LXI. *On Metallic Colouring in Birds and Insects.* *By* A. A. Michelson*.

[Plate IV.]

WITH the exception of bodies which shine by their own light, the appearance of colour in natural objects is due to some modification which they impart to the light which illuminates them. In the great majority of cases this modification is caused by the absorption of part of the light which falls on the object, and which, penetrating to a greater or smaller depth beneath the surface, is reflected, and finally reaches the eye. If the proportion of the various colours constituting white light which is absorbed by the illuminated body is the same for each, the light which reaches the eye has the same composition as before, and we say the body itself is white; but if this proportion be different, the resulting light is coloured, and the colour of the body itself corresponds to that colour or colour combination which is least absorbed; it is complementary to the colours which are most strongly absorbed.

Thus the light from a green leaf in the sunshine, after penetrating a short distance in the substance of the leaf, is either transmitted or reflected to the eye. In its passage through the substance of the leaf it has lost a considerable part of the red light it originally contained, and the resulting combination of the remaining colours produces the effect of the complementary colour or green, as can readily be shown by analysing the light into its elementary colours by a prism, and comparing the resulting spectrum before and after the reflexion from the leaf.

The same explanation holds for all the paints, dyes, and pigments in common use.

These colour effects occur in such an immensely greater proportion than all others combined, that the occasional appearance of an exception is all the more striking. The rainbow and the various forms of halo are almost the only instances of prismatic colours which occur in nature.

There remain only two other possible methods of producing colour.

A. Interference, including Diffraction.
B. "Metallic" Reflexion.

It has been abundantly proved that the usual "flat," "dead," "uniform" colouring, brilliant as this sometimes

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* Communicated by the Author.

can be, *e. g.*, in birds, butterflies, and flowers, finds its simple explanation in the existence of pigment cells; so that the same cause (doubtless with many modifications) is here effective as in the great majority of cases previously considered. But the lively, variable "metallic" glitter of burnished copper or gold; the reflexion from certain aniline dyes; the colours of certain pigeons, peacocks, humming-birds, as well as a number of butterflies, beetles, and other insects, requires another explanation.

While cases under A occur occasionally in nature—for example, in the colours of thin films, in the iridescence of mother of pearl, and (as an accessory) in the colours of the rainbow and of certain halos—they are so rare and so readily distinguished from the true metallic colours that they may be most conveniently treated as exceptions after the subject of surface-colour has been considered.

The designation "metallic" at once suggests that there may be some common property of all these colours which is typified by the metals themselves. But, as is well known, the principal characteristic which distinguishes the metals from all other substances in regard to their action on light, is their extraordinary opacity.

A very important consequence of such great opacity is that light is practically prevented from entering the substance at all, but is thrown back, thus giving the brilliant metallic reflexion so characteristic of silver, gold, copper, &c. In fact, the distance to which light can penetrate in most metals is only a small fraction of a light-wave; so that a wave-motion such as constitutes light, strictly speaking, cannot be propagated at all. Again, as this opacity may be different for different colours, some would be transmitted more freely than others, so that the resulting transmitted light would be coloured; and the reflected light would be approximately complementary to the transmitted colour.

For most metals the difference is not very great; so that the reflected light, except in the case of gold and copper and a few alloys, is nearly white. In the case of the aniline dyes, however, there is a marked difference, as is clearly shown by their absorption spectrum. In transmitted light, even a very small thickness of fuchsine shows no yellow, green, or blue, and gives as a resultant of the remaining colours a beautiful crimson. The light which it reflects, however, is just this yellow and green which it refuses to transmit, and it accordingly shimmers with a metallic golden green colour, which changes when the surface is inclined, becoming full green, or even bluish green when the illumination is sufficiently oblique*.

The chief characteristics by which "metallic" reflexion may be distinguished may be summarized as follows:—

1. The brightness of the reflected light is always a large fraction of the incident light, varying from 50 per cent. to nearly 100 per cent.

2. The absorption is so intense that metal films are quite opaque even when their thickness is less than a thousandth of a millimetre.

3. If the absorption varies with colour, that colour which is most copiously transmitted will be the part of the incident white light which is least reflected—so that the transmitted light is complementary to the reflected.

