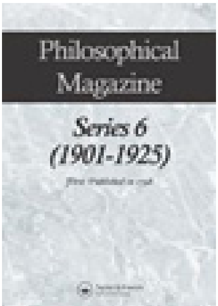

Philosophical Magazine Series 6
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tphm17

XXXVIII. On periodic irrotational waves at the surface of deep water

Lord Rayleigh O.M.F.R.S.

Available online: 08 Apr 2009

THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[SIXTH SERIES.]

MAY 1917.

XXXVIII. *On Periodic Irrotational Waves at the Surface of Deep Water.* *By* Lord RAYLEIGH, *O.M., F.R.S.**

THE treatment of this question by Stokes, using series proceeding by ascending powers of the height of the waves, is well known. In a paper with the above title† it has been criticised rather severely by Burnside, who concludes that "these successive approximations can not be used for purposes of numerical calculation...". Further, Burnside considers that a numerical discrepancy which he encountered may be regarded as suggesting the non-existence of permanent irrotational waves. It so happens that on this point I myself expressed scepticism in an early paper ‡, but afterwards I accepted the existence of such waves on the later arguments of Stokes, McCowan§, and of Korteweg and De Vries‖. In 1911¶ I showed that the method of the early paper could be extended so as to obtain all the later results of Stokes.

The discrepancy that weighed with Burnside lies in the fact that the value of β (see equation (1) below) found best to satisfy the conditions in the case of $\alpha=\frac{1}{4}$ differs by about

* Communicated by the Author.
† Proc. Lond. Math. Soc. vol. xv. p. 26 (1915).
‡ Phil. Mag. vol. i. p. 257 (1876); 'Scientific Papers,' vol. i. p. 261.
§ Phil. Mag. vol. xxxii. pp. 45, 553 (1891).
‖ Phil. Mag. vol. xxxix. p. 422 (1895).
¶ Phil. Mag. vol. xxi. p. 183 (1911).

50 per cent. from that given by Stokes' formula, viz. $\beta=-\frac{1}{2}\alpha^4$. It seems to me that too much was expected. A series proceeding by powers of $\frac{1}{4}$ need not be very convergent. One is reminded of a parallel instance in the lunar theory where the motion of the moon's apse, calculated from the first approximation, is doubled at the next step. Similarly here the next approximation largely increases the numerical value of β. When a smaller α is chosen $(\frac{1}{10})$, series developed on Stokes' plan give satisfactory results, even though they may not converge so rapidly as might be wished.

The question of the convergency of these series is distinct from that of the existence of permanent waves. Of course a strict mathematical proof of their existence is a desideratum; but I think that the reader who follows the results of the calculations here put forward is likely to be convinced that permanent waves of moderate height do exist. If this is so, and if Stokes' series are convergent in the mathematical sense for such heights, it appears very unlikely that the case will be altered until the wave attains the greatest admissible elevation, when, as Stokes showed, the crest comes to an edge at an angle of 120°.

It may be remarked that most of the authorities mentioned above express belief in the existence of permanent waves, even though the water be not deep, provided of course that the bottom be flat. A further question may be raised as to whether it is necessary that gravity be constant at different levels. In the paper first cited I showed that, under a gravity inversely as the cube of the distance from the bottom, *very long* waves are permanent. It may be that under a wide range of laws of gravity permanent waves exist.

Following the method of my paper of 1911, we suppose for brevity that the wave-length is 2π, the velocity of propagation unity*, and we take as the expression for the stream-function of the waves, reduced to rest,

$$\psi=y-\alpha e^{-y}\cos x-\beta e^{-2y}\cos 2x-\gamma e^{-3y}\cos 3x \\ -\delta e^{-4y}\cos 4x-\epsilon e^{-5y}\cos 5x, \quad . \quad . \quad (1)$$

in which x is measured horizontally and y vertically downwards. This expression evidently satisfies the differential equation

* The extension to arbitrary wave-lengths and velocities may be effected at any time by attention to dimensions.

to which ψ is subject, whatever may be the values of the constants α, β, &c. And, much as before, we shall find that the surface condition can be satisfied to the order of α^7 inclusive; β, γ, δ, ϵ being respectively of orders α^4, α^5, α^6, α^7.

