A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion.

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Contents.

Prefatory Note.

(This prefatory note is intended merely to point out those features of the article which are likely to be of most interest to the reader who is pressed for time, and who is already familiar with the literature of the subject. The article proper begins with the Introduction.)

The subject-matter of the present article is a new set of postulates for *ordinary Euclidean three-dimensional geometry*. The method employed can be readily extended to Euclidean geometry of more than three dimensions, but is not so readily adapted to the study of projective geometry.

The chief points of difference between the present set of postulates and the well known sets*) given by Pasch, Veronese, Peano, Pieri, Hilbert, Veblen, Schweitzer, and others, are: 1) the use of the solid body instead of the point as an undefined concept; 2) the extreme simplicity of the undefined relation of inclusion; 3) the systematic definitions of the straight line, the plane, and the 3-space, which can be readily extended, if desired, to space of n dimensions; and (4) the attempt to separate the 'existence postulates' from the postulates expressing 'general laws'.

In regard to this last point, a word of explanation may be desirable. By an 'existence postulate' we mean a postulate that demands the existence of some element satisfying certain conditions; as, for example, the proposition that a line passing through a vertex of a triangle and any interior point must intersect the opposite side; or the proposition that through any point outside a given line it is always possible to draw at least one parallel. By a 'general law' we mean a proposition of the form: '*if* such and such points, lines, etc. exist, *then* such and such relations will hold between them'; for example, the proposition that if B is between A and C, and X between A and B, then X is between A and C; or

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*) M. Pasch, Vorlesungen über neuere Geometrie, Leipzig 1882.

G. Veronese, Fondamenti di Geometria a più dimensioni, 1891, translated into German by A. Schepp, Grundzüge der Geometrie, 1894.

G. Peano, I Principii di Geometria, Turin 1889; also Sui fondamenti della geometria, in Rivista di Matematica 4, p. 51—90, 1894.

M. Pieri, Della geometria elementare come sistema ipotetico deduttivo; monographia del punto e del moto, Memorie della Reale Accademia delle Scienze di Torino, (2) 49, p. 173—222, 1899; also Sur la géométrie envisagée comme un système purement logique, Bibliothèque du Congrès international de Philosophie, Paris 1900 3, p. 367—404.

D. Hilbert, Grundlagen der Geometrie, Gauß-Weber Festschrift, 1899; translated into English by E. J. Townsend, The Foundations of Geometry, 1902; third German edition, as vol. 7 of the series called Wissenschaft und Hypothese, (Leipzig 1909). In this third edition, the numbering of the axioms is slightly altered, in view of an article by E. H. Moore, 1902; Axiom II 4 of the original list is now omitted, and what was originally Axiom II 5 is now numbered II 4.

O. Veblen, A system of Axioms for Geometry, Trans. Am. Math. Soc. 5 (1904), p. 343—384. Also, The Foundations of Geometry, in a volume called Monographs on Topics of Modern Mathematics relevant to the Elementary Field, edited by J. W. A. Young, p. 1—51, 1911.

Another set of postulates, based on concepts not so closely connected with the present work, has been recently given by A. R. Schweitzer, A theory of geometrical relations, Am. J. of Math., 31 (1909), p. 365—410.

the proposition that if two distinct lines are parallel to a third line they are parallel to each other. Now in the usual development of the subject, the demonstration of many of the general laws is made to depend upon the use of existence postulates; for example, in proving the simplest laws of order for points on a line, Hilbert and Veblen use a 'triangle transverse' postulate, concerning the points of intersection of the sides of a triangle with a line in the plane. In the present treatment, on the contrary, the attempt has been made to separate the general laws from the existence postulates, and to prove all general laws, as far as possible, without the aid of auxiliarly 'construction lines'. This restriction adds considerably to the difficulty of many of the proofs, but the attempt, though not completely successful, has a certain logical interest; and the lost simplicity of proof can be at once regained, if desired, by transposing the existence postulates to an earlier place in the list.

Among the *definitions*, the most important is the *new definition of linear segment* (Def. 5), for on this definition, and the analogous definitions of triangle and tetrahedron, the whole theory is based. Attention may also be called to the definition of the *mid-point* of a segment (Def. 17), the definition of *congruence* (Def. 21), and to the fact that all *metric properties* are obtained directly in terms of the fundamental concepts, without the intervention of Cayley's 'absolute'. Also, the word 'sphere' may be replaced by 'any convex solid', except in the parts of the paper that deal with congruence.

On the *consistency* of the postulates, see the end of Chapter II, where an interesting geometry of *points of finite size* is exhibited.

In regard to the *independence of the postulates*, the 'general laws' are shown to be independent of each other, and the existence postulates are shown to be independent of each other and of the general laws. By slight changes in wording, it would be easy to secure 'absolute' independence for the combined list of general laws and existence postulates; but such changes would tend to introduce needless artificialities, from which the postulates as they now stand are entirely free.

More important than the question of independence is the proof of the *sufficiency of the postulates to determine a unique type of system;* or, to use a phrase of Veblen's, the proof that the postulates form a *categorical set*. Little attention seems to have been paid to this question except by the present writer in connection with the foundations of algebra*), and

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*) E. V. Huntington, A complete set of postulates for the theory of absolute continuous magnitude, Trans. Am. Math. Soc., 3 (1902), p. 264—279, and later papers. Compare also the forthcoming Lehrbuch der Algebra by A. Loewy.

by Veblen in connection with the foundations of geometry; and yet there appears to be no other way of proving that *all* the propositions of a science are deducible from a given set of postulates, than by showing that the postulates form a 'sufficient' or 'categorical' set. In the present article, the deductions from the postulates are carried just far enough to establish this 'theorem of sufficiency', which forms, in fact, a natural stopping place in any study of 'foundations'.

The previous authors to which the writer is chiefly indebted are Peano, Russell*), Hilbert, Veblen, and Schur**).

The logical symbolism of Peano, although not employed in the paper as prepared for the press, has been of almost indispensable value in working out the details of the demonstrations. Without some such symbolism, it is almost impossible, in work of this sort, to avoid errors.

The term 'propositional function', and the emphasis on the notion of the 'variable' (in the Introduction) are due chiefly to Russell. The convenient notation for the prolongations of a segment, the extensions of a triangle, etc., is taken from Peano. The 'four-point postulate' (Postulate 11) is the first half of a proposition suggested by Schur. The very simple proof of the commutative law of multiplication, by means of the proposition about the three altitudes of a triangle, is also due to Schur. Postulate 14, the last of the postulates on congruence, was suggested by one of the assumptions in Veblen's later list. The construction for drawing a line perpendicular to a given line is due to Hilbert.

On account of the use of the solid body instead of the point as the fundamental variable, most of the pseudo-geometries given in the proofs of independence had to be new constructions. Example E 5, however, is taken from Hilbert.

The writer is also indebted to Mr. P. E. B. Jourdain, who has verified many of the demonstrations.

Introduction. Plan of the article.

This introduction is intended to explain the point of view adopted in the present article, and the nature of the principal results obtained.

Fundamental concepts. We agree to consider a certain set of *postulates* (namely, the postulates stated in Chapter II), involving, besides the sym-

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*) B. Russell, Principles of Mathematics, 1903; A. N. Whitehead and B. Russell, Principia Mathematica, 1, 1911. See also Whitehead's excellent popular Introduction to Mathematics, ('Home University Library', London 1911).

**) F. Schur, Zur Proportionslehre, Math. Ann. 57 (1903), p. 205—208.

bols which are necessary for all logical reasoning, only the following two *variables:*

1) the symbol K, which may mean any *class of elements* $A, B, C, \cdots$; and

2) the symbol R, which may mean any *relation* A R B, between two elements.

These postulates are not definite propositions — that is, they are not in themselves either true or false. Their truth or falsity is a function of the logical interpretation given to the variables K and R, just as the truth or falsity of a conditional equation in algebra is a function of the numerical values given to the variables in such an equation. They may therefore be called *'propositional functions'* (to use a term of Russell's), since they become definite propositions (true or false) only when definite 'values' are given to the variables K and R.

For example, if we give K and R the following values:

1) K = the class of ordinary spheres (including the null spheres);

2) R = the relation of inclusion (so that 'A R B' means 'A inside of B');

then all the postulates will be found to be true; but if K and R have the values assigned to them in any one of the examples given in Chapter IV, as for instance:

1) K = a class comprising nine ordinary numbers, namely: 2, 3, 5, 7, 10, 14, 15, 21, 210;

2) R = the relation 'factor of';

then at least one of the postulates (here Postulate 4) will be found to be false.

Having this set of postulates before us, we agree to consider their *logical consequences*, that is, the theorems which can be deduced from them by the processes of formal logic *without reference to any particular determination of the variables* K *and* R.

These derived theorems, like the original postulates, will be propositional functions, whose truth or falsity depends on the values given to the variables K and R. All that the process of logical deduction tells us, is that *if* the original postulates are true for certain values of the variables, *then* all the derived theorems will be true for the same values. In other words, *if* any given system (K, R) has all the properties stated in the postulates, *then* it will also have all the properties stated in the theorems.

*Definition of abstract geometry.**) The body of theorems thus deduced

*) We confine ourselves here to ordinary Euclidean three-dimensional geometry; sets of postulates for other varieties of geometry may be obtained by suitable modifications of the set here given.

from the given set of postulates constitutes an abstract deductive theory, which, in view of the familiar example already mentioned, may properly be called *abstract geometry;* and any given system (K, R) — for example, the system: K = the class of spheres, R = the relation of inclusion — which satisfies all the postulates may then be called a *concrete geometrical system*, or simply, a *concrete geometry.*

Applied geometry. Abstract geometry, as thus defined, is a purely mathematical theory, which will apply equally well to all 'geometrical systems', wherever such systems may be discovered.

For example, it is conceivable that in the field of Economics, or of Botany, a class K and a relation R might be discovered, which would satisfy all the postulates of Chapter II, and which would therefore form a geometrical system.