4. The change of colour of the reflected light with changing incidence has already been mentioned. It follows the invariable rule that the colour always approaches the violet end of the spectrum as the incidence increases. If the colour of the normal reflexion is violet the light vanishes (changing to ultra-violet), and if the normal radiation be infra-red it passes through red, orange, and yellow as the incidence increases.

While the criteria just considered are the simplest and most convenient for general observation, it is to the more rigorous results of more refined optical methods that we must look for the final test of the quality of reflexion in any given case; to determine whether or not a colour phenomenon may be due to "metallic" reflexion or to one of the other general causes.

Such optical tests are furnished by the effect of reflexion upon polarized light. The elements of the resulting elliptic vibrations may be expressed in terms of the amplitude ratio R of the components, and of the phase difference P corresponding to the angle of incidence I, as in the following tables for silver and for glass.

The very marked difference in the run of the numbers in these tables may be rendered still more striking by plotting the values as ordinates of the curves shown in Pl. IV. fig. 1, which gives at a glance the form of the "phase" curve (full line) and the "amplitude" curve (dotted line) for silver, steel, graphite, selenium, flint glass, crown glass, and quartz. It is evident

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* The change in colour is very much more marked when the light is polarized perpendicularly to the plane of incidence. As the angle of incidence approaches the angle corresponding to the "polarizing angle" the colour is a deep blue or even purple.

Silver.			*Glass.*		
I.	P.	R.	I.	P.	R.
0°	·00	45	0°	·00	45
10°	·01	45	10°	·00	40
20°	·02	44	20°	·00	30
30°	·03	43	30°	·00	25
40°	·04	41	40°	·00	20
50°	·05	40	50°	·00	12
60°	·10	38	60°	·50	7
70°	·20	37	70°	·50	25
80°	·30	40	80°	·50	40
90°	·50	45	90°	·50	45

that metals have a smoother phase curve than semi-metallic substances like graphite and selenium, and these show less abrupt changes than do transparent substances such as glass and quartz.

In fact we may take the steepness of the curve where it is steepest (better where the phase difference is 1/4) as a measure of the transparency of the substance; and theory shows that this *steepness* is in fact inversely proportional to the *absorbing power* of the substance.

Starting with the formula of Cauchy

$$\tan \Delta = \frac{2k \sin^2 i \cos i}{(\nu^2 \cos^2 r + k^2) \cos^2 i - \sin^4 i},$$

differentiating for i and putting in the resulting expression $\Delta = \pi/2$, and I for the corresponding angle of incidence, we have

$$\frac{d\Delta}{di} = \frac{\sin \mathrm{I}(\tan^2 \mathrm{I} + 2)}{k} \quad (k = \text{coeff. of absorption}).$$

On the other hand, if the phase change be the result of a *surface film*, and we start with the corresponding approximate formula

$$\tan \Delta_1 = \epsilon \tan (i + r)$$

we find

$$\frac{d\Delta_1}{di} = \frac{1}{\epsilon \sin^2 \mathrm{I}} \quad (\epsilon = \text{coeff. of ellipticity}).$$

In this case the steepness is inversely proportional to what Jamin has termed the "ellipticity."

In point of fact both causes are effective; and for semitransparent substances it is impossible to obtain results which agree even approximately with experiment by either formula. But the rigorous expression of Cauchy, which contains both k and i, is so unwieldy to be practically useless.

The difficulty may be obviated by making use of the empirical relation $E=E_\epsilon+E_k$, where $E=\frac{1}{d\Delta/di}$; which may be translated to mean that the actual "ellipticity" is made up of two parts which combine additively; one due to the surface film and the other resulting from absorption.

If the medium under observation be transparent $E_k=0$, hence $E_\epsilon=E$. If it be opaque E_ϵ is small compared with E_k, so that approximately $E_k=E$.

For semitransparent media it will be necessary to determine the absorption, k, by direct measurement, from which E_k may be calculated by means of the formula

$$E_k=\frac{k}{\sin I(\tan^2 I+2)},$$

and E_ϵ may then be determined by

$$E_\epsilon=E-E_k.$$

In the case of substances like fuchsine and diamond green, in which the medium is almost perfectly transparent for certain colours, we may find E_ϵ for this colour; and if it be correct to assume that E_ϵ does not vary with the colour, the value of $E_k=E-E_\epsilon$ may be accurately determined for the semi-transparent and the opaque portions of the spectrum.

