$ and $\aleph_\beta$ are any two Alephs,

$$\aleph_\alpha^{\aleph_\beta}=\aleph_\alpha\,.\,2^{\aleph_\beta}. \qquad (3)$$

In the first part † of Bernstein's proof, (3) is proved if

$$\aleph_\beta \geqq \aleph_\alpha,$$

* "Untersuchungen aus der Mengenlehre," *Gött. Diss.*, Halle-a.-S., 1901, pp. 49–50.

† A more general theorem than that of the *first* part was proved by me before I had seen Bernstein's memoir (Phil. Mag. March 1904, p. 302), namely: if $\aleph_\alpha \leqq 2^{\aleph_\beta}$, then $\aleph_\alpha^{\aleph_\beta}=2^{\aleph_\beta}$.

I take this opportunity of correcting two slips in this paper:

p. 303, line 7; *delete* "$\leqq 2^{\mathfrak{a}}$."

line 18; *for* "$\aleph_1^{\aleph_0}=\aleph_1$" *read* "$\aleph_1^{\aleph_0}=2^{\aleph_0}$."

and in the second part it is supposed that

$$\aleph_\alpha > \aleph_\beta\ ;$$

and an extended form of complete induction, which extends to all Alephs, is used, and is, in essentials, as follows.

By Cantor's definition of an exponential number,

$$\aleph_\alpha^{\aleph_\beta}$$

is the cardinal number of all coverings (Belegungen) of a manifold of cardinal number $\aleph_\beta$ with the elements of a manifold of cardinal number $\aleph_\alpha$, which we will suppose to be arranged in a series of type ω_α. Now every such covering is obtained by the covering of the manifold of cardinal number $\aleph_\beta$ with some (or all) elements of some segment of the above series of type ω_α; and the cardinal number of this segment is less than $\aleph_\alpha$. Hence, each of the coverings first-named is found among the aggregate of all the coverings of the manifold of cardinal number $\aleph_\beta$ with the elements of the various segments, taken in turn, of the series of type ω_α. Now, the cardinal number of all the coverings of a manifold of cardinal number $\aleph_\beta$ with the elements of a segment M_γ of the above series M_{ω_α} of type ω_α is

$$\mathfrak{m}_\gamma^{\aleph_\beta}, \quad \ldots \ldots \ldots \quad (4)$$

where $\mathfrak{m}_\gamma$ is the cardinal number of M_γ. Hence we can state, by the Schröder-Bernstein theorem, that

$$\aleph_\alpha^{\aleph_\beta} \leqq \sum_{\gamma<\omega_\alpha} (\mathfrak{m}_\gamma^{\aleph_\beta}), \quad \ldots \ldots \quad (5)$$

where the Σ means that the summation is to be extended over all the numbers (4) such that M_γ is a segment of M_{ω_α} (or such that $\gamma < \omega_\alpha$).

On the other hand,

$$\mathfrak{m}_\gamma^{\aleph_\beta} \leqq \aleph_\alpha^{\aleph_\beta},$$

and hence the right-hand side of (5) is less than or equal to

$$\sum_{\gamma<\omega_\alpha} (\aleph_\alpha^{\aleph_\beta}) = \aleph_\alpha \,.\, \aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}.$$

Comparing this result and (5), we conclude that

$$\aleph_\alpha^{\aleph_\beta} = \sum_{\gamma<\omega_\alpha} (\mathfrak{m}_\gamma^{\aleph_\beta}) \,. \quad \ldots \ldots \quad (6)$$

Hence, if we know a theorem for all Alephs less than $\aleph_\alpha$, we may, by substitution in the right-hand side of (6), find the same theorem for $\aleph_\alpha$. Such a theorem is

$$\aleph_\alpha^{\aleph_\beta}=2^{\aleph_\beta}=\aleph_\alpha \,.\, 2^{\aleph_\beta},$$

where

$$\aleph_\alpha \leqq \aleph_\beta\ ;$$

and (6) allows us to conclude the validity of the theorem (3), even if

$$\aleph_\alpha > \aleph_\beta.$$

From (3), we may deduce an interesting theorem concerning those cardinal numbers which are unaltered by exponentiation. In fact, from (3) and the laws of multiplication of Alephs *, it follows that if, and only if

$$2^{\aleph_\beta} \leqq \aleph_\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the right-hand side of (3) reduces to $\aleph_\alpha$.

Thus, that

$$\aleph_\alpha^{\aleph_\beta} = \aleph_\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

it is necessary and sufficient that (7) should hold. In particular, if, as is probable, we can assert (8) if only

$$\aleph_\beta < \aleph_\alpha,$$

it is necessary and sufficient that

$$2^{\aleph_\beta} = \aleph_{\beta+1}.$$

6.