It is important to notice, however, that the question whether any given system (K, R) does actually possess the properties enumerated in the postulates, is not (except in the arithmetical case mentioned below) a question of pure mathematics, but rather a matter for observation and experiment. Thus, the abstract theory of geometry as such gives us no information whatever about the nature of perceptual space, any more than it does about the facts of Economics or Botany: all that it tells us is that *if* any system (K, R), wherever it may be found, is rightly judged to possess the properties mentioned in the postulates, *then* it will necessarily possess also the properties mentioned in the theorems.

Furthermore, the only systems (K, R) about which such a judgment can be made with assured accuracy are the systems whose elements are purely numerical, that is to say, purely logical; in all other cases, our observations are only approximate, and in all such cases the conclusions reached by the application of geometric theory — even though the process of inference is perfectly rigorous — are of course no more accurate than the premisses on which they are based.

Consistency of the postulates. The first question to be asked concerning any set of postulates such as that which we are here considering, is this: Does any system exist which satisfies all the postulates? If the existence of any such system can be established, then the postulates are said to be *consistent*, and no theorems deduced from them can ever lead to contradiction.

In the present case, we have already mentioned one system (K, R) as satisfying all our postulates, namely, the system: K = the class of spheres, and R = the relation of inclusion; but if we mean by 'sphere' and 'inclusion' the common notions given by observation of perceptual space, then our judgment that this system satisfies all the postulates can be only approximate, and the system does not provide a satisfactory proof of the consistency of the postulates. If, however, we use 'sphere' and 'inclusion' in the sense in which these terms are used in analytical geometry (see the end of Chapter II), then we have a strictly numerical system, which certainly exists, and which can be shown to satisfy all the postulates with absolute accuracy. The consistency of the postulates is thus completely established.

Separation of general and existence postulates. The list of postulates given in Chapter II will be found to be divided into two groups, the first containing the 'general laws' (Postulates 1—18), and the second containing the 'existence postulates' (Postulates E 1—E 7).

These two groups of postulates play quite distinct rôles in the development of the subject, and the attempt to keep them separate from the start is not without logical interest. (Compare, however, the remarks under Theorem 15.)

Independence of the postulates. A second question to be asked is: Are the postulates independent? that is, are we sure that no one of them is merely a consequence of the rest?

To answer this question, we exhibit, in Chapter IV, a list of 'pseudo-geometries', by which it is shown that the 'general laws' are independent of each other, and also that the 'existence postulates' are independent of each other and of the general laws.

Sufficiency of the postulates to determine a unique type uf system. A third and most important part of our work is to show that any two systems (K, R) which satisfy all the postulates are *formally equivalent,* or *isomorphic,* with respect to the variables K and R. This means that if (K′, R′) and (K″, R″) are any two 'geometrical systems' — that is, any two systems that satisfy all the postulates — then it is possible to set up a one-to-one correspondence between the elements $A', B', C', \cdots$ of K′ and the elements $A'', B'', C'', \cdots$ of K″ in such a way that whenever A' and B' in one system satisfy the relation A' R′ B', then the corresponding elements A'' and B'' in the other system will satisfy the relation A'' R″ B''. By the establishment of this isomorphism, we show that *any theorem involving only the two variables* K *and* R, *which is true in one of the systems will be true in the other also;* hence *all the systems* (K, R) *which satisfy the postulates may be said to belong to a single type.*

With the proof of this theorem, the series of deductions which we draw from the postulates is brought to a natural conclusion; and in view of this theorem, we may then define *abstract geometry* as the *study of the properties of a particular type of system* (K, R), *namely, that type which is completely determined by the Postulates* 1—18 *and* E 1—E 7.

Chapter I.

Definitions.

As explained in the introduction, the only variables that are involved in our set of postulates are the class K and the relation R, and it is theoretically possible to express every theorem of geometry explicitly in terms of these variables.

In order, however, to avoid tedious repetitions of certain combinations of these symbols, it is necessary to replace these combinations by single terms, which are introduced by definition, for the sole purpose of abbreviation.

For convenience of reference, all the definitions that we shall require are collected together in the present chapter. In framing these definitions we have freely used the terminology suggested by the particular interpretation of the variables K and R with which we are most familiar, namely, K = the class of spheres, and R = the relation of inclusion; but it must be constantly borne in mind that the meaning of all words (such as 'point', 'segment', 'line', etc.) which are here defined in terms of K and R, will vary with varying interpretations of K and R, and that as far as the definitions are concerned, K and R may stand for any class and any relation that we please.

(In this connection the list of 'pseudo-geometries' in Chapter IV will be found instructive.)

Spheres, and the relation of inclusion.

Definition 1. If A is an element of the class K, then A shall be called an abstract sphere, or simply a *sphere.*

Definition 2. If $A \mathrm{R} B$, we shall say that the sphere A is *within* the sphere B, or that B *contains* A.

Definition 3. Suppose $A \mathrm{R} B$ and $B \mathrm{R} A$ are both false; two cases can then occur. 1) If there is no sphere X such that $X \mathrm{R} A$ and $X \mathrm{R} B$, then A and B are called *mutually exclusive,* or each *outside* the other; while 2) if there is some such sphere X, then A and B are said to *overlap.*

Points.

Definition 4. If A is a sphere, and if there is no other sphere X such that $X \mathrm{R} A$, then A is called an abstract point, or simply a *point.* That is, a point is any sphere which contains no other sphere within it.

It may be noticed that there is nothing in this definition, or in any of our work, which requires our 'points' to be *small*; for example, a perfectly good geometry is presented by the class of all ordinary spheres whose diameters are not less than one inch; the 'points' of this system are simply the inch-spheres. (Compare the example given at the end of Chapter II.)

Segments, and the straight line.

The following definition is of central importance in the entire theory.

Definition 5. Let A and B be any given points. If X is a point such that every sphere which contains A and B also contains X, then X is said to belong to the *segment* $[AB]$ or $[BA]$.

Fig. 1.

The segment $[AB]$ is thus a class of points, uniquely determined by A and B. The points A and B belong to the segment, and are called its *end-points*. If we exclude the end-points, the class that remains is called the *interior* of the segment $[AB]$, and may be denoted by (AB). If (AB) is the null class, the segment $[AB]$ may be said to be *hollow*.

Definition 6. Besides the class $[AB]$, two other classes, called the *prolongations* of $[AB]$, are determined by the points A and B, according to the following scheme.

If X is a point such that	belongs to the segment	then X is said to belong to a class denoted by	and called the *prolongation* of $[AB]$ beyond	Its *boundary* consists of the point
A	$[BX]$	$[AB']$	A	A
B	$[AX]$	$[BA']$	B	B

(This notation, which is due in its essential elements to Peano, is easily remembered if we observe that $[AB']$ contains A, and $[BA']$ contains B; that is, in each case the letter which is not modified by an accent represents a point which belongs to the class in question.)

Fig. 2.

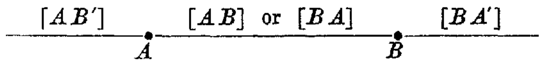
If we exclude from $[AB']$ and $[BA']$ the boundary points A and B, the classes that remain may be called the *interiors* of the prolongations, and may be denoted by (AB') and (BA'), respectively; and if either of these interiors is the null class, the corresponding prolongation may be said to be *hollow*.

The following special terminology will also be found to be convenient.

Definition 7. In a set of two or more classes like $[AB]$, $[AB']$, etc., if no two of the classes have any point in common (unless it be a common boundary point, represented by an unaccented letter that appears explicitly in both symbols), then the classes may be said to form a *simple set of non-overlapping* regions, and the logical sum of such a set may be called a *simple sum.*

We can now define the line AB as a class of points uniquely determined by A and B, as follows.

Definition 8. If A and B are two distinct points, the *line* AB is the class of all points which belong to the segment $[AB]$ or to either of its two prolongations. Three points are said to be *collinear*, if any one of them belongs to the line determined by the other two.



Definition 9. The line AB is said to be divided by the point A into two *half-lines,* or *rays*, one containing the regions $[AB]$ and $[BA']$ in which B occurs without accent, and the other containing the region $[AB']$ in which B' occurs with the accent. Two points of the line are said to be on the *same side* of A or on *opposite sides* of A according as they belong to the same half-line or to different half-lines. In a similar way, the line is said to be divided by the point B.

Triangles, and the plane.

The following definitions, 10—14, for the plane are precisely analogous to the definitions 5—9 for the straight line.

Definition 10. If X is a point such that every sphere which contains A, B, and C, also contains X, then X is said to belong to the *triangle* $[ABC]$.

The triangle includes the points A, B, C which are called its *vertices,* and the segments $[AB]$, $[AC]$, and $[BC]$, which are called its *sides.* The *interior* of the triangle, excluding the points of the sides, is denoted by (ABC). If the class (ABC) is a null class, the triangle is said to be *hollow.*

We next define certain 'extensions' of the triangle, which are to be analogous to the 'prolongations' of a segment. In order to exhibit this analogy in its clearest form, we first observe that the definition of a prolongation of a segment may be worded as follows: 'If X is a point such that one element of the boundary is incident with the figure formed by joining X with the opposite element of the boundary, then X is said to belong to a class denoted by' etc. Now in the case of a triangle, the elements of the boundary are of two kinds, the three vertices, A, B, C, and the three sides, $[AB]$, $[AC]$, $[BC]$; hence the desired definitions will be obtained by associating each vertex with the opposite side, and each side with the opposite vertex, as in the following scheme.

Definition 11.

