XVII.—Theorems relating to a Generalization of Bessel's Function. II.

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CONTENTS.

1.

Introduction.

The theory of the functions commonly known as q functions might perhaps be greatly developed, if investigators were to work on lines suggested by the functional notation of well-known analytic functions. For instance, the analysis connected with the circular functions $\sin x$, $\cos x$, . . . , might be regarded as the theory of certain infinite products without using any special functional notation. It need not be explained however, how great was the gain to elementary algebra by the introduction of the exponential function (regarded as the limit of a certain infinite product, or as the limit of a certain infinite series) denoted e^x, with certain characteristic properties, enabling the worker to make transformations easily and quickly. Of course, the vast store of interesting and in many cases useful results connected with the elementary functions of analysis might have been obtained without the introduction of any notation capable of rapid and easy transformations, but I think it unlikely that they would have been obtained.

In chapter xi. of Cayley's *Elliptic Functions* the identity

$$\frac{1}{1-q^2 \cdot 1-q^4 \ldots 1-q^{2n}}\left\{1+\frac{q^2}{1-q^2}\cdot\frac{q^{2n}}{1-q^{2n+2}}+\frac{q^8}{1-q^2\cdot 1-q^4}\cdot\frac{q^{4n}}{1-q^{2n+2}\cdot 1-q^{2n+4}}+\ldots\right\}$$
$$=\prod_{m=1}^{\infty}\frac{1}{(1-q^{2m})} \qquad (1)$$

is used in order to express Jacobi's Θ function, in the well-known form

$$1-2q\cos 2x+2q^4\cos 4x-2q^9\cos 6x+\ldots$$

The likeness of the series (1) to Bessel's series is very obvious. It is a very special case of the series which I have denoted $J_{[n]}$ in previous papers, and in itself might have suggested a theory of q functions analogous to Bessel's functions. In the discussion of q functions a great variety of notations has been used. I propose in this paper to bring before the Society a series of formulæ relating firstly to a function $\mathrm{E}_p(x)$ analogous to exp (x). These formulæ are supplementary to those given in *Trans. Roy. Soc. Edin.*, vol. xli. pp. 105–118, and will lead to one or two interesting properties of a function $\mathrm{J}^p_{m,n}(x)$, which may be termed a generalized Bessel-function of double order, and to various novel expressions of elliptic functions in terms of the generalized Bessel-function. For example, JACOBI's Θ function is expressed by the form

$$\Theta\left(\frac{2\mathrm{K}x}{\pi}\right) = q_0 \frac{\mathfrak{J}_{[n]}(u)\mathfrak{J}_{[n]}(v)}{\mathrm{J}_{[n]}(u)\mathrm{J}_{[n]}(v)} \qquad \text{(A)}$$

in which

$$q_0 = \prod_{m=1}^{\infty}(1-q^{2m})$$

$$u = \frac{i\sqrt{q}}{q-1}e^{ix}$$

$$v = \frac{i\sqrt{q}}{q-1}e^{-ix}$$

$$p = q = e^{-\pi\frac{\mathrm{K}'}{\mathrm{K}}}$$

It is noteworthy that n (the order of the J functions) is in (A) an arbitrary number. It appears only in the expression on the right side of that equation. A definite integral expression for the functions J will also be given.

2.

FUNCTION $\mathrm{E}_p(x)$.

The series

$$1 + \frac{x}{p-1} + p\frac{x^2}{p-1\cdot p^2-1} + \ldots\ldots$$

and its equivalent product

$$(1-x)(1-px)(1-p^2x)\ldots\ldots$$

are well known: we derive a function analogous to the exponential function. (Cf. *Trans. R.S.E.*, xli. p. 116; and *Proc. L.M.S.*, series 2, vol. ii. p. 194.)

$$\mathrm{E}_{\frac{1}{p}}(x) = 1 + \frac{x}{[1]} + p\frac{x^2}{[2]!} + \ldots + p^{r(r-1)/2}\frac{x^r}{[r]!} + \ldots\ldots$$

$$\mathrm{E}_p(x) = 1 + \frac{x}{[1]} + \frac{x^2}{[2]!} + \ldots\ldots + \frac{x^r}{[r]!} + \ldots\ldots$$

$$[r] = (p^r-1)/(p-1)$$

The function $\mathrm{E}_p(x)$ may be regarded, like the exponential function, either as the limit of a certain infinite series or a certain infinite product. The results numbered (2) . . . (26) are either easily obtained or are known in other forms.

$$E_p(x)\cdot E_{\frac{1}{p}}(-x) = 1 \qquad (2)$$

$$E_p(x)\cdot E_p(-x) = 1 + \frac{x^2}{1-p^2}(1-p)^2 + \frac{x^4}{1-p^2\cdot 1-p^4}(1-p)^4 + \dots\dots \qquad (3)$$

$$= E_{p^2}\left(x^2\frac{p-1}{p+1}\right) \qquad (4)$$

which reduce, when $p=1$, to $\exp(x) \times \exp(x) = 1$,

$$E_p\left(\frac{x}{p-1}\right)\cdot E_p\left(-\frac{x}{p-1}\right) = 1 + \frac{x^2}{1-p^2} + \frac{x^4}{1-p^2\cdot 1-p^4} + \dots \qquad (5)$$

$$= E_{p^2}\left(-\frac{x^2}{p^2-1}\right) \qquad (6)$$

$$= \prod_{m=1}^{\infty}(1-x^2p^{-2m}) \qquad (7)$$

The product is absolutely convergent if $|p| > 1$. The series are convergent, however, if $|p| \geqslant 1$, and also for $|p| < 1$ provided $x < \frac{1}{1-p}$.