In the extended principle of induction used above, which may be stated thus: If a certain proposition P holds of $\aleph_0$, and if, when it holds of all Alephs less than $\aleph_\alpha$, it holds of $\aleph_\alpha$, P holds of all Alephs; the proof of P for $\aleph_\alpha$ is reduced, by (6), to the proof of P for a sum (of cardinal number $\aleph_\alpha$) of numbers for which P is assumed to hold. This method cannot be applied to give a shorter proof of the equality †

$$\aleph_\gamma^\nu = \aleph_\gamma,$$

since we must have previously proved that

$$\aleph_\gamma^2 = \aleph_\gamma$$

in order to prove that the cardinal number of a series of type $\omega_{\gamma+1}$ is greater than that of a series of type ω_γ ‡. But if the exponent, instead of being ν, is transfinite, we can, as we

* Phil. Mag. March 1904, p. 301.

† *Ibid.* p. 300. This theorem seems, from an indication given by Bernstein (*op. cit.* p. 49), to have been known to Cantor.

‡ *Ibid.* Jan. 1904, p. 74.

can easily convince ourselves, apply the extended method of induction, provided that exponentiation with this transfinite number leaves *some* Aleph unaltered.

This extended principle of induction is very closely connected (through Cantor's "third principle of generation") with the question as to whether the ordinal number of the series of all the ordinal numbers defined by Cantor can be defined without contradiction, and hence with the argument of Burali-Forti *. I have returned, then, in the following section, to the considerations which I have advanced in the January (1904) number of this Magazine.

7.

In defining an aggregate which should serve as a criterion whether any given aggregate is "consistent" or "inconsistent" †, I have used the conception, mentioned by Schönflies, of the (well-ordered) series 𝔚 ‡ such that every well-ordered series is similar to it or to a segment of it.

This series 𝔚 was, now, stated by Schönflies § to be similar to the series (W) of all the ordinal numbers, as defined by Cantor by the help of his three generating principles ‖.

This statement appears to me to be incorrect; in fact, I shall now show that we must agree to regard the series of these "Cantor's ordinal numbers" as similar to a *segment*

* *Ibid.* p. 64.

† *Ibid.* p. 67, line 18. The wording in the definition of W is to be replaced by the slightly different wording given above.

‡ We consider in the criterion the *aggregate* which is the field of the generating-relation of the *series* 𝔚.

§ "Die Entwickelung . . . ." p. 41.

‖ The purpose of the third principle of generation is sometimes misunderstood. For example, in the in some respects excellent fourth Note ("Sur la théorie des ensembles et des nombres infinis") on pp. 617–655 of Couturat's book "De l'infini mathématique," Paris, 1896 (see esp. pp. 639–642), the object of this principle is taken to be to enable one to surpass the second number-class, just as the second principle has enabled one to surpass the first. This view seems to agree with that of Schönflies (*op. cit.* p. 48; *cf.* Phil. Mag. March 1904, p. 300); but rather further on, a different, and self-contradictory, view of this object is taken. The third principle shows, namely, the occasion for using the second principle to create a new number *after all* those generated by the application of the first two principles to a fundamental number $\omega, \Omega, \ldots$

The true view was clearly stated by Cantor in his 'Grundlagen.' The first two principles create an infinite series of ordinal numbers, while the third principle enables us to separate out various number-classes in this series (*cf.* §§ 7, 8).

merely of $\mathfrak{W}$. The ground of this lies in the fact that Cantor's third, or limiting principle, which applies to all ordinal numbers, does not apply to certain well-ordered *aggregates*, which transcend even the series of all the transfinite ordinal numbers of Cantor.

In order to state shortly what is contained in the third principle, it is convenient to single out the first number of each of the number-classes as the "class-characteristic" of all the other numbers of that class We thus define the "class-characteristic" of any ordinal number α as either α itself, if α is the first number of a number-class ($\alpha=\omega_\gamma$) *, or, if not, the first number (ω_γ) after α which is the first of a number-class.

Then the principle in question can be stated :—

The cardinal number of all the ordinal numbers preceding the class-characteristic ω_γ of a given ordinal number is $\aleph_\gamma$.

Let us now consider whether the series of all the ordinal numbers which are subject to the third principle has a type ; in other words, whether the assumption that it has a type leads to a contradiction, as was the case in Burali-Forti's argument. Let the type be β: then β is its own class-characteristic †, say $\beta=\omega_\alpha$. To find the cardinal number of all the ordinal numbers preceding β, we notice that every Aleph less than $\aleph_\beta$ (that is to say, every Aleph whose suffix is less than β) is the cardinal number of some segment of the series of type β, so that the cardinal number in question is at least equal to $\aleph_\beta$. That it is also at most equal to $\aleph_\beta$ is evident from the fact that $\aleph_\beta$ is the next greater Aleph to the series of Alephs of all the segments. Thus the cardinal number of the β ordinal numbers is

$$\aleph_\beta \text{ or } \aleph_{\omega_\alpha},$$

and, since it is not $\aleph_\alpha$, the third principle does not appear to be satisfied.