If X is a point such that	is incident with	then X is said to belong to a class denoted by	This class is called the *extension* of $[ABC]$ beyond the	Its *boundary* consists of the points of
A	$[BCX]$	$[AB'C']$	vertex A	$[AB']$, $[AC']$
B	$[ACX]$	$[BA'C']$	„ B	$[BA']$, $[BC']$
C	$[ABX]$	$[CA'B']$	„ C	$[CA']$, $[CB']$
$[AB]$	$[CX]$	$[ABC']$	side $[AB]$	$[AB]$, $[AC']$, $[BC']$
$[AC]$	$[BX]$	$[ACB']$	„ $[AC]$	$[AC]$, $[AB']$, $[CB']$
$[BC]$	$[AX]$	$[BCA']$	„ $[BC]$	$[BC]$, $[BA']$, $[CA']$

The first three of these classes, $[AB'C']$, $\cdots$, are called the *vertical* extensions of the triangle, and the last three, $[ABC']$, $\cdots$, the *lateral* extensions. The points named as the 'boundary' of each class will themselves belong to that class in any system in which Postulates 1—2 are valid. The *interiors* of the several regions, excluding the points of the boundary, are denoted by $(AB'C')$, $\cdots$, (ABC'), $\cdots$, respectively.

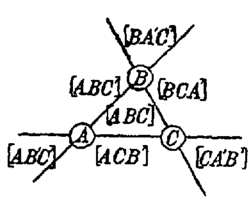


Fig. 3.

By an immediate extension of the terminology adopted in Definition 7, we have:

Definition 12. In a set of two or more classes like $[ABC]$, $[ABC']$, etc., if no two of the classes have any point in common (unless it be a point of a common boundary, represented by letters that appear explicitly in both symbols), then the set of classes may be called a *simple set* and their logical sum a *simple* sum.

For example, $[BA'C']$ and $[BCA']$ have the common boundary $[BA']$; if they have no other point in common, they form a simple set.

We now define the plane ABC as a class of points uniquely determined by A, B, and C, as follows.

Definition 13. If A, B, and C are three points not in the same line, the *plane* ABC is the class of all points that belong to the triangle $[ABC]$ or to any one of its six extensions. Four points are said to be *coplanar*, if any one of them belongs to the plane determined by the other three.

Definition 14. The plane ABC is said to be divided by the line AB into two *half-planes*, one containing the four regions in which C occurs without accent, the other containing the three regions in which C'

occurs with the accent. Similarly, the plane is divided by the line AC and by the line BC.

Before passing to the analogous definitions of a space $ABCD$, it will be convenient to introduce at this point definitions concerning parallel lines and the congruence of segments. On this arrangement, all the definitions required for the special case of *geometry of two dimensions* will be found in consecutive order.

Parallel lines.

Definition 15. Suppose we are dealing with a system (K, R) in which the 'planes' have all the properties demanded by Postulates 6—8. Then if two lines AB and CD lie in the same plane, and have no point in common, they are said to be *parallel*, and we write: $AB \parallel CD$.

To indicate that two lines AB and CD are either parallel or coincident, we shall use the notation $AB \sim CD$.

Definition 16. If $AB \parallel CD$ and $BC \parallel DA$, then the four points A, B, C, D are said to form a *parallelogram*, of which $[AC]$ and $[BD]$ are the *diagonals*.

By the aid of parallel lines, we now define the mid-point of a segment, as follows.

Definition 17. Let $[AB]$ be any given segment. If there is a parallelogram $AXBY$ of which $[AB]$ is one diagonal, and if the other diagonal intersects $[AB]$ in M, then M is called *a* middle point of the segment $[AB]$. If there is only one such point M (as will always be the case in every system in which Postulates 1—11 are valid), then M is called *the mid-point* of $[AB]$, and we write: $M = \text{mid}\, AB$. In this case the segment $[AB]$ is said to be *bisected* at M.

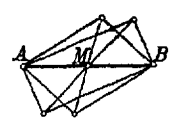


Fig. 4.

The center of a sphere.

In order to define the 'center' of an (abstract) sphere, we first define the points 'on the surface' of a sphere, as a subclass among the points that are within the sphere.

Definition 18. If A and B are within a sphere S, and if all points of the prolongations (AB') and (BA') are outside of S, then the segment $[AB]$ is called a *chord* of the sphere S.

Definition 19. If A is an end-point of any chord of a sphere S, then A is said to lie *on the surface* of the sphere.

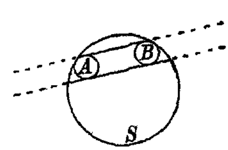


Fig. 5.

Definition 20. If O is a point within a sphere S, and if every pair of chords which intersect at O

are the diagonals of a parallelogram, then O is called the *center* of the sphere.

Any chord through the center is called a *diameter*, and is bisected at the center. Either half of a diameter is called a *radius.*

Congruence of segments.

By the aid of the two notions of the mid-point of a segment and the center of a sphere, we can now define the relation of congruence between two segments, as follows.

Definition 21. Two segments $[AB]$ and $[CD]$ are called *congruent* — in symbols, $AB \equiv CD$ — when and only when one of the following conditions is satisfied:

1) If the two segments $[AB]$ and $[CD]$ are on the same line, then we must have either $[AB] = [CD]$, or mid AC = mid BD, or mid AD = mid BC; and if they lie on parallel lines, then they must be opposite sides of a parallelogram.

2) If they have a common end point (or a common mid-point), but do not lie on the same line, then they must be radii (or diameters) of the same sphere.

3) If they do not lie on the same line or on parallel lines, and do not have a common end-point or a common mid-point, then there must be two segments $[OX]$ and $[OY]$ which are congruent to the given segments according to 1), and congruent to each other according to 2).

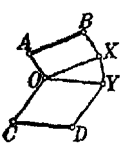


Fig. 6.

According to this definition, all the radii of a given sphere are obviously congruent; hence, all the points on the surface of a sphere may be said to be *equidistant* from the center.

It will be noticed that congruence according to 1) is connected with the idea of *translation*, and that according to 2) with the idea of *rotation* — these two ideas being necessarily involved in any adequate definition of congruence.

Perpendicular lines.

Definition 22. If the diagonals of a parallelogram are congruent, the parallelogram is called a *rectangle*, and a triangle which forms half of a rectangle is called a *right triangle.*

Definition 23. Two lines that intersect at a point O are called *perpendicular*, if every segment which joins a point of one line with a point of the other is the diagonal of a rectangle with one vertex at O.

The number line.

The following definitions will be useful when we come to introduce coordinates into our system.

We select any fixed line OU as a special line to which reference will be made in future operations, and we call this line the *number line*, the point O the *zero* point, and the point U the *unit* point.

It is to be understood that any line will answer the purpose of the number line; but when once chosen it must remain fixed during the course of any particular investigation.

Definition 24. If A and B are any two points on the number line OU, and if X is another point on that line such that mid OX = mid AB, then X is called the *sum* of A and B, and we write $X = A + B$.

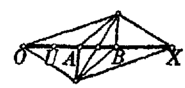


Fig. 7.

Definition 25. Let A and B be any two points on the number line OU, and let P be any convenient point not on that line. If a line through B parallel to UP meets OP in Q, and if a line through Q parallel to PA meets OU in Y, and if this point Y is independent of the particular choice of the auxiliary point P, then Y is called the *product* of A and B; and we write $Y = A \times B$. In particular, we write $A \times A = A^2$.

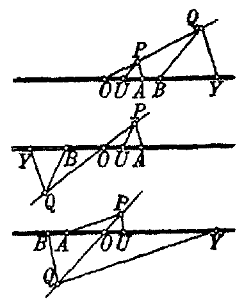


Fig. 8.

Definition 26. If A and B are any two points on the number line OU, and if (according to a readily understood meaning) the direction from A to B is the same as the direction from O to U, then we say that A *precedes* B, and write: $A < B$.

A point X on the number line is called *negative* or *positive* according as it precedes or follows the zero point.

Tetrahedra, and the space.

We now return to the Definitions 5—9 for the straight line, and Definitions 10—14 for the plane, and proceed to carry the analogy of these definitions one step further, into three dimensions. The following definitions 27—31 do not involve in any way the notions introduced in definitions 15—26.

Definition 27. If X is a point such that every sphere which contains A, B, C, and D, also contains X, then X is said to belong to the *tetrahedron* $[ABCD]$.

The tetrahedron includes the *vertices* A, B, C, D, the *edges* $[AB]$, $[AC]$, $[AD]$, $[BC]$, $[BD]$, $[CD]$, and the *faces* $[ABC]$, $[ABD]$, $[ACD]$, and

$[BCD]$, which form its *boundary*. The notation $(ABCD)$ is used as before to denote the *interior* of the class $[ABCD]$, exclusive of the points on the faces.

In order to follow the analogy explained in connection with Definition 11, we notice that the boundary of a tetrahedron consists of 14 elements, namely: four vertices, six edges, and four faces. By associating each of these elements with its 'opposite' element, we obtain the 14 'extensions' of a tetrahedron, as follows.

Definition 28.

If X is a point such that	is incident with	then X is said to belong to a class denoted by	This class is called one of the	Its *boundary* consists of the points of
A	$[BCDX]$	$[AB'C'D']$	4 *vertical* extensions of the tetrahedron	$[AB'C']$, $[AB'D']$, $[AC'D']$
. . .	. . .	. . .		. . .
. . .	. . .	. . .		. . .
. . .	. . .	. . .		. . .
$[AB]$	$[CDX]$	$[ABC'D']$	6 *edgewise* extensions of the tetrahedron	$[AC'D']$, $[BC'D']$, $[ABC']$, $[ABD']$
. . .	. . .	. . .		. . .
. . .	. . .	. . .		. . .
. . .	. . .	. . .		. . .
. . .	. . .	. . .		. . .
. . .	. . .	. . .		. . .
$[ABC]$	$[DX]$	$[ABCD']$	4 *facial* extensions of the tetrahedron	$[ABD']$, $[ACD']$, $[BCD']$
. . .	. . .	. . .		. . .
. . .	. . .	. . .		. . .
. . .	. . .	. . .		. . .