It follows that

$$E_p(\omega x)E_p(-\omega x)E_p(\omega^2x)E_p(-\omega^2x)\dots\dots E_p(\omega^n x)E_p(-\omega^n x) = E_{p^{2n}}\left(x^{2n}\frac{(p-1)^{2n}}{p^{2n}-1}\right). \qquad (8)$$

$$\text{if} \quad |p| > 1, \quad \omega = (-1)^{\frac{1}{n}}$$

The corresponding theorem in case $p < 1$ is easily obtained by inversion of the base p.

$$E_{\frac{1}{p}}\left(\frac{px}{p-1}\right)\cdot E_{\frac{1}{p}}\left(-\frac{px}{p-1}\right) = \sum_{n=0}^{\infty} p^{n^2+n}\frac{x^{2n}}{(p^2-1)(p^4-1)\dots\dots(p^{2n}-1)} \qquad (9)$$

$$= \prod_{m=1}^{\infty}\left\{1-x^2p^{2m}\right\} \qquad (10)$$

$$E_{\frac{1}{p}}\left(\frac{x\sqrt{p}}{p-1}\right)\cdot E_{\frac{1}{p}}\left(-\frac{x\sqrt{p}}{p-1}\right) = \prod_{m=1}^{\infty}\left\{1-x^2p^{2m-1}\right\} \qquad (11)$$

$$= E_{\frac{1}{p^2}}\left(-\frac{px^2}{1-p^2}\right) \qquad (12)$$

$$= \sum_{0}^{\infty} p^{n^2}\frac{x^{2n}}{(p^2-1)(p^4-1)\dots(p^{2n}-1)} \qquad (13)$$

$$E_{\frac{1}{p^2}}\left(\frac{px^2}{1-p^2}\right)\cdot E_{\frac{1}{p^2}}\left(\frac{px^{-2}}{1-p^2}\right) = \frac{1}{\prod_{m=1}^{\infty}(1-p^{2m})}\left\{1-2p(x^2+x^{-2})+2p^4(x^4+x^{-4})-2p^9(x^6+x^{-6})+\dots\right\} \qquad (14)$$

On putting $x=e^{iu}$, the series on the right becomes JACOBI'S function

$$\Theta\left(\frac{2Ku}{\pi}\right)$$

$$\text{in which} \quad p = e^{-\pi\frac{K'}{K}}$$

$$E_p(x)E_{\frac{1}{p}}(y) = 1 + \frac{(x+y)}{[1]} + \frac{(x+y)(x+py)}{[1][2]} + \dots\dots \qquad (15)$$

$$p<1,\ x<\frac{1}{1-p},\ y \text{ unrestricted}$$

(Cf. *Proc. Edin. Math. Soc.*, vol. xxii. (8).) or $p>1,\ y<\frac{1}{p-1},\ x$ unrestricted.

From this we derive

$$\mathrm{E}_p\left(\frac{x}{1-p}\right).\mathrm{E}_{\frac{1}{p}}\left(\frac{p^n x}{p-1}\right)=1+[n]x+\frac{[n][n+1]}{[2]!}x^2+\ldots\ldots\ldots \quad (16)$$

$$=\int_n^p(x) \quad (17)$$

$$\mathrm{E}_p\left(\frac{x}{1-p}\right)\cdot\mathrm{E}_{\frac{1}{p}}\left(\frac{px}{p-1}\right)=1+x+x^2+\ldots\ldots \quad (18)$$

$$=\frac{1}{1-x} \qquad (x<1) \quad (19)$$

In this expression we notice that inversion of the base p simply interchanges the E functions in the product on the left side of the equation (18).

$$\mathrm{E}_p\left(\frac{p^n x}{1-p}\right)\mathrm{E}_{\frac{1}{p}}\left(\frac{x}{1-p}\right)=1-[n]x+p\frac{[n][n-1]}{[2]!}x^2-p^3\frac{[n][n-1][n-2]}{[3]!}x^3+\ldots\ldots \quad (20)$$

$$=\phi_n^p(-x) \quad (21)$$

Hence

$$\int_n^p(x)=\int_{-n}^p(p^n x) \quad (22)$$

$$\int_n^p(x)\times\int_n^p(-x)=\int_n^{p^2}(x^2) \quad (23)$$

$$\phi_n^p(x)\times\phi_n^p(-x)=\phi_n^{p^2}(x^2) \quad (24)$$

$$\int_n^p(x)\times\phi_n^p(-x)=1 \quad (25)$$

The equations

$$e^x\cdot e^{-x}=1$$

$$\mathrm{E}_p(x)\cdot\mathrm{E}_{\frac{1}{p}}(-x)=1$$

are special cases of (25).