A fairly good test is that furnished by selenium. The incidence I corresponding to $\Delta=90°$ is nearly independent of the colour, being about 71°.

The value of $\frac{d\Delta}{di}$ calculated by the preceding formula is, very nearly,

$$\frac{d\Delta}{di}=\frac{1}{E_k}\doteq\frac{10}{k}.$$

Following are the values of $\frac{d\Delta}{di}$, E, E_k, and k*.

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* These last are taken from the results of Professor Wood, Phil. Mag. 1902, vol. iii.

Selenium.

λ.	$\frac{d\Delta}{di}$.	E.	Ek.	10Ek.	k.
6950	23	·043	·000	·000	·05
6410	21	·048	·005	·065	·09
6075	18	·056	·013	·13	·13
5745	16	·062	·019	·19	·20
5410	13	·077	·034	·34	·28
5070	11	·091	·048	·48	·40
4740	9	·111	·068	·68	·53
4405	8	·125	·082	·82	·61

Following are similar series of observations for fuchsine and for diamond green.

Fuchsine.

λ.	$\frac{d\Delta}{di}$.	E.	10Ek.	$\log\frac{1}{R}$.
670	50	·020	·00	·04
640	40	·025	·05	·05
615	18	·055	·35	·38
590	0	·11	·90	·75
560	7	·14	1·2	1·4
525	6	·16	1·4	1·7
500	4·5	·22	2·0	1·5
470	4·5	·22	2·0	1·0

Diamond Green.

λ.	$\frac{d\Delta}{di}$.	E.	10Ek.	$\text{Log}\frac{1}{R}$.
700	12	·08	0·7	0·8
680	8	·12	1·1	1·1
660	7	·14	1·3	1·3
640	5	·20	1·9	1·5
620	4	·25	2·4	1·5
600	4	·25	2·4	1·2
580	4	·25	2·4	1·0
560	6	·16	1·5	·65
540	25	·04	0·3	·23
520	60	·016	0·06	·08
500	80	·012	0·09	·02
480	70	·014	0·04	·01
460	45	·022	0·12	·25
440	15	·007	0·57	·40

$$R = \frac{I_0}{I} = \frac{\text{incident light}}{\text{transmitted light}}\,{}^{*}.$$

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* No account was taken of the loss by reflexion.

The quantities in the last column are proportional to k; but the actual values of k thus deduced from observations of transmitted light are considerably less (about 1/3 of the value given), possibly because of the unevenness of the film which makes the measurement of the actual thickness (of the order of one thousandth of a millimetre) uncertain.

The agreement in the last two columns of the tables, while somewhat imperfect, is still enough to show that the results are of the right order of magnitude—and if it be remembered that the properties of the specimens vary considerably with the method of preparation, it is probable that the outstanding differences may be thus accounted for.

In any case the agreement is much better than it would be, had the ellipticity been attributed to absorption alone.

In the aniline colours the absorption varies enormously with the colour, and we have all the gradations from metallic reflexion to almost perfect transparence combined in a single specimen. The measurement of the phase-change and the amplitude-ratio for these substances show changes in the form of the curves almost identical with those given in the preceding figure.

Pl. IV. fig. 2 shows the curves obtained for fuchsine and for diamond green. It may be noted that in both these figures the "phase" curve is much more characteristic in its changes than the "amplitude" curve.

These specimens are prepared by dissolving the aniline colour in hot alcohol, filtering hot, and covering a hot glass surface with the solution. The alcohol evaporates rapidly, leaving a mirror surface of thickness of the order of a thousandth of a millimetre.

A quite remarkable alteration occurs in the phase curves when the solution is diluted. The film deposited is then very much thinner than before (from one-tenth to one one-hundredth of the former thickness) and, for some colours, the thickness is so small that considerable light is reflected from the surface of the glass. The resulting phase curve may then be *negative*, as shown in Pl. IV. fig. 3, for the colours red, orange, and yellow*.

Such a result has been predicted from theoretical considerations†, but so far as I am aware, no attempt has been made to show that this depends on the colour of the incident

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* The lower curves show more clearly how the maximum value of k varies with the colour.