The points named as the 'boundary' of each class will themselves belong to that class in any system in which Postulates 1—2 are valid. The notations $(AB'C'D'), \cdots, (ABC'D'), \cdots, (ABCD'), \cdots$ are used as before to denote the *interiors* of the several regions.

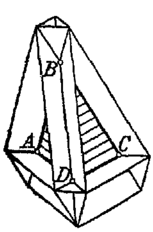


Fig. 9.

Definition 29. A *simple set* of non-overlapping regions is a set having properties analogous to those given in Definition 12.

Definition 30. If A, B, C, D are four points not in the same plane, the *space* $ABCD$ is the class of all points which belong to the tetrahedron $[ABCD]$ or to any of its fourteen extensions.

Definition 31. A space $ABCD$ is said to be divided into two *halfspaces* by each of the planes ABC, ABD, ACD, and BCD. (Compare Definition 14.)

Parallel planes and lines.

Definition 32. A line and a plane, or two planes, are said to be *parallel* if they belong to the same space and have no point in common.

All these definitions 1—32 are expressible directly in terms of the fundamental variables K and R.

Chapter II.

The Postulates.

In this chapter we enumerate the postulates which form the subject of discussion in the present article. Postulates 1—18 are 'general laws'; Postulates E1—E7 are 'existence postulates'.

All the postulates are expressible in terms of the two fundamental variables K and R, by Definitions 1—32.

General laws for spheres and points (see Defs. 1—4).

Postulate 1. Let A, B, C be any (abstract) spheres. If A is within B and B within C, then A is within C.

Postulate 2. If A is within B, then A and B are distinct.

Postulate 3. a) If the class of spheres which contain the point A is the same as the class of spheres which contain the point B, then $A = B$. b) If the class of points within a sphere S is the same as the class of points within a sphere T, then $S = T$.

General laws for the straight line (see Defs. 5—9).

Postulate 4. If X is a point of the segment $[AB]$, then $[AB]$ is the 'simple sum' of the two segments $[AX]$ and $[BX]$.

Postulate 5. If two lines have two distinct points in common, they coincide.

General laws for the plane (see Defs. 10—14).

Postulate 6. If X is a point of the triangle $[ABC]$, then $[ABC]$ is the 'simple sum' of the three triangles $[ABX]$, $[ACX]$, and $[BCX]$.

Postulate 7. If the segment $[XY]$ intersects the segment $[AB]$, then the triangles $[ABX]$ and $[ABY]$ have no point in common except the points of $[AB]$.

Postulate 8. If two planes have three non-collinear points in common, they coincide.

General laws for parallel lines (see Defs. 15—17).

Postulate 9. If two lines are parallel to a third line, they are either parallel or coincident.

Postulate 10. If AB and CD are parallel lines, then no one of the four points A, B, C, D lies within the triangle formed by the other three.

Postulate 11. ('Four-point postulate.') Let A, B, C, D be any set of four points, no three of which are collinear, and A', B', C', D' any other set of four points, no three of which are collinear; and consider the two sets of six lines,

$$AB, AC, AD, BC, BD, CD \text{ and } A'B', A'C', A'D', B'C', B'D', C'D',$$

which these points determine. If the first five lines of one set, taken in order, are parallel to (or coincident with) the first five lines of the other set, taken in the same order, then the remaining sixth line of the first set will be parallel to (or coincident with) the remaining sixth line of the other set. That is, if $AB \sim A'B'$, $AC \sim A'C'$, $AD \sim A'D'$, $BC \sim B'D'$ and $BD \sim B'D'$, then also $CD \sim C'D'$.

This postulate was suggested by a remark of Schur's (loc. cit.) and takes the place of the special form of Desargues' Theorem used by Hilbert.

General laws for congruence (see Defs. 18—21).

Postulate 12. If $AB \equiv CD$ and $CD \equiv EF$, then $AB \equiv EF$.

Postulate 13. If the surfaces of two concentric spheres are cut by one radius in A and X, and by another radius in B and Y, then $[AX] \equiv [BY]$.

In other words, the portions of two radii intercepted between the surfaces of two concentric spheres are congruent.

Postulate 14. Let A, B, C, X be four points of which the first three are collinear, and let A', B', C', X' be another set of four points of which the first three are collinear; and consider the two sets of six segments determined by these points. Then if

$$AB \equiv A'B',\ AC \equiv A'C',\ BC \equiv B'C',\ AX \equiv A'X', \text{ and } BX \equiv B'X',$$

we shall always have also $CX \equiv C'X'$.

This postulate was suggested by Assumption 11 in Veblen's latest list (1911); loc. cit.

General laws for space (see Defs. 27—32).

Postulate 15. If X is a point of the tetrahedron $[ABCD]$, then $[ABCD]$ is the 'simple sum' of the four tetrahedra $[ABCX]$, $[ABDX]$, $[ACDX]$, and $[BCDX]$.

Postulate 16. If the segment $[XY]$ intersects the triangle $[ABC]$, then the tetrahedra $[ABCX]$ and $[ABCY]$ have no point in common except the points of $[ABC]$.

These postulates 15 and 16 are precisely analogous to Postulates 6 and 7 for the plane. Instead of the exact analogue of Postulate 8, however, we choose the following stronger postulate, which limits the system to three dimensions.

Postulate 17. If $ABCD$ is a space, then every point belongs to this space.

The following postulate is analogous to Postulate 10.

Postulate 18. If a line XY is parallel to a plane ABC, then no one of the five points A, B, C, X, Y belongs to the tetrahedron formed by the other four.

Existence postulates (see Defs. 1—32).

Having thus completed the list of 'general laws', we now give the 'existence postulates'.

Postulate E1. There are, in the class K, at least two distinct points.

Postulate E2. If AB is a line, there is a point X not on that line.

Postulate E3. If AB is a line, and C a point not on that line, then there is a point X such that CX is parallel to AB.

A system in which this Postulate E3 is satisfied may be called a system in which parallel lines may be freely drawn. In general, it is only in such systems that the definitions relating to congruence (Defs. 18—23) have any meaning.

Postulate E4. If $[AB]$ is any segment in a system in which parallels can be freely drawn, then on any half line OP there is a point X such that the segment $[OX] \equiv [AB]$.

That is, any given segment can be 'laid off' on any given half-line.

Postulate E5. If S_1, S_2, S_3, $\cdots$ is an infinite sequence of spheres, each of which lies within the preceding one, then there is a point X which lies within them all.

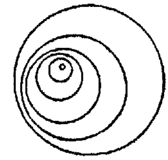


Fig. 10.

This is a simple modification of Dedekind's postulate concerning classes of points on a line.

The following postulate is made necessary by the fact that we have taken the solid sphere instead of the point as our fundamental variable.

Postulate E6. If any sphere has a center, then every sphere has a center (see Defs. 18—20), provided, of course, that it is not itself a point.

Finally, to give the system three dimensions we must have:

Postulate E7. If ABC is a plane, there is at least one point not in that plane.

As we shall show in Chapter III, *these Postulates* 1—18 *and* E1—E7, *are sufficient to determine completely the abstract theory of ordinary Euclidean three-dimensional geometry.*

To obtain a corresponding set of postulates for *two dimensions*, we have simply to *omit Postulates* 15—18, *and replace Postulate* E7 *by its negative*; in this case the most natural interpretation of 'abstract sphere' would be *'circle'* instead of 'sphere'.

Consistency of the postulates.

In order to give a rigorous proof of the consistency of these postulates, we must construct, *out of purely numerical materials*, a system (K, R) which will satisfy them all.

To do this, let $S(a, b, c, r)$ denote the class of all triads of real numbers x, y, z, which satisfy the equation

$$(x-a)^2+(y-b)^2+(z-c)^2 \leqq r^2,$$

where a, b, c, r are real numbers, *and r is not less than a certain fixed number g* (positive or zero).

We take as our class K the totality of all such S's, and we define the relation R between any two of these S's by agreeing that

$$S(a', b', c', r') \text{ R } S(a'', b'', c'', r'')$$

when and only when $r' \neq r''$ and every triad x, y, z which satisfies the relation

$$(x-a')^2+(y-b')^2+(z-c')^2 \leqq r'^2$$

satisfies also the relation

$$(x-a'')^2+(y-b'')^2+(z-c'')^2 \leqq r''^2.$$

In this system (K, R), the 'points' are the elements of the form $S(a, b, c, g)$; and it is not hard to show that all the postulates are satisfied

In the language of analytic geometry, this system is simply the *system of spheres whose radii are not less than g*, where, in the most familiar case, $g=0$. It is interesting to observe, however, that any other value of g is equally legitimate, so that we may speak of *a perfectly rigorous geometry in which the 'points', like the school-master's chalk-marks on the blackboard, are of definite, finite size, and the 'lines' and 'planes' of definite, finite thickness.*

Chapter III.

Theorems.

In this chapter we give such theorems as are necessary for the proof of the sufficiency of the postulates to determine a unique type of system. To avoid interruption in reading, the proofs of the theorems (when any proofs are needed) are given separately in the Appendix. After each theorem, the postulates on which the proof depends are stated in parentheses.

Spheres and points.

Theorem 1. (By 1, 2.) If a sphere A is within a sphere B, then B is not within A.

Theorem 2. (By 1, 2.) If A and B are any two distinct spheres, then one and only one of the following three relations will hold: 1) one of the spheres is within the other; or 2) they are mutually exclusive; or 3) they overlap.

Theorem 3. (By 1, 2.) If A is a point, and S is any sphere which is not a point, then A is either within S or outside of S; that is, the case of 'overlapping' cannot occur. Further, if A and B are two distinct points, they are mutually exclusive.

The straight line.

Theorem 4. (By 1—3.) If the endpoints of a segment coincide, the segment contains no other points; that is, $[AA] = A$.

Theorem 5. (By 1—4.) The segment $[AB]$ and its two prolongations $[AB']$ and $[BA']$ form a 'simple set of non-overlapping regions', so that the line AB is the 'simple sum' of these three regions.

