



## XXXVII. A general formula for the moments of the normal correlation function of any number of variates

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The experiments of Franck and Hertz show that in some cases, for example, argon and nitrogen, an electron may collide with a very large number of atoms without losing energy, and in cases where the electrons cannot make many collisions before being captured and attached to an atom, the effects seem to be reconcilable with the view that such collisions as occur are elastic, and that the alternatives are capture or escape with undiminished energy.

Since the frequency of vibrations of an electron is proportional to the magnetic force it is evident that it would be affected by an external magnetic field; it would also be affected by an external electric field, since this would displace the position of equilibrium. The consideration of these effects must, however, be left for another occasion.

XXXVII. *A General Formula for the Moments of the Normal Correlation Function of any Number of Variates.*  
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IF  $x_1, x_2, \dots, x_n$  be  $n$  variates reckoned from their respective means and expressed in their respective dispersions as units, and  $r_{pq}$  is the coefficient of correlation of  $x_p$  and  $x_q$ , the normal correlation function is given by

$$\phi(x_1, x_2, \dots, x_n) = \frac{1}{(2\pi)^{n/2} \sqrt{S}} e^{-\frac{1}{2S} \sum \sum S_{pq} x_p x_q}, \quad (1)$$

where

$$S = \begin{vmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ r_{21} & r_{22} & \dots & r_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ r_{n1} & r_{n2} & \dots & r_{nn} \end{vmatrix} = \begin{vmatrix} r_{pq} \end{vmatrix}$$

and

$$S_{pq} = \frac{\partial S}{\partial r_{pq}}.$$

The function  $\phi(x_1, x_2, \dots, x_n)$  may also be written in the form of a certain multiple integral which will be

\* Communicated by the Author.

found to be very useful in discussing the mathematical properties of the function. Indeed we have

$$\phi(x_1, x_2, \dots, x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} dw_1 \int_{-\infty}^{\infty} dw_2 \dots \int_{-\infty}^{\infty} dw_n e^{-\frac{1}{2} \sum r_{pq} w_p w_q} e^{-i \sum x_p w_p}. \quad (2)$$

This integral, which for  $n=1$  reduces to the well-known integral of Laplace, has been given by Charlier for the special case of  $n=2$ . A proof of the theorem for any value of  $n$  will be given in the Appendix.

By partial integration it is further easily verified that

$$\begin{aligned} & \cdot k_1 + k_2 + \dots + k_n x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} \phi(x_1, x_2, \dots, x_n) = \\ & = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} dw_1 \int_{-\infty}^{\infty} dw_2 \dots \int_{-\infty}^{\infty} dw_n \frac{\partial^{k_1+k_2+\dots+k_n} e^{-\frac{1}{2} \sum r_{pq} w_p w_q}}{\partial w_1^{k_1} \partial w_2^{k_2} \dots \partial w_n^{k_n}} e^{-i \sum x_p w_p}. \quad (3) \end{aligned}$$

Hence by the theorem of Fourier we have

$$\begin{aligned} & i^{k_1+k_2+\dots+k_n} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_n x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} \phi(x_1, x_2, \dots, x_n) = \\ & = \left[ \frac{\partial^{k_1+k_2+\dots+k_n} e^{-\frac{1}{2} \sum r_{pq} w_p w_q}}{\partial w_1^{k_1} \partial w_2^{k_2} \dots \partial w_n^{k_n}} \right]_{w_1=w_2=\dots=w_n=0}. \end{aligned}$$

Putting

$$\frac{\partial^{k_1+k_2+\dots+k_n} e^{-\frac{1}{2} \sum r_{pq} w_p w_q}}{\partial w_1^{k_1} \partial w_2^{k_2} \dots \partial w_n^{k_n}} = H_{k_1 k_2 \dots k_n}(w_1, w_2, \dots, w_n) e^{-\frac{1}{2} \sum r_{pq} w_p w_q},$$

we find, as

$$H_{k_1 k_2 \dots k_n}(0, 0, \dots, 0) = 0$$

when  $k_1 + k_2 + \dots + k_n$  is an odd number, for the moments of any order of  $\phi(x_1, x_2, \dots, x_n)$  the formula

$$\begin{aligned} & \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_n x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} \phi(x_1, x_2, \dots, x_n) \\ & = (-1)^{\frac{k_1+k_2+\dots+k_n}{2}} H_{k_1 k_2 \dots k_n}(0, 0, \dots, 0). \quad (4) \end{aligned}$$