† Drude, 'Theory of Optics,' p. 294.

light. This, however, follows, if we consider that the condition for such a negative phase curve is that the transition layer have an index of refraction greater than that of the second medium; and as the refractive index for magenta is low at the blue end of the spectrum and high at the red end, the inversion of sign is strictly in accord with the theory, of which indeed it furnishes a striking confirmation.

On applying the simpler general tests of metallic reflexion to the case of iridescent plumage of birds, scales of butterflies, and wing-cases of beetles, one is at once struck with the close resemblance these bear to the aniline colours, in every particular: for

1st. The intensity of the reflected light is much greater than for the "non-metallic" plumage, &c., in some cases approaching the value of the reflexion factor of the metals themselves.

2nd. The reflected light is always coloured, showing either a rapid change of index of refraction, or of coefficient of absorption with the wave-length or colour; and, indeed, it may perhaps be objected that these colours are far more vivid than any of the reflexion hues of the aniline dyes, or of any other case of "surface colour" hitherto observed.

3rd. In the cases which could be investigated for this relation (unfortunately rather few) the transmitted light is approximately complementary to that which is reflected.

4th. The change of colour with changing incidence strictly follows the law already mentioned—the colour always changing towards the blue end of the spectrum as the incidence increases.

This remarkable agreement has been pointed out by Dr. B. Walter in an admirable essay, "Die Oberflächen- oder Schiller-Farben," and it is shown that none of the other causes of colour phenomena (in particular interference and diffraction) can be effective; the laws which govern these last being totally different. It is, therefore, somewhat surprising to find that the contrary view is still held by many eminent naturalists, and it is hoped that the further evidence here presented may serve to emphasize the distinction between "metallic" or "surface" colour and the remaining classes of colour (due to pigments, interference, and diffraction).

In attempting to apply the more rigorous optical test of the measurement of the phase-difference and amplitude-ratios, one is met at the outset with the serious difficulty of the absence of true "optical" surface. In fact, the materials we have to deal with (feathers, butterfly scales, beetle wing-cases) are so irregular that the quantity of "regularly" reflected light which is brought to a focus by the observing telescope is insignificant, and is often masked by the light diffusely reflected. But by the simple device of replacing the objective of the collimator and of the observing telescope by low-power microscope objectives of small aperture, these difficulties are so far removed that it has been possible to obtain results which compare favourably with those obtained with the aniline films. In some of the measurements it has been found possible to deal with a single butterfly scale; and in these the irregularities of the surface were often insignificant, or of such nature that they could be taken into account.

Following is a diagram showing the results of a set of measurements on a beetle having a lustre resembling burnished copper. Beside it is a duplicate of the preceding observations on a *thin* film of magenta (Pl. IV. fig. 4).

The correspondence between the two sets of curves is so remarkable that it leaves no room to doubt that in this case the metallic coppery colour of the wing-case is due to an extremely thin film of some substance closely analogous in its optical qualities to the corresponding aniline dye*. The thickness of the magenta film was not very accurately determined, but from the fact that it was deposited from a solution of 1/20 of the concentration of that which produced the corresponding *thick* film (whose thickness is about 0·005 mm.), it is estimated that the thickness of the *thin* film is of the order of 0·00025 mm. It is, doubtless, unsafe from this to draw any more definite conclusion regarding the film of the wing-case, than to say it is probably of the same order.

An attempt was made to check this estimate by the following simple device.

A portion of the ellipsoidal wing-case of mean radius R was removed by passing over it very lightly a piece of the finest emery-paper fastened to a flat piece of wood. This left a clean elliptical hole of mean radius r showing the edges of the "metallic" film, whose width, h, could not be appreciated in a microscope with a half-inch objective. If this be estimated at less than 0·001 mm. the relation $\frac{t}{h}=\frac{r}{\mathrm{R}}=\frac{0\cdot 5 \text{ mm.}}{5 \text{ mm.}}$ gives t, the thickness of the film, less than a ten-thousandth of a millimetre.

A second specimen of the same general coppery lustre,

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* The character of the curves for the organic film is considerably more "metallic" than the corresponding curves for magenta.

gave a set of curves (Pl. IV. fig. 5) which showed a double reversal; the phase-curve being positive for crimson and red, negative for orange and orange yellow, and positive again for the yellow, green, and blue.