If n be infinite

$$\int_n^p(x)=\mathrm{E}_{\frac{1}{p}}\left(\frac{x}{p-1}\right) \quad (26)$$

FUNCTION $\mathrm{I}_{[n]}(x)$.

It is well known in the theory of BESSEL's function that

$$e^{-x}\cdot\mathrm{I}_n(x)=\frac{x^n}{2^n\cdot\Gamma(n+1)}\left\{1-x+\frac{2n+3}{2(2n+2)}x^2-\frac{2n+5}{2\cdot 3(2n+2)}x^3+\ldots\ldots\right\}$$

$$\mathrm{I}_n(x)=i^{-n}\mathrm{J}_n(ix).$$

In a paper on Basic numbers applied to BESSEL's function (*Proc. Lond. Math. Soc.*, series 2, vol. iii., 1905), I have extended this theorem in the form

$$\begin{aligned} &\mathrm{E}_p(-x)I_{[n]}(x) \\ &= \mathrm{E}_{\frac{1}{p}}(-x)\mathrm{I}_{[n]}(x) \\ &= \frac{x^n}{\{2n\}!}\left\{1 - x + \frac{[2n+3]}{[2][2n+2]}x^2 - \frac{[2n+5]}{[2][3][2n+2]}x^3 + \ldots\ldots\right\} \end{aligned} \qquad (27)$$

$$\mathrm{I}_{[n]}(x) = i^{-n}\mathrm{J}_{[n]}(ix)$$
$$I_{[n]}(x) = i^{-n}\mathfrak{J}_{[n]}(ix)$$

the conditions for convergence being as follows:

Case i $\quad |p| > 1 \quad$ $\mathrm{E}_p(x)$ and $\mathrm{I}_{[n]}(x)$ are absolutely convergent for all values of x.

$\mathrm{E}_{\frac{1}{p}}(x)$ and $I_{[n]}(x)$ are absolutely convergent if $x < \frac{p}{p-1}$.

Case ii $\quad |p| < 1 \quad$ $\mathrm{E}_{\frac{1}{p}}(x)$ and $I_{[n]}(x)$ are absolutely convergent for all values of x.

$\mathrm{E}_p(x)$ and $\mathrm{I}_{[n]}(x)$ are absolutely convergent if $x < \frac{1}{1-p}$.

The series (27) is convergent for all values of p.
It is easily deduced that

$$\mathrm{J}_{[n]}(x) = \mathrm{E}_p(ix)\mathrm{E}_p(-ix)\mathfrak{J}_{[n]}(x) \qquad (28)$$

$$\mathfrak{J}_{[n]}(x) = \mathrm{E}_{\frac{1}{p}}(ix)\mathrm{E}_{\frac{1}{p}}(-ix)\mathrm{J}_{[n]}(x) \qquad (29)$$

From these relations some interesting expressions for various elliptic functions may be found.

3.

Relations with Elliptic Functions.

By means of equation (29) we are able to write

$$\frac{\mathfrak{J}_{[n]}(xt)\mathfrak{J}_{[n]}(xt^{-1})}{\mathrm{J}_{[n]}(xt)\mathrm{J}_{[n]}(xt^{-1})} = \mathrm{E}_{\frac{1}{p}}(ixt)\mathrm{E}_{\frac{1}{p}}(-ixt)\mathrm{E}_{\frac{1}{p}}(ixt^{-1})\mathrm{E}_{\frac{1}{p}}(-ixt^{-1}) \qquad (30)$$

Replacing x by u, $(u = ix\sqrt{p}/(p-1))$, we obtain by means of result (11)

$$\frac{\mathfrak{J}_{[n]}(ut)\mathfrak{J}_{[n]}(ut^{-1})}{\mathrm{J}_{[n]}(ut)\mathrm{J}_{[n]}(ut^{-1})} = \prod_{m=1}^{\infty}\left\{(1 - x^2t^2p^{2m-1})(1 - x^2t^{-2}p^{2m-1})\right\} \qquad (31)$$

Using result (12), the right side of this equation may be written

$$\mathrm{E}_{\frac{1}{p^2}}\left(\frac{px^2t^2}{1-p^2}\right)\mathrm{E}_{\frac{1}{p^2}}\left(\frac{px^2t^{-2}}{1-p^2}\right) \qquad (32)$$

or

$$= \mathrm{E}_{\frac{1}{p^2}}\left(\frac{1}{[2]}\cdot\frac{px^2t^2}{1-p}\right) \times \mathrm{E}_{\frac{1}{p^2}}\left(\frac{1}{[2]}\cdot\frac{px^2t^{-2}}{1-p}\right) \qquad (33)$$

This expression, when expanded in a LAURENT series of ascending and descending powers of t, takes the form (*Trans. R.S.E.*, vol. xli. p. 117 (μ)),

$$I_{[0]}\left(\frac{x^2}{p-1}\right)+\sum_{m=1}^{\infty}(-1)^m(t^{2m}+t^{-2m})p^{m^2}I_{[m]}\left(\frac{x^2}{p-1}\right) \quad . \quad . \quad . \quad (34)$$