We suppose that the free surface is the stream-line $\psi=0$, and the constancy of pressure there imposed requires the constancy of U^2-2gy, where U, representing the resultant velocity, is equal to $\sqrt{\{(d\psi/dx)^2+(d\psi/dy)^2\}}$, and g is the constant acceleration of gravity now to be determined. Thus when $\psi=0$,

$$\begin{aligned}U^2-2gy=1+2(1-g)y+\alpha^2e^{-2y}+2\beta e^{-2y}\cos 2x\\ +4\gamma e^{-3y}\cos 3x+6\delta e^{-4y}\cos 4x+8\epsilon e^{-5y}\cos 5x\\ +4\alpha\beta e^{-3y}\cos x+6\alpha\gamma e^{-4y}\cos 2x+8\alpha\delta e^{-5y}\cos 3x \quad . \quad (2)\end{aligned}$$

correct to α^7 inclusive. On the right of (2) we have to expand the exponentials and substitute for the various powers of y expressions in terms of x.

It may be well to reproduce the process as formerly given, omitting δ and ϵ, and carrying (2) only to the order α^5. We have from (1) as successive approximations to y:—

$$y=\alpha e^{-y}\cos x=\alpha\cos x; \quad . \quad . \quad . \quad . \quad (3)$$

$$y=\alpha(1-y)\cos x=-\tfrac{1}{2}\alpha^2+\alpha\cos x-\tfrac{1}{2}\alpha^2\cos 2x; \quad . \quad (4)$$

$$\begin{aligned}y&=\alpha(1-y+\tfrac{1}{2}y^2)\cos x\\ &=-\frac{\alpha^2}{2}+\alpha\left(1+\frac{9\alpha^2}{8}\right)\cos x-\frac{\alpha^2}{2}\cos 2x+\frac{3\alpha^3}{8}\cos 3x, \quad (5)\end{aligned}$$

which is correct to α^3 inclusive, β being of order α^4. In calculating (2) to the approximation now intended we omit the term in $\alpha\gamma$. In association with $\alpha\beta$ and γ we take $e^{-3y}=1$; in association with β, $e^{-2y}=1-2y$; while

$$\alpha^2e^{-2y}=\alpha^2(1-2y+2y^2-\tfrac{4}{3}y^3).$$

Thus on substitution for y^2 and y^3 from (5)

$$\alpha^2e^{-2y}=\alpha^2\{1-2y+\alpha^2-4\alpha^3\cos x+\alpha^2\cos 2x-\tfrac{4}{3}\alpha^3\cos 3x\}.$$

In like manner

$$2\beta e^{-2y}\cos 2x=2\beta\cos 2x-2\alpha\beta(\cos x+\cos 3x).$$

Since the terms in $\cos x$ are of the fifth order, we may replace $\alpha\cos x$ by y, and we get

$$\begin{aligned}U^2-2gy=1+\alpha^2+\alpha^4+2y(1-g-\alpha^2-2\alpha^4+\beta)\\ +(\alpha^4+2\beta)\cos 2x+(-\tfrac{4}{3}\alpha^5+4\gamma-2\alpha\beta)\cos 3x. \quad . \quad (6)\end{aligned}$$

The constancy of (6) requires the annulment of the coefficients of y and of $\cos 2x$ and $\cos 3x$, so that

$$\beta=-\tfrac{1}{2}\alpha^4, \qquad \gamma=\tfrac{1}{12}\alpha^5, \quad . \quad . \quad . \quad . \quad (7)$$

and

$$g=1-\alpha^2-\tfrac{5}{2}\alpha^4. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The value of g in (8) differs from that expressed in equation (11) of my former paper. The cause is to be found in the difference of suppositions with respect to ψ. Here we have taken $\psi=0$ at the free surface, which leads to a constant term in the expression for y, as seen in (5), while formerly the constant term was made to disappear by a different choice of ψ.