However ‡, although ω_α can never be equal to α when α is a Cantor's ordinal number, it does not follow that β is not equal to ω_β. And, in fact, this is so, as the following considerations show.

The series of Cantor's ordinal numbers is known to be

* See the notation in Phil. Mag. March 1904, p. 295.

† For if β is not the first number of a class, there are predecessors of the same class. But every predecessor of β belongs to one of Cantor's number-classes which is itself surpassed by a Cantor's number-class.

‡ My attention was called to this point, which I had overlooked, by a remark of Mr. G. H. Hardy, Fellow of Trinity College, Cambridge.

ordinally similar to the series of Alephs, or, what is the same thing, to the series of class-characteristics :

$$\omega, \omega_1, \omega_2, \ldots \omega_\omega, \ldots \omega_\gamma, \ldots.$$

Hence, to every class-characteristic ω_γ of Cantor's ordinal numbers corresponds one, and only one, Cantor's ordinal number γ, and *vice versâ*. Thus, if

$$\beta = \omega_\alpha,$$

α cannot be a Cantor's ordinal number, for, if it were, β would be one too. Further, β or ω_α (if it exists) is the least ordinal number which is greater than all Cantor's ordinal numbers, so

$$\alpha = \beta.$$

Accordingly, if β exists, the third principle is satisfied, in spite of first appearances, by the series of all Cantor's ordinal numbers; and the (Burali-Forti's) contradiction resulting herefrom leads us to deny the existence of β, the type of W.

Now, the series W is well-ordered*, although it cannot have a type, and evidently other well-ordered series (having no types) transcending W, can be formed. So we must conclude that the series W is similar to a segment merely of the series ($\mathfrak{W}$) such that every well-ordered series is similar either to it or to a segment of it †.

We can define a series ordinally similar to W by positing one element and then positing successive elements according to Cantor's first and second principles. It results from our considerations that the ordinal number of every element thus formed is subject to Cantor's third principle ; that is to say, we cannot, without contradiction, speak of an ordinal number of an element which follows all those whose ordinal numbers obey the third principle. In other words, we cannot, as seemed possible if we assumed that

$$\omega_\gamma > \gamma$$

always, define ordinal numbers which transcend all Cantor's ordinal numbers. The name of "principle of limitation" may, then, convey the wrong impression that the series W is not, as we shall say in the next section, "absolutely" infinite ‡.

* Phil. Mag. Jan. 1904, pp. 65–66.

† This is the series described, in not quite such accurate terms as the above, in Phil. Mag. Jan. 1904, p. 67, lines 18–19. It follows from the above that $\mathfrak{W}$ can *not* be used as a substitute for W in a criterion of "consistency."

‡ The "absolute" infinity of W was *stated* by Cantor in 1882 ('Grundlagen . . .,' p. 44).

8.

The series of all ordinal numbers may, it seems to me, properly be called an "*absolutely*" infinite series. For, if a well-ordered series has a type, it is, in a certain sense, completed; while the above series W cannot, as is shown by Burali-Forti's contradiction, have a type.

This seems to be the most promising way of regarding Burali-Forti's contradiction, and the words "absolutely infinite" seem preferable to the equivalent word "inconsistent," which I, in common with Cantor, have used hitherto; because an "inconsistent" aggregate is not itself contradictory (it exists, in the mathematical sense of the word), but a cardinal number or type of it does not exist. However, I shall, in the next section, enter briefly into the history of the use of this word in the theory of aggregates.

9.

The conception and name of an "inconsistent" aggregate originated with Cantor *, but the only published reference to them occurs in two papers by Hilbert †.

With regard to Hilbert's statements, it does not seem to follow that if the axioms of arithmetic (which are, according to Hilbert, the laws of operation with real numbers and the axiom of continuity) do not contradict one another, then the real number-continuum is "consistent." For it does not appear to be doubtful that the laws of operation with ordinal numbers or Alephs form a system free from contradiction, and yet the aggregate of all ordinal numbers or Alephs is "inconsistent."

Further, Hilbert states that a "similar" method to that pursued by him for the axioms of real numbers, when applied to all Alephs, fails, so the totality of all Alephs is an "inconsistent" aggregate (a mathematically non-existent

* In a letter to me of January 6th, 1901, Professor Cantor said:—"Ich unterscheide auf's strengste zwischen unendlichen *Mengen* (consistenten Vielheiten) einerseits und den ihnen zukommenden abstracten unendlichen *Zahlen* andrerseits." There was no further explanation of the term "consistency," and I confused it with Schröder's requirement in the conception of a "common manifold" ('Vorlesungen über die Algebra der Logik (exakte Logik),' Bd. i. 1890, pp. 147–148). On finding that the aggregate of all ordinal numbers had no cardinal number, I applied the name "inconsistent" used by Schröder ('Algebra und Logik der Relative,' 1895, p. 4) to this aggregate (Phil. Mag. Jan. 1904, p. 67).