In case $x=1$, the product $\prod\limits_{m=1}^{\infty}\left\{(1-x^2t^2p^{2m-1})(1-x^2t^{-2}p^{2m-1})\right\}$ may be expressed as

$$\frac{1}{\prod\limits_{m=1}^{\infty}(1-p^{2m})}\left\{1-p(t^2+t^{-2})+p^4(t^4+t^{-4})-\dots\dots\right\} \quad . \quad . \quad . \quad (35)$$

(*Cf.* CAYLEY's *Elliptic Functions*, p. 297, ed. 1876.)
We see incidentally that

$$\frac{1}{\prod\limits_{m=1}^{\infty}(1-p^{2m})}=I_{[n]}\left(\frac{1}{p-1}\right) \quad . \quad . \quad . \quad . \quad . \quad (36)$$

for all positive integral values of n. Denoting the nature of the base by an index, we write

$$I^p_{[0]}\left(\frac{1}{p-1}\right)=I^p_{[1]}\left(\frac{1}{p-1}\right)=I^p_{[2]}\left(\frac{1}{p-1}\right)=\dots\dots \quad . \quad . \quad (37)$$

$$\mathrm{I}^{\frac{1}{p}}_{[0]}\left(\frac{p}{1-p}\right)=p^{-1}\,\mathrm{I}^{\frac{1}{p}}_{[1]}\left(\frac{p}{1-p}\right)=p^{-4}\,\mathrm{I}^{\frac{1}{p}}_{[2]}\left(\frac{p}{1-p}\right)=\dots\dots \quad . \quad . \quad . \quad (38)$$

which is the expression in generalized Bessel-function-notation of the well-known result

$$\frac{1}{1-q^2\cdot1-q^4\ \dots\ 1-q^{2n}}\left\{1+\frac{q^{2n+2}}{(1-q^2)(1-q^{2n+2})}+\dots\right\}=\prod_{m=1}^{\infty}\frac{1}{(1-q^{2m})}.$$

On replacing t by $e^{i\theta}$, the equation (31) becomes

$$\frac{\mathfrak{J}_{[n]}\left(ix\frac{p^{\frac12}e^{i\theta}}{p-1}\right)\mathfrak{J}_{[n]}\left(ix\frac{p^{\frac12}e^{-i\theta}}{p-1}\right)}{\mathrm{J}_{[n]}\left(ix\frac{p^{\frac12}e^{i\theta}}{p-1}\right)\mathrm{J}_{[n]}\left(ix\frac{p^{\frac12}e^{-i\theta}}{p-1}\right)}=\prod_{m=1}^{\infty}(1-2x^2\cos2\theta p^{2m-1}+x^4p^{4m-2}) \quad . \quad . \quad (39)$$

Using now JACOBI's notation, and writing $u=\frac{i\sqrt{p}}{p-1}e^{ix}, v=\frac{i\sqrt{p}}{p-1}e^{-ix}, q_0=\prod\limits_{m=1}^{\infty}(1-q^{2m})$, $q=p$, we obtain

$$q_0\frac{\mathfrak{J}_{[n]}(u)\mathfrak{J}_{[n]}(v)}{\mathrm{J}_{[n]}(u)\mathrm{J}_{[n]}(v)}=\Theta\left(\frac{2\mathrm{K}x}{\pi}\right) \quad . \quad . \quad . \quad (40)$$

$$2q_0q^{\frac14}\sin x\cdot\frac{\mathfrak{J}_{[n]}(p^{\frac12}u)\mathfrak{J}_{[n]}(p^{\frac12}v)}{\mathrm{J}_{[n]}(p^{\frac12}u)\mathrm{J}_{[n]}(p^{\frac12}v)}=\mathrm{H}\left(\frac{2\mathrm{K}x}{\pi}\right) \quad . \quad . \quad . \quad (41)$$

$$2q^{\frac14}k^{-\frac12}\sin x\frac{\mathrm{J}_{[n]}(u)\mathfrak{J}_{[m]}(p^{\frac12}u)\mathrm{J}_{[n]}(v)\mathfrak{J}_{[m]}(p^{\frac12}v)}{\mathrm{J}_{[m]}(p^{\frac12}u)\mathfrak{J}_{[n]}(u)\mathrm{J}_{[m]}(p^{\frac12}v)\mathfrak{J}_{[n]}(v)}=sn\left(\frac{2\mathrm{K}x}{\pi}\right) \quad . \quad . \quad . \quad (42)$$

$$2q^{\frac14}\cos x\sqrt{\frac{k'}{k}}\cdot\frac{\mathrm{J}_{[n]}(u)\mathfrak{J}_{[m]}(ip^{\frac12}u)\mathrm{J}_{[n]}(v)\mathfrak{J}_{[m]}(ip^{\frac12}v)}{\mathrm{J}_{[m]}(ip^{\frac12}u)\mathfrak{J}_{[n]}(u)\mathrm{J}_{[m]}(ip^{\frac12}v)\mathfrak{J}_{[n]}(v)}=cn\left(\frac{2\mathrm{K}x}{\pi}\right) \quad . \quad . \quad . \quad (43)$$

$$\sqrt{k'}\frac{\mathrm{J}_{[n]}(u)\mathfrak{J}_{[m]}(iu)\mathrm{J}_{[n]}(v)\mathfrak{J}_{[m]}(iv)}{\mathrm{J}_{[n]}(iu)\mathfrak{J}_{[m]}(u)\mathrm{J}_{[n]}(iv)\mathfrak{J}_{[m]}(v)}=dn\left(\frac{2\mathrm{K}x}{\pi}\right) \quad . \quad . \quad . \quad . \quad (44)$$