We have now, remembering that in the case under consideration  $r_{pq} = r_{qp}$  and  $r_{pp} = 1$ ,

$$\left. \begin{aligned} H_{200} \dots 0 &= [\Sigma r_{1q} w_q]^2 - 1, \\ H_{110} \dots 0 &= \Sigma r_{1q} w_q \Sigma r_{2q} w_q - r_{12}, \\ H_{400} \dots 0 &= [\Sigma r_{1q} w_q]^4 - 6[\Sigma r_{1q} w_q]^2 + 3, \\ H_{310} \dots 0 &= [\Sigma r_{1q} w_q]^3 \cdot \Sigma r_{2q} w_q - 3 \Sigma r_{1q} w_q \cdot \Sigma r_{2q} w_q \\ &\quad - 3[\Sigma r_{1q} w_q]^2 r_{12} + 3r_{12}, \\ H_{220} \dots 0 &= [\Sigma r_{1q} w_q]^2 [\Sigma r_{2q} w_q]^2 - [\Sigma r_{2q} w_q]^2 - [\Sigma r_{1q} w_q]^2 \\ &\quad - 4r_{12} \Sigma r_{1q} w_q \cdot \Sigma r_{2q} w_q + 1 + 2r_{12}^2, \\ H_{2110} \dots 0 &= [\Sigma r_{1q} w_q]^2 \Sigma r_{2q} w_q \cdot \Sigma r_{3q} w_q - \Sigma r_{2q} w_q \Sigma r_{3q} w_q \\ &\quad - 2r_{12} \Sigma r_{1q} w_q \Sigma r_{3q} w_q - 2r_{13} \Sigma r_{1q} w_q \Sigma r_{2q} w_q \\ &\quad - r_{23} [\Sigma r_{1q} w_q]^2 + r_{23} + 2r_{21} r_{13}, \\ H_{11110} \dots 0 &= \Sigma r_{1q} w_q \cdot \Sigma r_{2q} w_q \cdot \Sigma r_{3q} w_q \Sigma r_{4q} w_q \\ &\quad - r_{12} \Sigma r_{3q} w_q \Sigma r_{4q} w_q - r_{13} \Sigma r_{2q} w_q \Sigma r_{4q} w_q \\ &\quad - r_{23} \Sigma r_{1q} w_q \Sigma r_{4q} w_q - r_{14} \Sigma r_{2q} w_q \Sigma r_{3q} w_q \\ &\quad - r_{24} \Sigma r_{1q} w_q \Sigma r_{3q} w_q - r_{34} \Sigma r_{1q} w_q \Sigma r_{2q} w_q \\ &\quad + r_{12} r_{34} + r_{13} r_{24} + r_{23} r_{14}. \end{aligned} \right\} \cdot (5)$$

Putting

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_n x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} \phi(x_1, x_2, \dots, x_n) = m_{k_1 k_2 \dots k_n},$$

we find consequently

$$\left. \begin{aligned} m_{20} \dots 0 &= 1, \\ m_{110} \dots 0 &= r_{12}, \\ m_{40} \dots 0 &= 3, \\ m_{310} \dots 0 &= 3r_{12}, \\ m_{220} \dots 0 &= 1 + 2r_{12}^2, \\ m_{2110} \dots 0 &= r_{23} - 2r_{12} r_{13}, \\ m_{11110} \dots 0 &= r_{12} r_{34} + r_{13} r_{24} + r_{23} r_{14} \end{aligned} \right\} \dots (6)$$

and any other moment of the second and fourth orders may be obtained by permutation of indices.

Without deducing the polynomials  $H$  of the sixth order, the moments of the sixth order may be determined in the following way:—

Evidently we have

$$H_{k_1 k_2 \dots k_{s+1} \dots k_n} = \frac{\partial H_{k_1 k_2 \dots k_n}}{\partial w_s} - \Sigma r_{sq} w_q H_{k_1 k_2 \dots k_n},$$

Using this formula twice, we find, putting

$$w_1 = w_2 = \dots = w_n = 0,$$

$$\begin{aligned} & H_{k_1 k_2 \dots k_s+1 \dots k_t+1 \dots k_n} (0, 0, \dots, 0) \\ &= r_{st} H_{k_1 k_2 \dots k_n} (0, 0, \dots, 0) - (-1)^{\frac{k_1+k_2+\dots+k_n}{2}} \left[ \frac{\partial^2 H_{k_1 k_2 \dots k_n}}{\partial w_s \partial w_t} \right]_{w_1=w_2=\dots=w_n=0}; \end{aligned}$$

or

$$\begin{aligned} & m_{k_1 k_2 \dots k_s+2 \dots k_n} \\ &= m_{k_1+k_2 \dots k_n} - (-1)^{\frac{k_1+k_2+\dots+k}{2}} \left[ \frac{\partial^2 H_{k_1 k_2 \dots k_n}}{\partial w_s^2} \right]_{w_1=w_2=\dots=w_n=0}, \\ & m_{k_1 k_2 \dots k_s+1 \dots k_t+1 \dots k_n} \\ &= r_{st} m_{k_1 k_2 \dots k_n} - (-1)^{\frac{k_1+k_2+\dots+k_n}{2}} \left[ \frac{\partial^2 H_{k_1 k_2 \dots k_n}}{\partial w_s \partial w_t} \right]_{w_1=w_2=\dots=w_n=0}. \end{aligned} \quad (7)$$