A series of curves for a very thin film of magenta (estimated thickness 0·00005 mm.) gave results surprisingly resembling those of the beetle. The second point of inversion being, however, in the green instead of the yellow, and the "metallic" character of the film being much less marked than in the beetle wing-case. The resemblance in the lower curves, showing the variation of maximum steepness with the colour, is even more striking. It can scarcely be doubted, therefore, that here again the "metallic" colour is produced in a film whose thickness is of the order of a ten thousandth of a millimetre or less.

A third example (Pl. IV. fig. 6) is added, in which the correspondence is less marked, for the purpose of illustrating the general character of the curves for a case of green metallic lustre. There is in fact no aniline colour which shows an accurate correspondence, but the same magenta curves may be referred to for comparison.

The beetle wing-cases furnish in many cases a fairly smooth surface, and the difficulties in obtaining the necessary measurements are far less than when working with feathers of birds or with butterfly scales. Nevertheless, as Pl. IV. fig. 7 shows, the same general characteristics obtain in these, in both the phase-curves and the amplitude-ratios. It may be noted that the two curves do not always correspond*, but it is probable that the difference may be explained by the difficulty in obtaining accurate results with surfaces so irregular.

It is worthy of note that in all of these curves (except that furnished by a *red* humming-bird feather) the curves are negative; from which it is fair to conclude that the film which produces the surface colour is very thin.

The total number of specimens which have been examined is perhaps not so large as it should be to draw general conclusions, and it is clearly desirable that it be extended; but so far the evidence for *surface film*, as the effective source

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* If we take the approximate formula $k = \tan 2\psi$, it is at once apparent that the dotted curves in our diagrams should have the highest minimum value for all the cases of great opacity. Thus the opacity may be inferred from the dotted curve for ψ as well as from the full curve for $d\Delta/di$, and in general the indications in the two cases show a rough agreement, the steeper full curves corresponding to the lower dotted curves, and *vice versa*.

of the metallic colours in birds and insects, is entirely conclusive.

It is clear that in all of these curves the descriptive colour corresponds in general to that colour for which the full curve is least steep, and for which the dotted curve is highest; and is complementary to the colour for which the full curve is steepest and the dotted curve is lowest, as we should expect; since the former corresponds to high reflective power, while the latter is characteristic of transparent substances with but moderate reflecting power.

EXCEPTIONS.

Morpho alga.

The measurement of the phase-difference in the light reflected from the blue-winged butterfly (*Morpho alga*), instead of being zero at normal incidence, had values which ranged from 0·15 to —0·15, and which were found to vary with the orientation of the specimen. There were also corresponding changes in the general character of the phase and amplitude curves, all of which showed clearly that the whole phenomenon is considerably complicated by a structure of the scales.

An examination under the microscope revealed the presence of exceedingly fine hairs (which can only be seen in reflected light) arranged without much regularity with their length parallel with that of the scale*.

It was at first natural to attribute the blue colour to the light diffracted from these hairs; and it is not impossible that some of the silky sheen which these butterflies exhibit, is at least in part due to these hairs, whose diameter is much less than a light-wave, and which are therefore in the same relation to the light-waves as the small particles which cause the blue colour of the sky. But the changes in colour with varying incidence, so characteristic of true "surface colours," were precisely the same in this specimen, and were practically independent of the orientation; whereas the changes with the angle of incidence, which should result on the hypothesis that the colour is due to diffraction, should follow an entirely different law.

Another species of butterfly (*Papilio Ulysses*) was also examined and found to yield normal surface-colour curves, as

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* There are three varieties of scales, of different shapes. These are arranged in overlapping layers, the outer layer being quite transparent and the lower one opaque. The middle layer is the one showing blue by reflexion and brownish-yellow in transmitted light.

shown in Pl. IV. fig. 7 (No. 6). There is in this case no such minute linear structure as in the case of *Morpho alga* ; and as here the phenomenon is clearly a case of "surface-colour," so it is highly probable that the same cause is effective in the case of *Morpho*.

Many other specimens were subsequently examined, but all fell into one or other of the two classes typified by these two.