We notice that in the expressions for sn, cn, dn, two arbitrary constants (orders of the functions) n, m appear:

JACOBI's function $Z(x) = \frac{\Theta'(x)}{\Theta(x)}$ is related to the J functions as follows,

$$\frac{2K(p-1)}{i\pi\sqrt{p}} \cdot Z\left(\frac{2Kx}{\pi}\right) = e^{ix}\left\{\frac{\mathfrak{J}'_{[n]}(u)}{\mathfrak{J}_{[n]}(u)} - \frac{J'_{[n]}(u)}{J_{[n]}(u)}\right\} \quad e^{-ix}\left\{\frac{\mathfrak{J}'_{[n]}(v)}{\mathfrak{J}_{[n]}(v)} - \frac{J'_{[n]}(v)}{J_{[n]}(v)}\right\} \quad . \quad . \quad (45)$$

It is plain that WEIERSTRASS's functions σ, ζ, $\wp$, may be expressed by similar formulæ.

For convenience of printing, the order of the functions will sometimes be expressed by n instead of $[n]$. The known formulæ of JACOBI's functions will, it is evident, give rise to corresponding forms in the case of the J functions: for example

$$sn^2 + cn^2 = 1$$

gives rise to

$$\sin^2 x \frac{\mathfrak{J}^2_m(p^{\frac{1}{4}}u)\mathfrak{J}^2_m(p^{\frac{1}{4}}v)}{J^2_m(p^{\frac{1}{4}}u)J^2_m(p^{\frac{1}{4}}v)} + k' \cos^2 x \frac{\mathfrak{J}^2_m(ip^{\frac{1}{4}}u)\mathfrak{J}^2_m(ip^{\frac{1}{4}}v)}{J^2_m(ip^{\frac{1}{4}}u)J^2_m(ip^{\frac{1}{4}}v)} = \frac{k}{4q^{\frac{1}{4}}}\frac{\mathfrak{J}^2_n(u)\mathfrak{J}^2_n(v)}{J^2_n(u)J^2_n(v)} \quad . \quad . \quad (46)$$

$$= u\frac{\sqrt{p}}{p-1}e^{ix}, \; v = \frac{\sqrt{p}}{p-1}e^{-ix}, \; p = q = e^{-\pi\frac{K'}{K}}.$$

Using (11) and (12) it is easily found by the method of § 10, p. 116, vol. xli., *Trans. R.S.E.*, that

$$\prod_{m=1}^{\infty}\left\{1 - 2ap^{2m-1}\cos x + a^2p^{4m-2}\right\} = I_{[0]}\left(\frac{a}{p-1}\right) + 2\sum_{n=1}^{\infty} p^{n^2}\cos nx\, I_{[n]}\left(\frac{a}{p-1}\right).$$

By FOURIER's theorem we write therefore

$$p^{n^2}I_{[n]}\left(\frac{a}{p-1}\right) = \frac{1}{2\pi}\int_0^{2\pi}\left\{\prod_1^{\infty}(1 - 2ap^{2m-1}\cos x + a^2p^{4m-2})\right\}\cos nx \cdot dx \quad . \quad . \quad (47)$$

In case $a = 1$, this reduces to

$$p^{n^2}I_{[n]}\left(\frac{1}{p-1}\right) = \frac{1}{2\pi q_0}\int_0^{2\pi}\cos nx \cdot \Theta\left(\frac{Kx}{\pi}\right)dx \quad . \quad . \quad . \quad . \quad (48)$$

4.

FUNCTION $J^p_{n,m}(x)$.

Forming in a series, according to powers of t, the product

$$J_{[n]}(xt) \times J_{[n]}(xt^{-1})$$

we obtain

$$J^p_{n,}(x) + \sum_{m=1}^{+\infty}(-1)^m(t^{2m} + t^{-2m})J^p_{n,m}(x) \quad . \quad . \quad . \quad . \quad (49)$$

in which

$$J^p_{n,m}(x) = \sum_{r=0}^{\infty}(-\;)^r\frac{x^{2m+2n+4r}}{\{2m+2n+2r\}!\{2m+2r\}!\{2n+2r\}!\{2r\}!} \quad . \quad . \quad . \quad (50)$$

$$\{2r\}! = [2][4] \; . \; . \; . \; . \; [2r]$$

In the same way if we take

$$\mathfrak{J}^{p}_{n,m}(x)=\sum_{r=0}^{\infty}(-1)^{r}\frac{x^{2m+2n+4r}}{\{2m+2n+2r\}!\{2m+2r\}!\{2n+2r\}!\{2r\}!}p^{4r(m+n+r)} \quad . \quad . \quad (51)$$

which is related to $J_{n,m}$ by inversion of the base p, since

$$J^{\frac{1}{p}}_{n,m}(x)=p^{2(m^2+n^2+mn)}\mathfrak{J}^{p}_{n,m}(x)$$

Taking the product of two $\mathfrak{J}$ series we find

$$\mathfrak{J}_{[n]}(xt)\times\mathfrak{J}_{[n]}(xt^{-1})=\mathfrak{J}^{p}_{n,0}(x)+\sum_{m=1}^{\infty}(-1)^{m}p^{2m(m+n)}(t^{2m}+t^{-2m})\mathfrak{J}^{p}_{n,m}(x) \quad . \quad . \quad . \quad (52)$$