Theorem 6. (By 1—5.) If X and Y are two distinct points of a line AB, then the line XY will contain A and B, and hence be identical with the line AB.

Theorem 7. (By 1—5.) If three points are on a line AB, then one of them is on the segment formed by the other two.

This theorem expresses the necessary and sufficient condition that three points shall be collinear.

From these postulates and theorems, all the 'general laws' of order for points on a straight line can be deduced; that is, in so far as points exist at all on a given line, they will have the proper relations of order.

The plane.

Theorem 8. (By 1—3.) If the vertices of a triangle coincide, the triangle contains no other points; that is $[AAA] = A$.

[Theorem 9a. (By 1—6.) The triangle $[ABC]$ and its three *vertical* extensions form a simple set of non-overlapping regions.]

Theorem 9. (By 1—7.) The triangle $[ABC]$ and all its six extensions, $[AB'C']$, $\cdots$, $[ABC']$, $\cdots$, form a simple set of non-overlapping regions, so that the plane ABC is the simple sum of these seven regions. (Proof by considering six possible cases.)

Theorem 10. (By 1—8.) If X, Y, and Z are three non-collinear points of a plane ABC, then the plane XYZ will contain A, B, and C, and hence be identical with the plane ABC.

Theorem 11. (By 1—8.) If four points are in a plane ABC, then either: 1) one of them belongs to the triangle formed by the other three; or 2) the segment joining two of them intersects the segment formed by the remaining two.

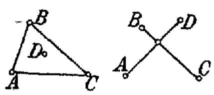


Fig. 11.

This theorem expresses the necessary and sufficient condition that four points shall be coplanar. (Peano, loc. cit.)

Theorem 12. (By 1—8.) If X and Y are two distinct points in a plane ABC, then every point of the line XY will belong to the plane ABC.

From these postulates 1—8 the following further theorems can be deduced, without the aid of any of the existence postulates.

Theorem 13. (By 1—8.) In the triangle $[ABC]$, if X is on the side opposite A, and Y on the side opposite B, then the segments $[AX]$ and $[BY]$ will have a common point.

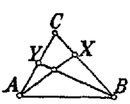


Fig. 12.

To establish this theorem 13, we consider the point X in relation to the triangle $[ABY]$, and show, by a process of exclusion, that X must lie in the lateral extension $[BYA']$; hence, by Definition 11, the segments $[AX]$ and $[BY]$ must intersect.

Theorem 14. (By 1—8.) If X and Y are points in the interior of any one of the seven compartments of a plane ABC, then every point of the segment $[XY]$ is in the interior of the same compartment.

Parallel lines.

Theorem 15. (By 1—9.) Through a given point, there is not more than one line parallel to a given line.

This theorem shows that Postulate 9, although stated in the form of a 'general law', nevertheless enables us to infer the 'existence' of points of intersection of many lines. It might perhaps be called an existence postulate in disguise. The same remark applies to several other postulates, as, for example, to Postulate 10, which gives us the following theorem.

Theorem 16. (By 1—10.) If two parallel lines are given, and if A, B are any two points of one, and X, Y any two points of the other, then either $[AX]$ and $[BY]$, or else $[AY]$ and $[BX]$, will have a common point.

The following theorems depend upon the 'four-point postulate'.

Theorem 17. (By 1—11.) A segment cannot have more than one middle point. Hence, the diagonals of every parallelogram bisect each other.

Theorem 18. (By 1—11.) If mid AX = mid AY, then $X = Y$.

That is, if one end of a segment is changed while the other end is held fast, then the middle point will be changed.

The proof of this theorem 18 may be made to depend on the following lemma.

Lemma for Theorem 18. (By 1—11.) If a given plane contains at least one parallelogram $ABCD$, with its middle point M, and at least one other point E distinct from A, B, C, D, and M, then every segment in the plane will be the diagonal of a parallelogram in the plane, and hence will have a middle point; and further, throughout the plane, a parallel can always be drawn to any given line through any given point not on that line.

All the preceding theorems have been obtained from Postulates 1—11, without the use of any of the existence postulates. If now we add Postulates E1—E3, we have the following theorems concerning existences.

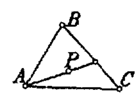


Fig. 13.

Theorem 19. (By 1—11, E1—E3.) If a point P is in the interior of a triangle $[ABC]$, then the line AP intersects the opposite side (BC).

This Theorem 19, together with Theorem 13, were used as postulates by Peano.

Theorem 20. (By 1—11, E1—E3.) If $[ABC]$ is a triangle, and E is on (BC), and D on the prolongation (BA'), then the line DE will meet the side (AC) in a point F.

This is the 'triangle transverse axiom' in the form used by Veblen.

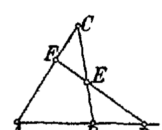


Fig. 14.

Theorem 21. (By 1—11, E1—E3.) If $[AB]$ is a segment, there are points on the prolongations (AB') and (BA').

Theorem 22. (By 1—11, E1—E3.) The following constructions are always possible:

1) to draw a line parallel to a given line through any point not on that line;

2) to bisect a given segment;

3) to extend a given segment to double its length.

Congruence, and perpendicularity.

Theorem 23. (By 1—12.) The relation of congruence is reflexive, symmetrical, and transitive.

Theorem 24. (By 1—12.) If X is on $[OA]$, and $[OX] \equiv [OA]$, then $X = A$.

That is, on a given half-line OA, not more than one segment $[OX]$ can be laid off congruent to a given segment.

Theorem 25. (By 1—13.) If A, B, C are collinear, and if A', B', C' are collinear and in the same order as A, B, C, then whenever $AB \equiv A'B'$ and $AC \equiv A'C'$, we shall always have $BC \equiv B'C'$.

Briefly stated, this theorem tells us that the sums of congruent segments are congruent.

Theorem 26. (By 1—14.) Let two given lines meet in O. If *any* segment $[XY]$, joining a point of one of the lines with a point of the other, is the diagonal of a rectangle having one vertex at O, then *every* such segment will have the same property, and the lines will be perpendicular.

Theorem 27. (By 1—14.) If two lines in the same plane are perpendicular to the same line, then they are either parallel or coincident. Also, if, in any plane, a line is perpendicular to one of two parallel lines, it will be perpendicular to the other also.

Theorem 28. (By 1—14.) If X is in a line perpendicular to $[AB]$ at its middle point M, then X is equidistant from A and B.

Theorem 29. (By 1—14.) If X is equidistant from A and B, and $M =$ mid AB, then XM is perpendicular to AB.

These two theorems give us the important properties of the isosceles triangle.

Theorem 30. (By 1—14.) If through the vertices of a triangle lines are drawn perpendicular to the opposite sides, these three lines will have a common point. (Orthocenter.)

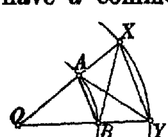


Fig. 15.

Theorem 31. (By 1—14.) If the surfaces of two concentric spheres are cut by one radius in A and X, and by another radius in B and Y, then the cross segments, $[AY]$ and $[BX]$, will be congruent, and the chords $[AB]$ and $[XY]$ will be parallel.

By the aid of the first four existence postulates, we have also the following theorem.

Theorem 32. (By 1—14, E1—E4.) The following constructions are always possible:

1) to draw a sphere with any given point O as center, and any given segment $[OA]$ as radius;

2) to find the point of intersection of the surface of a sphere with any line through its center;

3) to 'lay off' on any given half-line a segment congruent to any given segment;

4) to draw a perpendicular to any given line through any given point.

The number line.

Theorem 33. (By 1—5.) Let A, B, C be any points on the number line (see Def. 26). Then we have:

1) If A and B are distinct, then either $A < B$ or $B < A$.

2) If $A < B$, then A and B are distinct.

3) If $A < B$ and $B < C$, then $A < C$.

The points on the number line therefore form a series*), or ordered class, with respect to the relation $<$.

Theorem 34. (By 1—11.) Let A, B, C be any points on the number line. Then, in so far as the sums in question exist (see Def. 24), we shall have the following laws of addition**):

1) $(A+B)+C=A+(B+C)$. (Associative law.)

2) $A+B=B+A$. (Commutative law.)

3) If $A+X=A+Y$, then $X=Y$.

4) If A is not zero, then $A+A+\cdots+A$ is not zero.

We shall also have the following law connecting $+$ and $<$:

5) If $X<Y$, then $A+X<A+Y$.

Theorem 35. (By 1—11.) The product of two points on the number line is independent of the position of the auxiliary point used in the construction (see Def. 25).

Theorem 36. (By 1—11.) Let A, B, C be any points on the number

*) G. Vailati, Sui principî fondamentali della Geometria della retta, Rivista di Matematica, 2 (1892), p. 71—75; E. V. Huntington, The Continuum as a Type of Order, reprinted from the Annals of Mathematics, 1905 (Publication Office of Harvard University).

**) E. V. Huntington, The Fundamental Laws of Addition and Multiplication in Elementary Algebra, reprinted from the Annals of Mathematics, 1906 (Publication Office of Harvard University). Also, The Fundamental Propositions of Algebra, in the volume of Mathematical Monographs edited by J. W. A. Young, p. 150—207, 1911.

line, and let it be possible to draw parallel lines at pleasure (so that products of such points can always be obtained). Then we have the following laws of multiplication*):

1) $(A \times B) \times C = A \times (B \times C)$. (Associative law.)

2) $A \times (B + C) = A \times B + A \times C$, and

3) $(B + C) \times A = B \times A + C \times A$. (Distributive laws.)

4) If $A \times X = A \times Y$, or $X \times A = Y \times A$, and A not zero, then $X = Y$.

Also we have the following law connecting $\times$ and $<$:

5) If $X < Y$ and A positive, then $A \times X < A \times Y$ and $X \times A < Y \times A$.

Theorem 37. (By 1—14.) If A and B are points on the number line, as in Theorem 36, we have the commutative law for multiplication:

$$A \times B = B \times A.$$

Of these theorems, 33—37, the first four are obtained by the aid of Postulates 1—11; for the last, however, we assume also the postulates of congruence.