There is no essential difficulty in carrying the approximation to y two stages further than is attained in (5). If δ, ϵ are of the 6th and 7th order, they do not appear. The longest part of the work is the expression of e^{-y} as a function of x. We get

$$e^{-y}=1+\frac{3\alpha^2}{4}+\frac{125\alpha^4}{64}-\cos x\{\alpha+2\alpha^3\}$$

$$+\cos 2x\left\{\frac{3\alpha^2}{4}+\frac{125\alpha^4}{48}-\beta\right\}-\frac{2\alpha^3}{3}\cos 3x+\frac{125\alpha^4}{192}\cos 4x, \quad (9)$$

and thence from (1)

$$y=-\tfrac{1}{2}\alpha^2-\alpha^4+\cos x\left\{\alpha+\frac{9\alpha^3}{8}+\frac{625\alpha^5}{192}-\frac{3\alpha\beta}{2}\right\}$$

$$-\cos 2x\left\{\tfrac{1}{2}\alpha^2+\frac{4\alpha^4}{3}-\beta\right\}+\cos 3x\left\{\frac{3\alpha^3}{8}+\frac{625\alpha^5}{384}-\frac{3\alpha\beta}{2}+\gamma\right\}$$

$$-\frac{\alpha^4}{3}\cos 4x+\frac{125\alpha^5}{384}\cos 5x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

When we introduce the values of β and γ, already determined in (7) with sufficient approximation, we have

$$y=-\tfrac{1}{2}\alpha^2-\alpha^4+\cos x\left\{\alpha+\frac{9\alpha^3}{8}+\frac{769\alpha^5}{192}\right\}$$

$$-\cos 2x\left\{\frac{\alpha^2}{2}+\frac{11\alpha^4}{6}\right\}+\cos 3x\left\{\frac{3\alpha^3}{8}+\frac{315\alpha^5}{128}\right\}$$

$$-\frac{\alpha^4}{3}\cos 4x+\frac{125\alpha^5}{384}\cos 5x, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

in agreement with equations (13), (18) of my former paper when allowance is made for the different suppositions with respect to ψ, as may be effected by expressing both results in terms of a, the coefficient of $\cos x$, instead of α.

The next step is the further development of the pressure equation (2), so as to include terms of the order α^7. Where β, γ, &c. occur as factors, the expression for y to the third order, as in (5), suffices; but a more accurate value is required in $\alpha^2 e^{-2y}$. Expanding the exponentials and replacing products of cosines by cosines of sums and differences, we find in the first place

$$U^2-2gy=2(1-g-\alpha^2)y+1+\alpha^2+\alpha^4+\frac{19\alpha^6}{4}-4\alpha^2\beta$$
$$+\cos x\left\{-4\alpha^5+2\alpha\beta-\frac{37\alpha^7}{2}+\frac{97\alpha^3\beta}{6}-\frac{9\alpha^2\gamma}{2}\right\}$$
$$+\cos 2x\left\{\alpha^4+2\beta+\frac{19\alpha^6}{3}-2\alpha^2\beta\right\}$$
$$+\cos 3x\left\{-\frac{4\alpha^5}{3}-2\alpha\beta+4\gamma-\frac{37\alpha^7}{4}+\frac{13\alpha^3\beta}{4}+3\alpha^2\gamma-4\alpha\delta\right\}$$
$$+\cos 4x\left\{\frac{19\alpha^6}{12}+2\alpha^2\beta-6\alpha\gamma+6\delta\right\}$$
$$+\cos 5x\left\{-\frac{37\alpha^7}{20}-\frac{25\alpha^3\beta}{12}+\frac{15\alpha^2\gamma}{2}-12\alpha\delta+8\epsilon\right\}. \quad (12)$$