In a previous paper (*Trans. R.S.E.*, vol. xli. p. 106) it has been shown that

$$J_{[n]}(xt)\times\mathfrak{J}_{[\nu]}(xt)=\sum_{0}^{\infty}(-1)^{r}\frac{\{2n+2\nu+4r\}!}{\{2n+2\nu+2r\}!\{2n+2r\}!\{2\nu+2r\}!\{2r\}!}x^{n+\nu+2r} \quad . \quad . \quad (53)$$

There is a certain similarity of form among the series (50), (51), (53).

Consider now the product of four J functions

$$J_{[n]}(xt)\cdot J_{[n]}(xt)\cdot\mathfrak{J}_{[\nu]}(xt^{-1})\cdot\mathfrak{J}_{[\nu]}(xt^{-1}) \quad . \quad . \quad . \quad . \quad (54)$$

This expression may be written in two other forms. Firstly, by (49) and (52) we write it

$$\left\{J^{p}_{n,0}(x)+\sum_{1}^{\infty}(-1)^{m}(t^{2m}+t^{-2m})J^{p}_{n,m}(x)\right\}\cdot\left\{\mathfrak{J}^{p}_{\nu,0}(x)+\sum(-1)^{m}p^{2m(m+\nu)}(t^{2m}+t^{-2m})\mathfrak{J}^{p}_{\nu,m}(x)\right\} \quad . \quad (55)$$

Secondly, by means of (53) we express (54) as

$$\left\{\sum_{r=0}^{\infty}(-1)^{r}\frac{\{2n+2\nu+4r\}!}{\{2n+2\nu+2r\}!\{2n+2r\}!\{2\nu+2r\}!\{2r\}!}(xt)^{m+\nu+2r}\right\}\cdot$$
$$\left\{\sum_{r=0}^{\infty}(-1)^{r}\frac{\{2n+2\nu+4r\}!}{\{2n+2r+2\nu\}!\{2n+2r\}!\{2\nu+2r\}!\{2r\}!}(xt^{-1})^{n+\nu+2r}\right\} \quad . \quad . \quad . \quad . \quad (56)$$

Equating coefficients of powers of t in (55) and (56), we find from the terms independent of t

$$\sum_{r=0}^{\infty}\left(\frac{\{2n+2\nu+4r\}!}{\{2n+2\nu+2r\}!\{2n+2r\}!\{2\nu+2r\}!\{2r\}!}\right)^{2}x^{2n+2\nu+4r}=J^{p}_{n,0}\mathfrak{J}^{p}_{\nu,0}+2\sum_{m=1}^{\infty}p^{2m(m+\nu)}J^{p}_{n,m}\mathfrak{J}^{p}_{\nu,m} \quad . \quad (57)$$

The terms in the series on the left side of (57) are the squares of the terms in (53). Generally

$$\sum_{r=0}^{\infty}\frac{\{2n+2\nu+4r\}!\{2n+2\nu+4m+4r\}!\,x^{2m+2\nu+2n+4r}}{\{2m+2n+2\nu+2r\}!\{2m+2\nu+2r\}!\{2m+2n+2r\}!\{2n+2\nu+2r\}!\{2m+2r\}!\{2n+2r\}!\{2\nu+2r\}!\{2r\}!}$$
$$=\sum_{r=1}^{\infty}p^{2m(m+\nu)}\left\{J^{p}_{n,m+r}\mathfrak{J}^{p}_{\nu,m}+p^{4mr+2r(r+n)}J^{p}_{n,m}\mathfrak{J}^{p}_{\nu,m+r}\right\} \quad . \quad . \quad . \quad . \quad (58)$$

5.

In this section of the paper I propose to state briefly some results which may be deduced by means of (53), (28), (29),

$$\left.\begin{aligned} &\mathrm{E}_p(ix)\mathrm{E}_p(-ix) \\ &= 1 - \frac{x^2}{1-p^2}(1-p)^2 + \frac{x^4}{1-p^2\cdot 1-p^4}(1-p)^4 - \ldots\ldots \\ &= \mathrm{J}_{[0]}(px)\mathrm{J}_{[0]}(x) + 2p\mathrm{J}_{[1]}(px)\mathrm{J}_{[1]}(x) + 2p^4\mathrm{J}_{[2]}(px)\mathrm{J}_{[2]}(x) + \ldots\ldots \end{aligned}\right\} \quad . \quad . \quad (59)$$