By availing ourselves of the first three existence postulates, which give us the existence of a plane in which parallel lines can be drawn at pleasure, we obtain also the following theorems.

Theorem 38. (By 1—11, E1—E3.) If A and B are any two points of the number line, then

1) their *sum* $A + B$, and

2) their *product* $A \times B$, will exist, and be uniquely determined by A and B. Further, there exists

3) the zero point 0 for which $0 + 0 = 0$, and

4) the *unit* point U for which $U \times U = U$. Also,

5) for every point A, there is an *opposite* point X, such that $A + X = 0$, and

6) for every point A which is different from zero, there is a *reciprocal* point Y, such that $A \times Y = U$.

If, finally, we add the postulate of continuity, we have:

Theorem 39. (By 1—14, E1—E5.) The points on the number line form a 'system of real numbers', with respect to the operations $+$ and $\times$, and the relation $<$.**)

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*) See preceding footnote.

**) For a set of independent postulates for real numbers, see E. V. Huntington, A Set of Postulates for ordinary Complex Algebra, Trans. Am. Math. Soc. 6 (1905), p. 209—229, especially p. 219—220.

Coordinates in the plane.

In view of the foregoing theorems, it is easy to see how we may introduce coordinates in the plane.

Take two intersecting lines as axes, with origin at O, and on each of these axes reproduce the 'number line' by laying off segments according to Theorem 32, 3. Then by drawing parallels to the axes, we see that to every point in the plane, there correspond two points, one on each axis; and to every pair of points on the axes, there corresponds one point in the plane.

As far as plane geometry is concerned, therefore, it remains only to prove that the equation of a circle (that is, the class of points which are common to the surface of a sphere and a plane) has the usual form, when the axes are rectangular. This is at once evident from the following form of the *Pythagorean Theorem:*

Theorem 40. (By 1—14, E1—E4.) If the sides and hypotenuse of a right triangle are 'laid off' along the positive half of the 'number line', the sum of the squares of the points representing the two sides will be equal to the square of the point representing the hypotenuse.

For the case of plane geometry, in which all the points in the system are confined to a single plane, the theorem of isomorphism mentioned in the Introduction is then readily established.

The Space.

The following theorems for space are analogous to Theorems 8—12 for the plane.

Theorem 41. (By 1—3.) If the vertices of a tetrahedron coincide, the tetrahedron contains no other points; that is, $[AAAA] = A$.

[Theorem 42a. (By 1—8, 15.) The tetrahedron $[ABCD]$ and its four *vertical* extensions form a 'simple set'.]

Theorem 42. (By 1—8, 15—16.) The tetrahedron $[ABCD]$ and all its fourteen extensions form a simple set of non-overlapping regions, so that the space $ABCD$ is the simple sum of these fifteen regions.

(Proof by considering thirteen possible cases.)

Theorem 43. (By 1—8, 15—17.) If X, Y, Z, W are four non-coplanar points of a space $ABCD$, then the space $XYZW$ will be identical with the space $ABCD$.

Theorem 44. (By 1—8, 15—17). If five points are in a space $ABCD$, either: 1) one of them belongs to the tetrahedron formed by the other four; or 2) the segment joining two of them intersects the triangle formed by the remaining three.

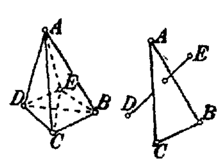

Fig. 16.

This theorem expresses the necessary and sufficient condition that five points shall be cospacial.

Theorem 45. (By 1—8, 15—17.) If X and Y are two distinct points in a space $ABCD$, then every point of the line XY belongs to that space; and if X, Y, Z are three non-collinear points in a space $ABCD$, then every point in the plane XYZ will belong to that space.

These theorems have been stated at length, because they still remain valid if we replace Postulate 17 by the weaker postulate exactly analogous to Postulate 8, as we should do if we wished to extend the theory to the geometry of more than three dimensions.

By the aid of Postulate 18 we obtain the following theorem:

Theorem 46. (By 1—11, 15—18, E1—E3.) If two planes have a point in common, then they have another point, and hence a line, in common.

From this point on, the usual theorems concerning lines and planes in space can be obtained by the usual methods of proof, and the introduction of coordinates in space presents no further difficulty. Further, if the coordinate axes are rectangular, the equation of a spherical surface is obtained in the usual way (by the aid of Theorem 40), and finally, by Postulate E6, we can assign a definite spherical surface to every sphere, and a definite sphere to every spherical surface. Hence:

Theorem 47. (By 1—18, E1—E7.) If two systems (K, R) satisfy all the postulates of Chapter II, they will be isomorphic with respect to the variables K and R, in the sense explained in the introduction.

The proof consists simply in showing that every system (K, R) which satisfies all the postulates is isomorphic with the special numerical system exhibited at the end of Chapter II.

With the proof of this theorem, the main part of our work is completed. It remains to consider the independence of the postulates, as is done in the following chapter.

Chapter IV.

Independence of the postulates.

In this chapter we exhibit a list of 'pseudo-geometries' which establish the independence of the postulates of Chapter II to the following extent: 1) the 'general laws', Postulates 1—18, are independent of each other, as shown by examples 1—18; and 2) the 'existence postulates', Postulates E1—E7, are independent of each other and of the general laws, as shown by examples E1—E7.

Each of these pseudo-geometries is a system (K, R) in which the variables K and R have such values that all but one of the postulates in question are satisfied, while the remaining one is not satisfied.

Example 1. Let K be a class consisting of three persons, a man A, his father B, and his grandfather C; and let R be the relation 'son of'. Then $A \mathrm{R} B$ and $B \mathrm{R} C$ are true, while $A \mathrm{R} C$ is false, so that Postulate 1 is not satisfied. In this system, since there is only one 'point' (see Def. 4), namely A, all the other general laws, 2—18, are satisfied 'vacuously'; that is, the conditions under which these postulates become effective do not occur.

Example 2. Let K be a class of any number of ordinary spheres, and let '$A \mathrm{R} B$' mean: 'A within or equal to B'. Here Postulate 2 fails, while Postulate 1 holds. Postulates 3—18 are satisfied vacuously, since there are no 'points' in the system.

Example 3a. Let K be a class of three ordinary spheres, A, B, and S, of which A and B are separate, but both included within S; and let R be the ordinary relation of inclusion. Here all the general laws 1—18, are satisfied, except Postulate 3a.

Example 3b. Let K be a class including all ordinary spheres, which we shall mark red, together with a duplicate set of spheres, in the same space, which we shall mark blue. Let R be the usual relation of inclusion, except in the case of two spheres which occupy the same position; in this case, we agree that the red sphere shall always be 'within' the blue.

This system satisfies all the general laws 1—18 except 3b. Incidentally, it also satisfies all the existence postulates E1—E7.

Example 4. Let K be a class consisting of nine numbers: A, B, X, Y; AX, AY, BX, BY; $ABXY$; where A, B, X, Y are any (distinct) primes, $AX =$ the product of A and X, $AY =$ the product of A and Y, etc.; let R be the relation 'factor of'. For example: $A \mathrm{R} AX$; $AX \mathrm{R} ABXY$; etc.

In this system there are four 'points', namely, the prime numbers A, B, X, Y. The 'segments' $[AB]$, $[AX]$, etc. are 'classes of points' defined according to Definition 5; thus, $[AB] = A, B, X, Y$; $[AX] = A, X$; etc. Postulate 4 fails, since the segment $[AB]$ contains X and Y, while the segment $[XY]$ contains A and B. All the other general laws are satisfied (most of them vacuously).

Example 5. Let K be a class of eleven numbers: A, B, X, Y; AB, AX, BY, XY; ABX, ABY; $ABXY$; where A, B, X, Y are primes $AB = A \times B$, etc., as in Example 4; and let R be the relation 'factor of', as before.

Here $[AY]$ includes B, and $[BX]$ includes A, while all the other segments are 'hollow'. Postulate 5 fails, since the 'line AB' (Def. 8) includes the points A, B, X, and Y, while the 'line XY' includes only X and Y.

Example 6. Let K be a class of 18 numbers: A, B, C, X, Y; AY, BY, CY, AB, AC, BC; ABX, ACX, BCX; $ABXY$, $ACXY$, $BCXY$; $ABCXY$; and let R = 'factor of'. (Same notation as in Example 4.)

Here every segment is 'hollow', so that no three points are collinear. The triangle $[ABC]$ (Def. 10) contains X and Y, and each of the triangles $[ABY]$, $[ACY]$, and $[BCY]$ contains X. All the other triangles contain no points other than their vertices. Postulate 6 fails, since X and Y belong to $[ABC]$, but Y does not belong to any of the triangles $[ABX]$, $[ACX]$, $[BCX]$. Since there is no plane which has all the properties 6, 7, 8, the definition of parallel lines is without meaning in this system, and the postulates concerning parallel lines are inoperative.

Example 7. Let K be a class of 19 numbers: A, B, C, P, Q, X; ABP, ACQ, APX, AQX, BCP, BCQ, BQX, CPX; $ABCPQ$, $ABPQX$, $ACPQX$, $BCPQX$; $ABCPQX$; and let R = 'factor of'. (Same notation as in Example 4.)

Here $[AB]$ and $[CX]$ contain P, and $[AC]$ and $[BX]$ contain Q. All other segments are hollow. Postulate 7 fails, since $[AB]$ and $[CX]$ intersect in P, while $[ABC]$ and $[ABX]$ contain the common point Q, which does not belong to their common boundary $[AB]$.

Example 8. Let K be a class comprising the following numbers:

1) six primes: A, B, C, X, Y, Z;
2) every number which is the product of two of these primes, as AB, AC, etc.;
3) every number which is the product of three of these primes, *except* ABZ, ACY, and BCX;
4) the following products of four primes: $ABCX$, $ABCY$, $ABCZ$, $ABXY$, $ACXZ$, $AXYZ$, $BCYZ$, $BXYZ$, $CXYZ$;
5) the following products of five primes: $ABCXY$, $ABCXZ$, $ABCYZ$; and
6) the number $ABCXYZ$;

and let R be the relation 'factor of', as in Example 4.