From the terms in $\cos x$ we now eliminate $\cos x$ by means of

$$\alpha\cos x=y\left(1-\frac{9\alpha^2}{8}\right)+\frac{\alpha^2}{2}+\frac{\alpha^2}{2}\cos 2x,$$

thus altering those terms of (12) which are constant, and which contain y and $\cos 2x$. Thus modified, (12) becomes

$$U^2-2gy=1+\alpha^2+\alpha^4+\frac{11\alpha^6}{4}-3\alpha^2\beta$$
$$+2y\left\{1-g-\alpha^2-2\alpha^4+\beta-7\alpha^6+\frac{167\alpha^2\beta}{24}-\frac{9\alpha\gamma}{4}\right\}$$
$$+\cos 2x\left\{\alpha^4+2\beta+\frac{13\alpha^6}{3}-\alpha^2\beta\right\}$$
$$+\cos 3x\left\{-\frac{4\alpha^5}{3}-2\alpha\beta+4\gamma-\frac{37\alpha^7}{4}+\frac{13\alpha^3\beta}{4}+3\alpha^2\gamma-4\alpha\delta\right.$$
$$+\cos 4x\left\{\frac{19\alpha^6}{12}+2\alpha^2\beta-6\alpha\gamma+6\delta\right\}$$
$$+\cos 5x\left\{-\frac{37\alpha^7}{20}-\frac{25\alpha^3\beta}{12}+\frac{15\alpha^2\gamma}{2}-12\alpha\delta+8\epsilon\right\}. \quad (13)$$

The constant part has no significance for our purpose, and the term in y can be made to vanish by a proper choice of g.

If we use only α, none of the cosines can be made to disappear, and the value of g is

$$g=1-\alpha^2-2\alpha^4-7\alpha^6. \quad . \quad . \quad . \quad . \quad . \quad (14)$$

When we include also β, we can annul the term in $\cos 2x$ by making

$$\beta=-\frac{\alpha^4}{2}\left(1+\frac{29\alpha^2}{6}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

and with this value of β

$$g=1-\alpha^2-\frac{5\alpha^4}{2}-\frac{619\alpha^6}{48}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

But unless α is very small, regard to the term in $\cos 3x$ suggests a higher value of β as the more favourable on the whole.

With the further aid of γ we can annul the terms both in $\cos 2x$ and in $\cos 3x$. The value of β is as before. That of γ is given by

$$\gamma=\frac{\alpha^5}{12}\left(1+\frac{139\alpha^2}{8}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and with this is associated

$$g=1-\alpha^2-\frac{5\alpha^4}{2}-\frac{157\alpha^6}{12}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The inclusion of δ and ϵ does not alter the value of g in this order of approximation, but it allows us to annul the terms in $\cos 4x$ and $\cos 5x$. The appropriate values are

$$\delta=-\frac{\alpha^6}{72}, \qquad \epsilon=\frac{\alpha^7}{480}, \quad . \quad . \quad . \quad . \quad (19)$$

and the accompanying value of γ is given by

$$\gamma=\frac{\alpha^5}{12}\left(1+\frac{413\alpha^2}{24}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

while β remains as in (15).

We now proceed to consider how far these approximations are successful, for which purpose we must choose a value for α. Prof. Burnside took $\alpha=\frac{1}{4}$. With this value the second term of β in (15) is nearly one-third of the first (Stokes') term, and the second term of γ in (20) is actually larger than the first. If the series are to be depended upon, we must clearly take a smaller value. I have chosen $\alpha=\frac{1}{10}$, and this makes by (15), (18), (20)

$$\beta=-\cdot000{,}052{,}42, \quad \gamma=\cdot000{,}000{,}976, \quad g=\cdot989{,}736{,}92. \quad (21)$$

The next step is the calculation of approximate values of y from (11), which now takes the form

$$y = -\cdot 0051 + \cdot 101{,}165{,}0 \cos x - \cdot 005{,}183{,}3 \cos 2x + \cdot 000{,}399{,}6 \cos 3x - \cdot 000{,}033{,}3 \cos 4x + \cdot 000{,}003{,}3 \cos 5x. \quad . \quad . \quad . \quad (22)$$

For example, when $x=0$, $y=\cdot 091{,}251{,}3$. The values of y calculated from (22) at steps of $22\frac{1}{2}°$ (as in Burnside's work) are shown in column 2 of Table I.