† "Ueber den Zahlbegriff," *Jahresber. d. d. M.-V.* Bd. viii. (1900) pp. 180–184; "Mathematische Probleme," *Gött. Nachr.* 1900, pp. 253–297, see especially pp. 264–266.

conception). There is, however, so I contend, no reason for thus denying existence to the totality of Alephs, but only for denying the existence of the cardinal number of this aggregate. This indicates the difference between my conception of "inconsistency" and that of Hilbert.

Cantor * has defined a "consistent" aggregate (consistente Vielheit) as such that the supposition of a collection by the mind of all its elements to one thing leads to no contradiction. Since this collection was considered by Cantor as the essential thing in his definition of "Menge," and hence of cardinal number †, this definition tends to agree with mine, in opposition to Hilbert's. But Cantor's definition is not of the nature of the (nominal) definitions in the symbolic logic of Peano and Russell, but rather a "phrase indicating what is to be spoken of" ‡.

So I replaced Cantor's definition, in my first paper §, by a formal definition, and I contend that the necessary limitation, noticed by Russell ||, but not discovered by him, in the notion of a "class" is supplied by introducing the postulate of "consistency." For Russell's contradiction seems to arise solely from the use of Cantor's inequality

$$2^{\mathfrak{a}} > \mathfrak{a},$$

where $\mathfrak{a}$ is supposed to be the cardinal number of an *inconsistent* class, such as the class of all propositions ¶.

Although we have thus arrived at the formulation of the restricted concept of "class," the "search with a mental telescope" ** for this concept appears difficult, and Cantor's "definitions" are, I think, to be regarded as attempts in such a search.

The idea of an inconsistent aggregate as an *absolutely* infinite one (§ 8)—a term also used by Cantor—appears to me to be suggestive. For then finite and transfinite aggregates (which are now both subject to mathematical operations) appear, after suitable rearrangement, as segments of an infinite whole (which is not thus subject). And thus the relation of this infinite to the transfinite aggregates has a

* In a letter of November 4th, 1903, referred to in Phil. Mag. Jan. 1904, pp. 67–70.

† See *Math. Ann.* Bd. xlvi. 1895, pp. 481–482, 497.

‡ Russell, 'The Principles of Mathematics,' vol. i., Cambridge, 1903, p. 304. *Cf.* Russell's definition of a cardinal number as a class, pp. 305, 111–116.

§ Phil. Mag. Jan. 1904, p. 67.

|| *Op. cit.* p. 20; *cf.* pp. 366–368, 101–107.

¶ This is also the opinion of Prof. Cantor (letter of July 9th, 1904).

** Russell, *op. cit.* preface, p. v.

certain analogy with the relation of the transfinite aggregate of type ω to the finite aggregates.

10.

I will now sum up the results of my investigations on the transfinite numbers, published in three papers in this Magazine.

The main result is that any aggregate, the cardinal number (or type) of which is not self-contradictory, can be well-ordered. A closer consideration of the proof given in my first paper led (in the present paper) to a proof of the universal validity of Cantor's third principle, and hence of the non-existence of ordinal numbers and Alephs which transcend all those defined or indicated by Cantor; and (in the first and second papers) the main theorem led to final forms that are to be given to the results of adding and multiplying any two transfinite cardinal numbers. The results on exponential numbers are not final, but one of the theorems on exponential numbers (§ 1 of the present paper) has been shown to have an important bearing on the theory of functions.

The Manor House, Broadwindsor, Dorset.
September 6th, 1904.

IV. *The Molecular Weights of Radium and Thorium Emanations. By* Walter Makower, *B.A., B.Sc.**

Part I.

The Molecular Weight of Radium Emanation.

1. *Introduction.*

Rutherford and Brooks (Trans. Roy. Soc. Canada, 1901; Chem. News, 1902) have determined the rate of diffusion of the emanation from radium into air by a method similar to that employed by Loschmidt in his investigations on the coefficient of interdiffusion of gases, and deduce that the molecular weight lies between 40 and 100. It is important to know the molecular weight with greater accuracy, and it was with the object of solving this problem that the present investigation was undertaken. During the course of the work, Curie and Danne (*C. R.* cxxxvi. p. 1314, 1903) have published some observations upon the rate of diffusion of the emanation from radium through capillary tubes of different lengths and diameters, and find for the coefficient of interdiffusion between the emanation and air

* Communicated by Prof. J. J. Thomson.