Hence we find, with the aid of formulæ (5) and (6),

$$\begin{aligned} m_{60} \dots 0 &= 15, \\ m_{510} \dots 0 &= 15r_{12}, \\ m_{420} \dots 0 &= 3 + 12r_{12}^2, \\ m_{4110} \dots 0 &= 3r_{23} + 12r_{12}r_{13}, \\ m_{330} \dots 0 &= 9r_{12} + 6r_{12}^3, \\ m_{3210} \dots 0 &= 3r_{13} + 6r_{12}r_{23} + 6r_{12}^2r_{13}, \\ m_{31110} \dots 0 &= 3r_{13}r_{24} + 3r_{12}r_{34} + 3r_{23}r_{14} + 6r_{12}r_{14}r_{13}, \\ m_{2220} \dots 0 &= 1 + 2r_{12}^2 + 2r_{13}^2 + 2r_{23}^2 + 8r_{12}r_{13}r_{23}, \\ m_{22110} \dots 0 &= r_{34} + 2r_{13}r_{14} + 2r_{23}r_{24} + 2r_{12}^2r_{34} + 4r_{12}r_{13}r_{24} \\ &\quad + 4r_{12}r_{14}r_{23}, \\ m_{211110} \dots 0 &= r_{23}r_{45} + r_{24}r_{35} + r_{25}r_{34} + 2r_{12}r_{13}r_{45} + 2r_{23}r_{14}r_{15} \\ &\quad + 2r_{13}r_{14}r_{25} + 2r_{13}r_{15}r_{24} + 2r_{12}r_{14}r_{35} + 2r_{12}r_{15}r_{34}, \\ m_{1111110} \dots 0 &= r_{23}r_{14}r_{56} + r_{13}r_{24}r_{56} + r_{12}r_{34}r_{56} + r_{34}r_{15}r_{26} + r_{34}r_{16}r_{25} \\ &\quad + r_{24}r_{15}r_{36} + r_{24}r_{16}r_{35} + r_{14}r_{25}r_{36} + r_{14}r_{26}r_{35} \\ &\quad + r_{23}r_{15}r_{46} + r_{23}r_{16}r_{45} + r_{13}r_{25}r_{46} + r_{13}r_{26}r_{45} \\ &\quad + r_{12}r_{35}r_{46} + r_{12}r_{36}r_{45}, \end{aligned} \quad (8)$$

and any other moment of the sixth order is obtained by permutation of indices.

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The moments of the variates expressed in any units being denoted by  $\nu_{k_1 k_2 \dots k_n}$ , we have further

$$\nu_{k_1 k_2 \dots k_n} = \sigma_1^{k_1} \sigma_2^{k_2} \dots \sigma_n^{k_n} m_{k_1 k_2 \dots k_n},$$

where  $\sigma_1, \sigma_2, \dots, \sigma_n$  are the respective dispersions of the variates.

On account of the reciprocity of the determinants  $S$  and  $\Delta$  we have finally, putting

$$\frac{S_{pq}}{S} = a_{pq}; \quad \Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}; \quad \Delta_{pq} = \frac{\partial \Delta}{\partial a_{pq}}$$

and

$$\frac{\partial^{k_1+k_2+\dots+k_n}}{\partial w_1^{k_1} \partial w_2^{k_2} \dots \partial w_n^{k_n}} e^{-\frac{1}{2\Delta} \Sigma \Sigma \Delta_{pq} w_p w_q} = R_{k_1 k_2 \dots k_n}(w_1, w_2, \dots, w_n) e^{-\frac{1}{2\Delta} \Sigma \Sigma \Delta_{pq} w_p w_q},$$

the general formula

$$\begin{aligned} \frac{\sqrt{\Delta}}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \dots \int_{-\infty}^{\infty} dx_n x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} e^{-\frac{1}{2} \Sigma \Sigma a_{pq} x_p x_q} \\ = (-1)^{\frac{k_1+k_2+\dots+k_n}{2}} R_{k_1 k_2 \dots k_n}(0, 0, \dots, 0). \end{aligned}$$

Here  $a_{pq}$  may be any real quantities, and up to the sixth order the value of the integral is obtained if in the respective equations (6) and (8)  $r_{pq}$  is exchanged for  $\frac{\Delta_{pq}}{\Delta}$ .