We have next to examine how nearly the value of y afforded by (22) really makes ψ vanish, and if necessary to calculate corrections. To this δ and ϵ in (1) do not contribute sensibly and we find $\psi = +\cdot 000{,}015{,}4$ for $x=0$. In order to reduce ψ to zero, we must correct the value of y With sufficient approximation we have in general

$$\delta\psi = \delta y(1 + \tfrac{1}{10} e^{-y} \cos x)^*,$$

or in the present case

$$\delta y = -\frac{\cdot 000{,}015{,}4}{1\cdot 091} = -\cdot 000{,}014{,}1,$$

so that the corrected value of y for $x=0$ is $\cdot 091{,}237{,}2$. If we repeat the calculation, using the new value of y, we find $\psi=0$.

TABLE I.

x.	y from (22).	y corrected.	$U^2-2gy-1$.	Corrected by $\delta\beta$.
0	$+\cdot 091{,}251{,}3$	$+\cdot 091{,}237{,}2$	$\cdot 010{,}104{,}9$	45
$22\frac{1}{2}$	$+\cdot 084{,}839{,}7$	$+\cdot 084{,}841{,}9$	 4,7	44
45	$+\cdot 066{,}182{,}8$	$+\cdot 066{,}181{,}8$	 4,3	43
$67\frac{1}{2}$	$+\cdot 036{,}913{,}1$	$+\cdot 036{,}915{,}1$	 4,1	44
90	$+\cdot 000{,}050{,}0$	$+\cdot 000{,}052{,}4$	 4,2	46
$112\frac{1}{2}$	$-\cdot 039{,}782{,}7$	$-\cdot 039{,}780{,}2$	 4,4	47
135	$-\cdot 076{,}316{,}2$	$-\cdot 076{,}317{,}5$	 4,3	43
$157\frac{1}{2}$	$-\cdot 102{,}381{,}1$	$-\cdot 102{,}395{,}1$	 4,7	44
180	$-\cdot 111{,}884{,}7$	$-\cdot 111{,}907{,}9$	$\cdot 010{,}105{,}1$	47

In the fourth column are recorded the values of $U^2 - 2gy - 1$, calculated from (1) with omission of δ and ϵ, and with the corrected values of y. $d\psi/dx$, $d\psi/dy$ were first found separately, and then U^2 as the sum of the two squares. The values of β, γ, g employed are those given in (15), (18), (20). The form of ψ in (1) with these values of the constants vanishes when y takes the values of the third column, and the pressure

* The double use of δ will hardly cause confusion.

at the surface is also constant to a high degree of approximation. The greatest difference is (·000,001,0), which may be compared with ·4, the latter amount representing the corresponding statical difference at the crest and trough of the wave. According to this standard the pressure at the surface is constant to $2\frac{1}{2}$ parts in a million.

The advantage gained by the introduction of β and γ will be better estimated by comparison with a similar calculation where only α (still equal to $\frac{1}{10}$) and g are retained. By (2) in this case

$$U^2-2gy-1=\alpha^2e^{-2y}+2(1-g)y. \quad . \quad . \quad . \quad (23)$$

Table II. shows the values of y and of α^2e^{-2y} corresponding to the same values of x as before. The fourth column gives (23) when g is so determined as to make the values equal at 0° and 180°. It appears that the discrepancy in the values of U^2-2gy is reduced 200 times by the introduction of β and γ, even when we tie ourselves to the values of β, γ, g prescribed by approximations on the lines of Stokes.