In this system every segment is hollow. The triangle $[BCX]$ contains A; $[ACY]$ contains B; $[ABZ]$ contains C; all the other triangles contain no points except their vertices. Postulate 8 fails, since the 'plane ABC' contains all the points A, B, C, X, Y, Z, while the 'plane XYZ' contains only X, Y, and Z.

Example 9. Let K be the class of all ordinary circles (including the null circles) whose centers lie within a given convex closed curve in an ordinary plane; and let R be the ordinary relation of inclusion.

In this system the 'abstract points' are the ordinary points of the plane and Postulates 1—8 are clearly satisfied. Postulate 9 fails.

Example 10. Let K be a class of eleven numbers: A, B, C, D; BC, BD, CD; ABC, ABD, ACD; $ABCD$; where the notation has the same meaning as in Example 4; and let R be the relation 'factor of'.

Here all the segments are hollow, and all the triangles are hollow except $[BCD]$, which contains A. According to the definition of parallel lines, we have $AB \parallel CD$ and $BC \parallel AD$; but Postulate 10 fails, since A belongs to the triangle $[BCD]$. All the other general laws are satisfied.

Example 11. To construct the example for Postulate 11, consider first an ordinary plane, with all its circles, points, and lines, and suppose the interior of a part of this plane — say a square $ABCD$ — is stretched or deformed in such a way that all the points within the square are crowded towards one corner C, without altering their relations of order, or causing any break in continuity. In this deformed plane, by a circle or line we mean, of course, a figure which was a true circle or line before the deformation.

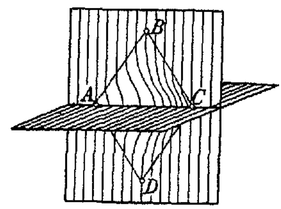

Fig. 17.

Secondly, consider another plane, containing a square $APCQ$ without any deformation, and place the two planes so that they intersect along the line AC.

Then as our class K we take all the circles that lie in these two planes, and as our relation R, the ordinary relation of inclusion.

In this system, Postulates 1—10 are clearly satisfied, but not Postulate 11. To see that Postulate 11 is not true, let PQ meet AC in M, while the deformed line BD meets AC in a different point N; then B, P, D, Q cannot be coplanar; but if Postulate 11 were true, we should have a right to infer, from a consideration of the 'four-points' $ACBP$ and $CADQ$, that $BP \parallel DQ$. All the other general laws, 12—18, are satisfied (many of them vacuously).

Example 12. Let K be the class of all ellipses (including the null ellipses) whose centers lie in a plane; and let R be the usual relation of inclusion.

To see that Postulate 12 fails in this system, consider two concentric ellipses that intersect at four real points, and let a line through the center O cut one of the ellipses in A and the other in B. Then $[OA]$ and $[OB]$ are both congruent to the common radii of the ellipses, but are not congruent to each other.

Example 13. Consider a system of concentric ellipses, no two of which intersect, such that through every point of the plane one ellipse will pass; and suppose that the ellipses are not all similar. Let K be the class of all ellipses (including the null ellipses) whose centers lie in a given plane, provided each of the ellipses can be obtained from one of the ellipses of the system just considered by a motion of translation. Let R be the usual relation of inclusion.

Here Postulate 13 is not satisfied. All the other general laws are satisfied.

Example 14. Let K be the class of all ellipses (including the null ellipses) whose centers lie in a plane, and which are obtainable by a motion of translation from a given system of concentric, *similar* and similarly placed ellipses.

Here Postulate 13 is satisfied, but not Postulate 14.

Example 15. Let K be a class comprising the following numbers (where the single letters denote primes, as in Example 4):

1) six primes: A, B, C, D, X, Y;
2) every product of two of these primes, as AB, AC, etc.;
3) every product of three of these primes, as ABC, ABD, etc.;
4) every product of four of these primes, *except* $ABCD$, $ABCY$, $ABDY$, $ACDY$ and $BCDY$;
5) the following products of five primes: $ABCXY$, $ABDXY$, $ACDXY$, $BCDXY$;
6) the number $ABCDXY$;

and let R be the relation 'factor of'.

Here no segment contains more than its end-points, and no triangle contains more than its vertices. The tetrahedron $[ABCD]$ contains X and Y; and each of the tetrahedra $[ABCY]$, $[ABDY]$, $[ACDY]$, and $[BCDY]$ contains X. Postulates 4—14 are satisfied vacuously; Postulate 15 fails.

Example 16. Let K be a class comprising the following numbers (for the notation, compare Example 4):

1) seven primes: A, B, C, D, X, Y, Z;
2) every product of two of these primes, *except* XY;
3) every product of three of these primes, *except* ABC, AXY, BXY, CXY and XYZ;
4) every product of four of these primes, *except* the following: $ABCX$, $ABCY$, $ABCZ$, $ABDX$, $ABXY$, $ACDY$, $ACXY$, $AXYZ$, $BCXY$, $BXYZ$, and $CXYZ$;
5) the following products of five primes: $ABCDZ$, $ABDXZ$, $ABDYZ$, $ACDXZ$, $ACDYZ$, $ADXYZ$, $BCDXY$, $BCDXZ$, $BCDYZ$, $BDXYZ$, $CDXYZ$;

6) the following products of six primes: $ABCDXZ$, $ABCDYZ$, $ABDXYZ$, $ACDXYZ$, $BCDXYZ$;

7) the number $ABCDXYZ$;

and let R be the relation 'factor of'.

Here all segments are hollow except $[XY]$, which contains D, and all triangles are hollow, except $[ABC]$, which contains D. All the tetrahedra are hollow, except $[ABDX]$, $[ABCX]$, $[ABXY]$, $[ACDY]$, $[ABCY]$ and $[ACXY]$, each of which contains Z. Postulate 16 fails, since Z lies on both sides of the plane ABC.

Example 17. Let K be a class containing the five prime numbers A, B, C, D, E, and every number which is the product of two, three, four, or five of these primes; and let R be the relation 'factor of'.

Here no segment contains more than its end-points, and no triangle or tetrahedron contains more than its vertices. Postulate 17 fails, since the point E does not belong to the 'space $ABCD$'.

Example 18. Let K be the same class as in Example 17, omitting the number $ABCD$; and let R be the relation 'factor of'.

Here the 'line AE' is 'parallel' to the 'plane BCD'; but E is within the tetrahedron $[ABCD]$, so that Postulate 18 clearly fails.

By these examples 1—18 we have thus proved the independence of all the 'general laws'. We now give a similar set of examples for the 'existence postulates'.

Example E1. Let K be a class consisting of a single sphere.

Example E2. Let K be the class of all spheres (including null spheres) whose centers lie along a given straight line; and let R = inclusion.

Example E3. Let K be a class including the following numbers (compare, for notation, example 4):

A; B; C; D; AB, etc.; ABC, etc., including all the combinations of the letters; and let R = 'factor of'.

This system contains only four points, which are situated at the vertices of a tetrahedron.

Example E4. Let K be the class of all egg-shaped convex solids (including the null solids, or points); and let R be the usual relation of inclusion.

In this system no 'sphere' has a center in the sense of the definition, so that the idea of congruence between segments that have a common end point has no meaning, and the general laws concerning congruence are inoperative.

Example E5. Let K be the class of all spheres (including null spheres) such that their radii and the coordinates of their centers can be expressed in terms of 1 by a finite number of applications of the four algebraic operations of addition, subtraction, multiplication, division, and the fifth operation $\sqrt{1+x^2}$, where x is any number already obtained by means of these five operations. (Hilbert.)

Example E6. Let K be the class of all spheres in space (including null spheres), together with a number of detached spheres to which the relation R (inclusion) does not apply.

These detached spheres will then have no centers, so that Postulate E6 is not satisfied.

Example E7. Let K be the class of all spheres (including null spheres) whose centers lie in a given plane; and let R be the usual relation of inclusion.

Here Postulates 1—18 are all satisfied (15—18 vacuously), and also E1—E6; but E7 is not.

The proofs of independence are thus complete.

Appendix.

In this appendix we give the demonstrations of such of the theorems in Chapter III as are likely to present any difficulty to the reader.

Proof of Theorem 17. (A segment cannot have more than one mid-point.)

Let $AXBY$ and $APBQ$ be two parallelograms on the same diagonal $[AB]$, and let $[AB]$ be met by $[XY]$ in M and by $[PQ]$ in N.

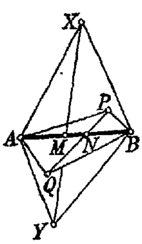

Fig. 18.

By considering the 'four-points' $ABPX$ and $BAQY$ (unless X is on the line AP), we find $PX \sim QY$, by Postulate 11. Then by considering $AQNY$ and $BPNX$ (unless Y is on the line QN), we find $NY \sim NX$. Hence N is on the line XY, and hence $N = M$.

In the special case when X is on the line AP, consider $BPNX$ and $AQNY$. In the other special case, when Y on the line QN, consider $ABPX$ and $BAQY$.

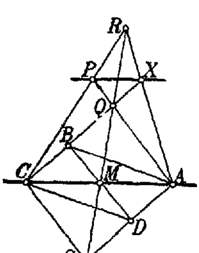

Fig. 19.

Proof of Lemma for Theorem 18. We divide the proof of this lemma into two steps.

1) Through any point P in the plane, not coincident with B or D, and not on the diagonal AC, a line can be drawn parallel to AC. (Steiner.)

Proof. Let PA meet BC in Q, let MQ meet PC in R and AD in S, and let CQ meet AR in X. Then, by Postulate 11, from $MBAQ \sim MDCS$, we find $AQ \sim CS$, and hence from $RQPX \sim RSCA$ we have $PX \parallel AC$.