Diamond Beetle.

If a specimen of the beetle popularly known as the Diamond Beetle be examined with a low power under the microscope, the bright green dots on the wing-case are seen to consist of depressions from which spring brilliant and exquisitely coloured scales; the colours varying throughout the range of the spectrum (green, however, predominating).

The colours exhibited by these scales are so vivid and varied, and the changes so rapid with varying incidence, that it was at once evident that the effect must be due to diffraction from regular striations, which were accordingly looked for under a magnification of about 1000 diameters. There were occasionally faint indications of striated structure, but so uncertain that if other indications had been less decided it might have been concluded that some other cause must have been effective. But on putting the microscope out of focus a moderately pure *spectrum* was observed, and by measuring the angles of incidence and diffraction of the various colours, the "grating" space could be determined, and was found to be of the order of a thousandth to a two thousandth of a millimetre.

The specimen was next examined by reflected light* and the striations at once appeared, the count of the striations giving numbers agreeing very well with the calculated values. Frequently a single scale showed two or even three series of striations, giving corresponding spectra in three different directions. Another important feature of these "gratings" was shown in the fact that the light is all concentrated in a single spectrum, showing that the striations must have an unsymmetrical saw-tooth shape †.

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* The observation is somewhat difficult on account of the very small working space when using high powers.

† It may be noted that the objection that the colours of birds and insects cannot be due to diffraction on account of the equalizing effects of the varying angles of incidence and diffraction, would not apply if the striations are so fine as to give practically a single spectrum extending over a range of 45°.

On immersing the specimen in oil or other liquid little or no change is observed, except in those specimens in which a small communicating aperture exists in the neck (point of support) of the scale. The oil can be seen to gradually fill the interior, and simultaneously all trace of colour vanishes*. It appears, then, that the colour in this case is due to fine striations on the interior surface of the scale.

Plustiotis resplendens.

This is a beetle whose whole covering appears as if coated with an electrolytic deposit of metal, with a lustre resembling brass. Indeed, it would be difficult for even an experienced observer to distinguish between the metal and the specimen.

On examination with the Babinet compensator it was found that the reflected light was *circularly polarized* even at normal incidence, whether the incident light was polarized or natural. The proportion of circularly polarized light is greatest in the blue, diminishing gradually in the yellow portion of the spectrum and vanishing in the orange-yellow—for which colour the light appears to be completely depolarized. On progressing towards the red end of the spectrum traces of circular polarization *in the opposite sense* appear, the proportion increasing until the circular polarization is nearly complete in the extreme red.

It was at first suspected that the phase difference (not always as great as one quarter, but varying between ·15 and ·25) was due to linear structure, as in the case of *Morpho alga*; but on rotating the specimen about the normal no change resulted. The effect must therefore be due to a "screw structure" of ultra microscopic, probably of molecular dimensions. Such a structure would cause a separation of natural incident light into two circularly polarized pencils travelling with different speeds, and having different coefficients of absorption.

Such cases have been observed in some absorbing crystals; but whereas in these the difference in absorption between the two circularly polarized pencils is quite small compared with the total absorption—here one of the two is almost totally reflected, while there is scarcely a trace of the other.

If this hypothesis be correct, however, the selective absorption (or reflexion) being reversed at the other end of the

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* Sometimes a faint indication of colour remains (usually greenish) which shows the characteristics of surface colour. It is probable that this surface colour acts conjointly with the effect of diffraction, and indeed the character of the spectrum indicates an excess of green which may be thus accounted for.

spectrum—then for the orange-yellow the resulting light should be compounded of these two; and the resulting light should be plane-polarized, not depolarized.

The depolarization is in fact only apparent; for on using a moderately high power objective it is at once evident that there is a structure in the wing-case which causes a difference of phase between the components varying very rapidly from point to point; and the resulting plane of the plane-polarized light varies with corresponding rapidity, leaving no trace of polarization when the observation is made with a telescope.

The absorption coefficient for this specimen is quite of the order of that of the metals; and the thickness of the "metallic" film is of the order of a ten thousandth of a millimetre.

I take this opportunity of expressing my appreciation of the courtesy of Messrs. R. M. Strong, V. E. Shelford, W. L. W. Field, and H. B. Ward, to which I am indebted for bringing the literature of the subject to my notice, and for the specimens on which these observations are based.