Table II.

x.	y.	α^2e^{-2y}.	$U^2-2gy-1$.
0	+·091,276,5	·008,331,4	·010,207,7
$22\frac{1}{2}$	·084,870,5	·008,438,8	. . . 183,4
45	·066,182,4	·008,760,2	. . . 120,7
$67\frac{1}{2}$	·036,882,6	·009,288,9	. . . 047,1
90	0	·010,000,0	. . . 000,0
$112\frac{1}{2}$	−·039,823,1	·010,829,0	. . . 010,4
135	−·076,318,5	·011,649,0	. . . 080,2
$157\frac{1}{2}$	−·102,344,1	·012,271,4	. . . 167,6
180	−·111,832,6	·012,506,5	·010,207,7

A cursory inspection of the numbers in column 4 of Table I. suffices to show that an improvement can be effected by a slight alteration in the value of β. For small corrections of this kind it is convenient to use a formula which may be derived from (2). We suppose that while α and ψ are maintained constant, small alterations $\delta\beta$, $\delta\gamma$, δg are incurred. Neglecting the small variations of β, γ, g when multiplied by α^2 and higher powers of α, we get

$$\delta y=\delta\beta\{\cos 2x-\tfrac{3}{2}\alpha\cos x-\tfrac{3}{2}\alpha\cos 3x\}$$
$$+\delta\gamma\{\cos 3x-2\alpha\cos 2x-2\alpha\cos 4x\}, \quad . \quad . \quad (24)$$

and

$$\delta(U^2-2gy)=2\alpha(\delta\beta-\delta g)\cos x+2\delta\beta\cos 2x$$
$$+2(2\delta\gamma-\alpha\delta\beta)\cos 3x-6\alpha\delta\gamma\cos 4x. \quad . \quad . \quad (25$$

For the present purpose we need only to introduce $\delta\beta$, and with sufficient accuracy we may take

$$\delta(U^2 - 2gy) = 2\delta\beta \cos 2x. \quad . \quad . \quad . \quad . \quad (26)$$

We suppose $\delta\beta = -\cdot 000{,}000{,}2$, so that the new value of β is $-\cdot 000{,}052{,}6$. Introducing corrections according to (26) and writing only the last two figures, we obtain column 5 of Table I., in which the greatest discrepancy is reduced from 10 to 4—almost as far as the arithmetic allows—and becomes but one-millionth of the statical difference between crest and trough. This is the degree of accuracy attained when we take simply

$$\psi = y - \alpha e^{-y} \cos x - \beta e^{-2y} \cos 2x - \gamma e^{-3y} \cos 3x, \quad . \quad (27)$$

with $\alpha = \frac{1}{10}$, g and γ determined by Stokes' method, and β determined so as to give the best agreement.

XXXIX. *The Application of Solid Hypergeometrical Series to Frequency Distributions in Space.* *By* S. D. Wicksell, *Dr. Phil., Lund, Sweden* *.

IN the number of this Journal issued in September 1914, Dr. L. Isserlis, under the above title, published a paper on the fitting of hypergeometrical series to correlation surfaces. The problem to describe curves of variation by aid of hypergeometrical series was treated as long ago as 1895 by Prof. Pearson in his classical Memoir: "Skew Variation in Homogeneous Material," Phil. Trans. vol. clxxxvi. Later, in 1899, Prof. Pearson gave a fuller discussion of the hypergeometrical series (Phil. Mag. vol. xlvii.). It is this paper that is the starting-point and chief place of reference of Dr. Isserlis. On the whole, the hypergeometrical series and its special case for $n = \infty$, the binomial series, play a dominating part in Prof. Pearson's celebrated theory of variation of one variate. As a consequence hereof, it was natural that the attempt should be made to employ solid hypergeometrical series as a means to describe also surfaces of correlation. Hereby, however, a fact has evidently been overlooked that greatly limits the range of applicability of any hypergeometrical or multinomial types of correlation functions. Of course, there must be some identical relations between the moments that should be more or less fulfilled in all cases of application. Dr. Isserlis also produces several such relations. But it is evident that he has not ascribed too much importance to these limitations. In the theory of variation of one variate there are similar conditions, but they

* Communicated by the Author.