#### APPENDIX.

In order to prove equation (2) we put, using the theorem of Fourier,

$$\begin{aligned} \phi(x_1, x_2, \dots, x_n) &= \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} dw_1 \int_{-\infty}^{\infty} dw_2 \dots \int_{-\infty}^{\infty} dw_n e^{-i \Sigma x_p w_p} \\ &\times \int_{-\infty}^{\infty} d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2 \dots \int_{-\infty}^{\infty} d\lambda_n \phi(\lambda_1, \lambda_2, \dots, \lambda_n) e^{i \Sigma w_p \lambda_p}. \end{aligned}$$

It will thus be required to show that

$$(2\pi)^n \sqrt{S} \int_{-\infty}^{\infty} d\lambda_1 \int_{-\infty}^{\infty} d\lambda_2 \dots \int_{-\infty}^{\infty} d\lambda_n e^{-\frac{1}{2S} \sum \sum S_{pq} \lambda_p \lambda_q + i \sum w_p \lambda_p} = e^{-\frac{1}{2} \sum \sum r_{pq} w_p w_q}. \quad (1)$$

The evaluation of this integral will be possible if the exponent  $-\frac{1}{2S} \sum \sum S_{pq} \lambda_p \lambda_q + i \sum w_p \lambda_p$  is reduced to the form  $-\frac{1}{2S} \sum \sum S_{pq} (\lambda_p + \alpha_p)(\lambda_q + \alpha_q) + \frac{1}{2S} \sum \sum S_{pq} \alpha_p \alpha_q$  and the integral (1) is then equal to

$$e^{\frac{1}{2S} \sum \sum S_{pq} \alpha_p \alpha_q}.$$

Thus it remains only to show that

$$\frac{1}{2S} \sum \sum S_{pq} \alpha_p \alpha_q = -\frac{1}{2} \sum \sum r_{pq} w_p w_q.$$

Putting  $i\lambda_p = \lambda_p'$  and  $i\alpha_p = \alpha_p'$ , we have

$$\begin{aligned} & \frac{1}{2S} \sum \sum S_{pq} \lambda_p' \lambda_q' + \sum w_p \lambda_p' \\ &= \frac{1}{2S} \sum \sum S_{pq} (\lambda_p' + \alpha_p') (\lambda_q' + \alpha_q') - \frac{1}{2S} \sum \sum S_{pq} \alpha_p' \alpha_q' \\ &= \frac{1}{2S} \sum \sum S_{pq} \lambda_p' \lambda_q' + \frac{1}{S} \sum \sum S_{pq} \lambda_p' \alpha_q'. \end{aligned}$$

Hence we have

$$w_s = \frac{1}{S} \sum_t S_{st} \alpha_t'.$$

This is a system of linear equations to determine the different  $\alpha_t'$ .

Putting

$$\Delta = \begin{vmatrix} \frac{S_{st}}{S} \end{vmatrix}; \quad \Delta_{st} = \frac{\partial \Delta}{\partial \frac{S_{st}}{S}},$$

the solution is

$$\alpha_s' = \frac{1}{\Delta} \sum_t \Delta_{st} w_t.$$

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But according to well-known theorems on determinants, we have

$$\frac{\Delta_{st}}{\Delta} = r_{st} ;$$

and hence

$$\alpha_s' = \sum_t r_{st} w_t.$$

For the sum

$$\frac{1}{2S} \sum_p \sum_q S_{pq} \alpha_p' \alpha_q'$$

we may now write

$$\frac{1}{2S} \sum_p \sum_q S_{pq} \sum_t r_{pt} w_t \sum_s r_{sq} w_s.$$

Changing the order of summation, we have this equal to

$$\frac{1}{2S} \sum_s \sum_t w_s w_t \sum_p r_{pt} \sum_q S_{pq} r_{sq}.$$

But

$$\sum_q S_{pq} r_{sq} = \begin{cases} 0 & \text{when } p \neq s, \\ S & \text{,, } p = s. \end{cases}$$

Thus the quadruple sum is equal to

$$\frac{1}{2S} \sum_s \sum_t w_s w_t r_{st} S,$$

or

$$\frac{1}{2S} \sum_p \sum_q S_{pq} \alpha_p' \alpha_q' = \frac{1}{2} \sum_p \sum_q r_{pq} w_p w_q.$$

As  $\alpha_p' = i\alpha_p$ , we have finally

$$\frac{1}{2S} \sum_p \sum_q S_{pq} \alpha_p \alpha_q = -\frac{1}{2} \sum_p \sum_q r_{pq} w_p w_q.$$

Q. E. D.