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LXII. *On a Physical Interpretation of Schlömilch's Theorem in Bessel's Functions.* *By* Lord Rayleigh, *O.M., F.R.S.*[*]

THIS theorem teaches that any function $f(r)$ which is finite and continuous for real values of r between the limits $r=0$ and $r=\pi$, both inclusive, may be expanded in the form

$$f(r)=a_0+a_1J_0(r)+a_2J_0(2r)+a_3J_0(3r)+\ldots, \quad (1)$$

J_0 being the Bessel's function usually so denoted; and Schlömilch's demonstration has been reproduced with slight variations in several text-books[†]. So far as I have observed, it has been treated as a purely analytical development. From this point of view it presents rather an accidental appearance; and I have thought that a physical interpretation, which is not without interest in itself, may help to elucidate its origin and meaning.

The application that I have in mind is to the theory of

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[*] Communicated by the Author.

[†] See, for example, Gray & Mathews' 'Bessel's Functions,' p. 30; Whittaker's 'Modern Analysis,' §165.

FIG. 1.

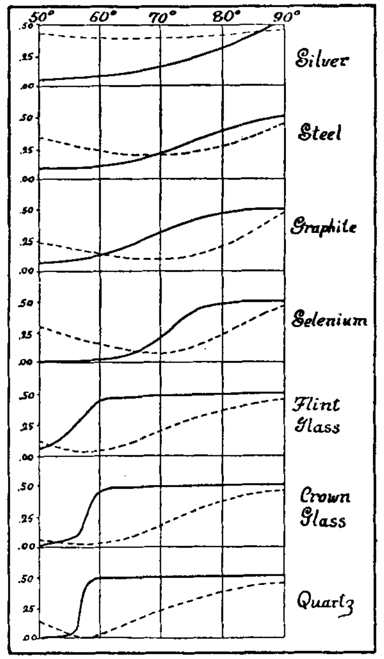

FIG. 2.

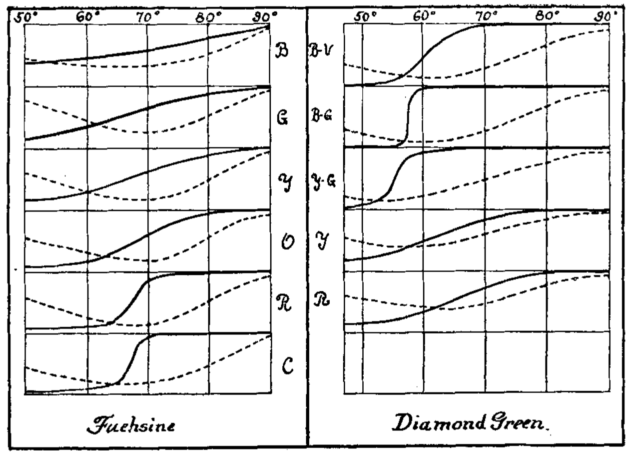

FIG. 3.

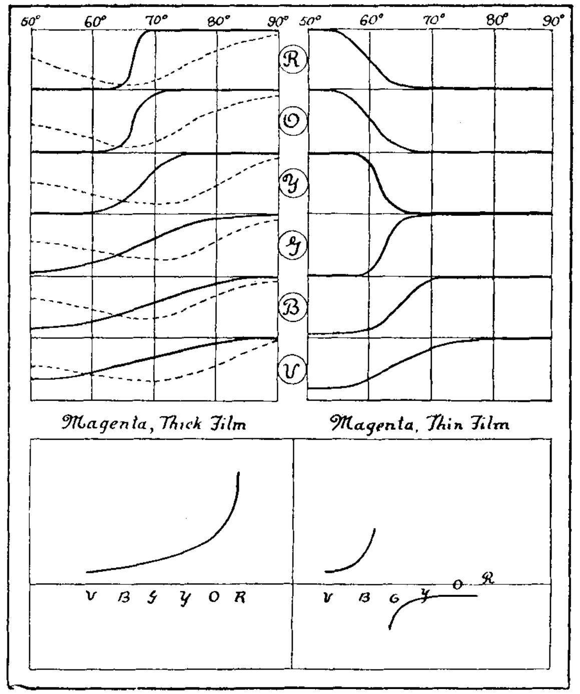

FIG. 4.