Indicating the nature of the base of each function by an index, we write

$$\mathrm{J}^p_{[0]}\left(\frac{x}{p+1}\right)\cdot\mathfrak{J}^{p^2}_{[0]}\left(\frac{x^2}{p^2+1}\right) + 2\sum_{m=1}^{\infty}\mathfrak{J}^{p^2}_{[m]}\cdot\mathrm{J}^p_{[2m]} = \mathfrak{J}^p_{[0]}\left(\frac{x}{p+1}\right) \quad . \quad . \quad . \quad (60)$$

whence by (29)

$$\left.\begin{aligned} &\mathrm{J}^p_{[0]}\left(\frac{x}{p+1}\right)\mathrm{J}^{p^2}_{[0]}\left(\frac{x^2}{p^2+1}\right) + 2\sum_{m=1}^{\infty}\mathrm{J}^p_{[2m]}\cdot\mathrm{J}^{p^2}_{[m]} = \mathrm{E}_{p^2}\left(\frac{ix^2}{p^2+1}\right)\mathrm{E}_{p^2}\left(-\frac{ix^2}{p^2+1}\right)\mathfrak{J}^p_{[0]}\left(\frac{x}{p+1}\right) \\ &\text{by (29)} \quad = \mathrm{E}_{p^2}\left(\frac{ix^2}{p^2+1}\right)\mathrm{E}_{p^2}\left(-\frac{ix^2}{p^2+1}\right)\mathrm{E}_{\frac{1}{p}}\left(\frac{ix}{p+1}\right)\mathrm{E}_{\frac{1}{p}}\left(-\frac{ix}{p+1}\right)\mathrm{J}^p_{[0]}\left(\frac{x}{p+1}\right) \\ &\text{by (4) and 12)} \quad = \mathrm{E}_{p^4}\left(x^4\frac{1-p^2}{(p^2+1)^3}\right)\mathrm{E}_{\frac{1}{p^2}}\left(x^2\frac{p-1}{(1+p)^3}\right)\mathrm{J}^p_{[0]}\left(\frac{x}{p+1}\right) \end{aligned}\right\} \quad . \quad (61)$$

From (19) we find

$$\frac{1}{(1-x^2t^{-2})(1-x^2t^2)} = \left\{\mathrm{I}_{[0]}\frac{x^2}{1-p} + \sum(t^{2m}+t^{-2m})\mathrm{I}_{[m]}\left(\frac{x^2}{1-p}\right)\right\}$$
$$\left\{I_{[0]}\left(\frac{px^2}{1-p}\right) + \sum(-1)^m p^{m^2}(t^{2m}+t^{-2m})I_{[m]}\left(\frac{px^2}{1-p}\right)\right\} \quad . \; . \; . \quad (62)$$

$$\frac{1}{1-x^4} = \mathrm{I}_{[0]}I_{[0]} + 2\sum(-1)^r p^{r^2}\mathrm{I}_{[r]}I_{[r]} \quad . \quad . \quad . \quad . \quad (63)$$

$$\frac{\mathrm{E}_{p^2}\left(\frac{px^2}{1-p}\right)\mathrm{E}_{p^2}\left(\frac{px^2}{p-1}\right)}{1-x^4} = \mathrm{I}_{[0]}\left(\frac{x^2}{1-p}\right)\mathrm{I}_{[0]}\left(\frac{px^2}{1-p}\right) + 2\sum(-1)^r p^{r^2}\mathrm{I}_{[r]}\left(\frac{x^2}{1-p}\right)\mathrm{I}_{[r]}\left(\frac{px^2}{1-p}\right) \quad . \quad (64)$$

$$\frac{\mathrm{E}_{p^4}\left(x^4\frac{(1+p)^2}{p^4-1}\right)}{1-x^4} = \mathrm{I}_{[0]}\left(\frac{x^2}{1-p}\right)\mathrm{I}_{[0]}\left(\frac{px^2}{1-p}\right) + 2\sum(-1)^r p^{r^2}\mathrm{I}_{[r]}\left(\frac{x^2}{1-p}\right)\mathrm{I}_{[r]}\left(\frac{px^2}{1-p}\right) \quad . \quad (65)$$

$$\frac{1}{1-2x^2\cos 2\theta + x^4} = \left\{\mathrm{I}_{[0]} + 2\cos 2\theta\mathrm{I}_{[1]} + \ldots\right\}\left\{I_{[0]} - 2p\cos 2\theta I_{[1]} + \ldots\right\} \quad . \quad (66)$$

$$\left\{\mathrm{J}_{[1]}\right\}^2 = \frac{[4]}{[2]}\mathrm{J}_{[0]}\mathrm{J}_{[2]} + p^2\frac{[8]}{[4]}\mathrm{J}_{[1]}\mathrm{J}_{[3]} + \ldots\ldots + p^{r(r-1)}\mathrm{J}_{[r-1]}\mathrm{J}_{[r+1]} + \ldots\ldots \quad . \quad . \quad (67)$$

$$\left\{\mathfrak{J}_{[1]}\right\}^2 = \frac{[4]}{[2]}\mathfrak{J}_{[0]}\mathfrak{J}_{[2]} + p^2\frac{[8]}{[4]}\mathfrak{J}_{[0]}\mathfrak{J}_{[3]} + \ldots\ldots + p^{r(r-1)}\mathfrak{J}_{[r+1]}\mathfrak{J}_{[r-1]}\ldots\ldots \quad . \quad . \quad (68)$$

$$\mathrm{E}_p(ix)\mathrm{E}_p(-ix) = \left\{\mathrm{J}_{[0]}\right\}^2 + \frac{[4]}{[2]}\left\{\mathrm{J}_{[1]}\right\}^2 + \ldots\ldots + p^{r(r-1)}\frac{[4r]}{[2r]}\left\{\mathrm{J}_{[r]}\right\}^2 + \ldots\ldots \quad . \quad (69)$$

$$\mathrm{E}_{\frac{1}{p}}(ix)\mathrm{E}_{\frac{1}{p}}(-ix) = \left\{\mathfrak{J}_{[0]}\right\}^2 + \frac{[4]}{[2]}\left\{\mathfrak{J}_{[1]}\right\}^2 + \ldots\ldots + p^{r(r-1)}\frac{[4r]}{[2r]}\left\{\mathfrak{J}_{[r]}\right\}^2 + \ldots \quad . \quad . \quad (70)$$

$$\left\{J_{[0]}\right\}^2 - 2p\left\{J_{[1]}\right\}^2 + 2p^4\left\{J_{[2]}\right\}^2 \quad \ldots\ldots = \left\{\sum(-1)^r \frac{x^{2r}}{(1-p^2)(1-p^4)\ldots(1-p^{2r})}(p-1)^{2r}\right\}$$
$$\left\{\sum \frac{(1+p)(1+p^3)\ldots(1+p^{2r-1})x^{2r}}{(1-p^2)(1-p^4)\ldots(1-p^{2r})}(p-1)^{2r}\right\} \qquad (71)$$

$$\left\{J_{[0]}\right\}^2 - \frac{[4]}{[2]}\left\{J_{[1]}\right\}^2 + \ldots\ldots + p^{r(r-1)}\frac{[4r]}{[2r]}\left\{J_{[r]}\right\}^2 +$$
$$= \left\{\sum(-1)^r \frac{x^{2r}}{(1-p^2)(1-p^4)\ldots(1-p^{2r})}(p-1)^{2r}\right\}$$
$$\left\{\sum \frac{2(1+p^2)(1-p^4)\ldots(1+p^{2r-r})\cdot(1+p^2)(1+p^4)\ldots(1+p^{2r})}{\{(1-p^2)(1-p^4)\ldots(1-p^{2r})\}^2}x^{2r}(p-1)^{2r}\right\} \qquad (72)$$

(Cf. *Trans. R.S.E.*, p. 110 (26).)

$$J_{[0]}(x)J_{[0]}(y) - 2pJ_{[1]}(x)J_{[1]}(y) + 2p^4J_{[2]}(x)J_{[2]}(y) - \ldots\ldots$$
$$= \left\{1 - \frac{y^2}{1-p^2}(p-1)^2 + \frac{y^4}{1-p^2\cdot 1-p^4}(p-1)^2 - \ldots\right\}\left\{1 - \frac{(x+py)^2}{[2]^2} + \frac{(x+py)^2(x+{}^3y)^2}{[2]^2[4]^2} - \ldots\right\} \qquad (73)$$

$$J_{[n]}J_{[-n]} + J_{[-n]}J_{-1+n]} = \frac{1}{x\Gamma_{p^2}([n])\Gamma_{p^2}([1-n])}\left\{[2]J^2_{[0]} + [4]J^2_{[1]} + [8]J^2_{[2]} + \ldots\ldots\right\} \qquad (74)$$

$$E_p(ix)E_p(-ix)\frac{x^{2n-1}}{\{2n\}!\{2n-2\}!} = J_{[n-1]}J_{[n]} + \frac{[4n+2]}{[2]}J_{[n]}J_{[n+1]}$$
$$+ p^{s(s-1)}[4n+4s-2]\frac{[4n+2][4n+4]\ldots\ldots[4n+2s-2]}{[2][4]\ldots[2s]}J_{[m+s]}J_{[m+s-1]} \qquad (75)$$

and a similar form for x^{2n}. (Cf. *Proc. Lond. Math. Soc.*, series 2, vol. iii.)

$$\sum_{r=0}^{\infty}(-1)^r\frac{[4r+2]}{[2]}p^{r(r-1)}J^2_{\left[\frac{2r+1}{2}\right]} = \frac{1}{[2]^2\cdot\Gamma_{p^2}(\frac{1}{2})}\left\{\sum(-1)^r\frac{x^{2r}}{(1-p^2)(1-p^4)\ldots(1-p^{2r})}(p-1)^{2r}\right\}$$
$$\left\{\sum \frac{2(1+p^2)\ldots(1+p^{2r-2})\cdot(1+p)\ldots(1+p^{2r-1})}{\{2r+1\}!}x^{2r-1}\right\} \qquad (76)$$

(Cf. *Proc. Edin. Math. Soc.*, Theorem of LOMMEL, vol. xxii.)

It is plain that great numbers of such theorems may be found and expressed in various forms by means of the transformations belonging to $E_p(x)$, but the examples given above will suffice to illustrate the notation.