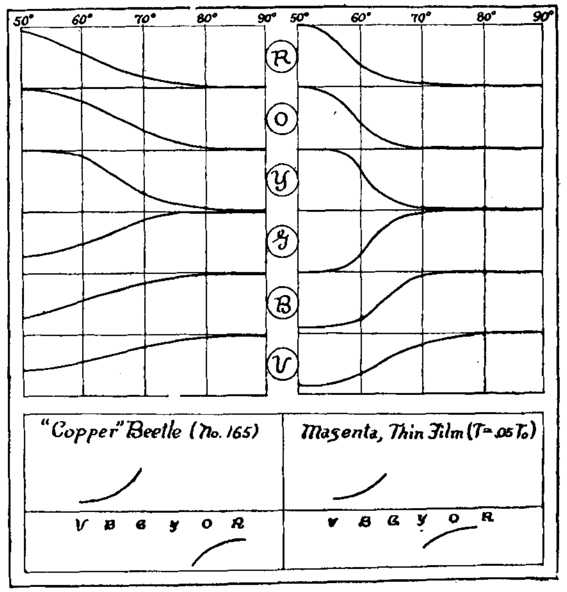

FIG. 5.

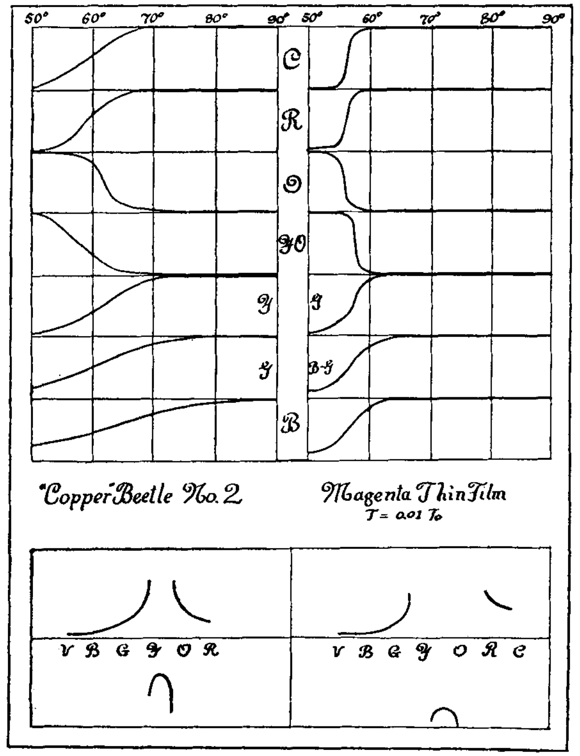

FIG. 6.

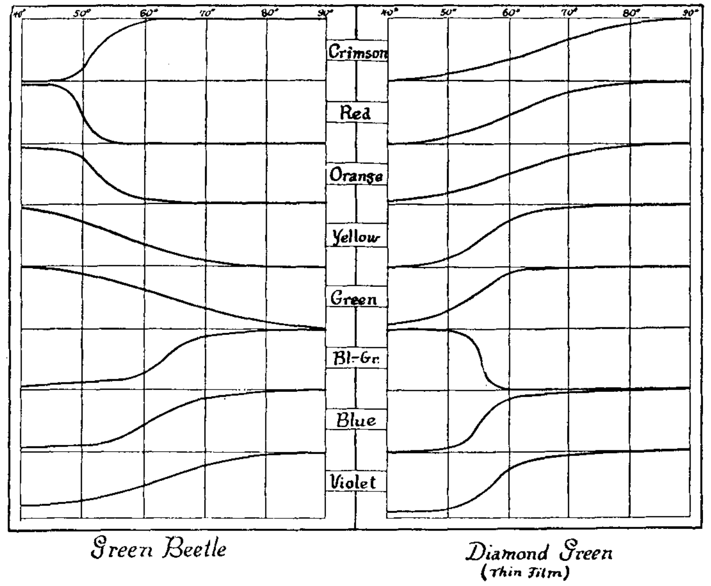

FIG. 7.