But if $MQ \parallel PC$, the proof requires modification, thus: Let AD meet PC in T and MQ in U, and let TM meet CQ in Y. Then from $MDTC \sim MBYA$ we find $TC \sim YA$, and from $MTUC \sim MYQA$ we find $UC \sim QA$; hence, from $TUCM \sim TAPY$, we have $PY \parallel CM$, that is, $PY \parallel CA$.

If PA does not meet BC, then PC will meet BA, and we have merely to interchange C and A in the proof.

2) To any given line in the plane of the parallelogram, a parallel can be drawn through any given point P not on that line.

Proof. Let the given line meet the diagonals AC and BD in E and F, respectively. Through E and F, by 1), draw parallels to BD and AC, meeting in G. Then $MEGF$ will be a parallelogram with the given line EF as one of its diagonals. Then through P draw a line parallel to EF by 1).

The demonstration of this lemma would of course be much simpler if we chose to avail ourselves of the existence postulates E1—E3.

Proof of Theorem 26. Let $OAPB$ be a rectangle, X any point on OA, and Y any point on OB. The conditions of Lemma 18 are clearly satisfied. First, let $Y = B$. Through X draw a parallel to OB, meeting BP in Q. Then $OXQB$ is a parallelogram. By applying Postulate 14 to O, A, X, B and B, P, Q, O, we find $XB \equiv QO$. Therefore $OXQB$ is a rectangle. Secondly, if Y is any point on line OB, treat Y in relation to $OBQX$ just as X was treated in relation to $OAPB$.

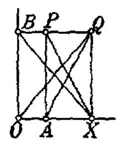

Fig. 20.

Proof of Theorem 30. Through the vertices of the given triangle, draw lines parallel to the opposite sides, which is clearly possible by the Lemma to Theorem 18. The altitudes of the given triangle are then the perpendicular bisectors of the sides of the new triangle, and therefore meet in a point, by Theorems 28 and 29.

Proof of Theorem 32. Part 3) is simply Postulate E4, and parts 1) and 2) are immediate implications therefrom. For part 4), we may use the following construction, given by Hilbert (loc. cit.). Let A, B, C be points on the given line, such that $AB \equiv AC$, and let P be the given point, outside that line. On any lines through A in the plane ABP, lay off AD and AE, congruent to AB. Then BDC and BEC will be right triangles. Let BD meet CE in F, and let BE meet CD in H. Then FH will be perpendicular to AB, by Theorem 30 (on the intersection of the three altitudes of a triangle). Then through P draw a line parallel to FH, which will be the required perpendicular.

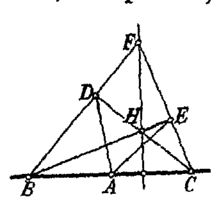

Fig. 21.

Proof of Theorem 34, 1. (Associative law of addition.) Let $A+B=X$, $B+C=Y$. By definition, there are two points, P and R, such that $PORX$ and $PARB$ are parallelograms; also P and S such that $POSY$ and $PBSC$ are parallelograms. Take T so that $PATY$ is a parallelogram. We are to prove that $PXTC$ is a parallelogram. Now we can show that RS and RT are both $\parallel$ to the given line, so that R, S, T are collinear. Then from

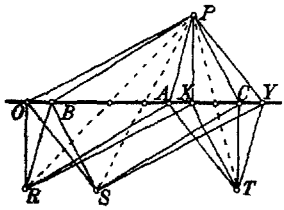

Fig. 22.

$$RXBS \sim YSTX,$$

we find $BS \parallel TX$, and from $ASCT \sim SARO$ we find $CT \parallel RO$. Hence, $TX \parallel CP$ and $CT \parallel PX$.

Proof of Theorem 35. (The product of two points is independent of the auxiliary point P.) We have A and B on the number line OU, and two points P and P', not on that line. By Def. 25, take Q on OP and Y on OU so that $BQ \parallel UP$ and $QY \parallel PA$. Also take Q' on OP', so that $BQ' \parallel UP'$; then, to prove $Q'Y \parallel P'A$, apply the four-point postulate to $OUPP' \sim OBQQ'$, and to $BQQ'Y \sim UPP'A$.

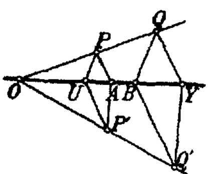

Fig. 23.

Proof of Theorem 36, 1. (Associative law of multiplication.) By Def. 25, we have A, B, C, X, Y, Z on the number line OU, where $X=A\times B$, $Y=B\times C$ and $Z=X\times C$; and P, Q, R, S on another line through O. We are to prove $Z=A\times Y$. That is, we have

$$UP \parallel BQ \parallel CR \parallel YS, \quad PA \parallel QX,$$
$$PB \parallel RY, \quad PX \parallel RZ;$$

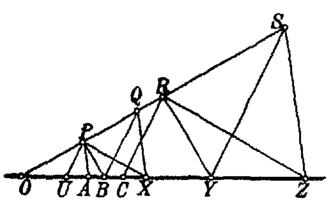

Fig. 24.

and we are to prove $PA \parallel SZ$. This follows by Postulate 11 from

$$BPQX \sim YRSZ.$$

Proof of Theorem 36, 2. (First distributive law for multiplication: $A\times(B+C)=A\times B+A\times C$.) We readily establish first the following lemmas: (1) If $M=\text{mid } OA$ and $N=\text{mid } OB$, then $MN \parallel AB$; and (2) If $AB \parallel XY$ and $M=\text{mid } AX$ and $N=\text{mid } BY$, then

$$MN \parallel AB \parallel XY.$$

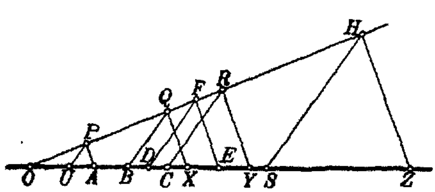

Fig. 25.

In the figure for the theorem, on the line OU we have A, B, C, X, Y,

S, Z, D, E, where $X = A \times B$, $Y = A \times C$, $S = B + C$, $Z = X + Y$, $D = \text{mid } BC = \text{mid } OS$, $E = \text{mid } XY = \text{mid } OZ$; and on the line OP we have Q, R, F and H where $UP \parallel BQ \parallel CR$, $PA \parallel QX \parallel RY$, $F = \text{mid } QR$, and H is such that $F = \text{mid } OH$. Then by lemma 2, $DF \parallel BQ \parallel UP$, and $FE \parallel QX \parallel PA$; and hence, by lemma 1, $SH \parallel DF \parallel UP$, and $HZ \parallel FE \parallel PA$. Therefore $Z = A \times S$.

Proof of Theorem 36, 3. (Second distributive law for multiplication: $(B+C) \times A = B \times A + C \times A$.) We have on the line OU, the points A, B, C, X, Y, S, Z, where $X = B \times A$, $Y = C \times A$, $S = B + C$, $Z = X + Y$, and on another line through O, the points P and Q, so that $UP \parallel AQ$, $PB \parallel QX$, $PC \parallel QY$. We are to prove $PS \parallel QZ$. Since $S = B + C$, there is an R such that $PB \parallel CR$, $PC \parallel BR$, $OP \parallel RS$; and since $Z = X + Y$, there is a T such that $QX \parallel YT$, $QY \parallel XT$, $OQ \parallel TZ$. Then from $BCPR \sim XYQT$, we have $PR \parallel QT$; from $CPOR \sim YQOT$ we have $OR \parallel OT$; and from

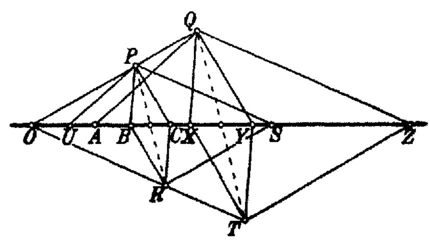

Fig. 36.

$$ORPS \sim OTQZ$$

we have $PS \parallel QZ$. That is, $Z = S \times A$.

Proof of Theorem 37. (Commutative law of multiplication.) By Th. 35, we may take the auxiliary point P so that $OP \perp OU$. (It is not necessary that $OP \equiv OU$.) We have on the line OU the points A, B, and $X = A \times B$; and on the line OP we have Q and R such that $UP \parallel BQ \parallel AR$ and $PA \parallel QX$. We are to prove that $X = B \times A$, that is, that $PB \parallel RX$. Following the method of Schur (loc. cit.), draw through A a line perpendicular to PB, meeting the line OP in S. Then in the triangle PAS, $PB \perp SA$, and $AB \perp PS$; hence, by Th. 30, $SB \perp PA$. But $PA \parallel QX$. Therefore $SB \perp QX$. In triangle SXQ, we have $SB \perp QX$ and $XB \perp QS$; hence $QB \perp SX$. But $QB \parallel UP \parallel RA$. Therefore $RA \perp SX$. In triangle RXS, we have $RA \perp SX$ and $XA \perp RS$; hence $SA \perp RX$. But SA is also $\perp$ to PB. Therefore RX and PB must be parallel, by Th. 27.

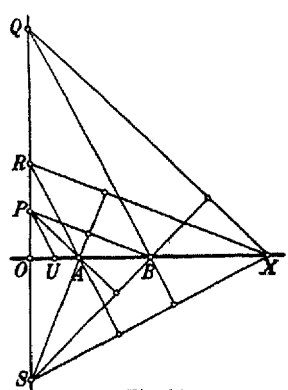

Fig. 37.

Proof of Theorem 40. (Pythagorean Theorem.) Let $OAPB$ be a rectangle, with mid-point at C. On the line OA take $[OX] \equiv [OP]$ and on the line OB take $[OY] \equiv [OP]$. On OP take $[OE] \equiv [OA]$ and

$[OF] \equiv [OB]$. Through A draw a parallel to XE, meeting OP in S; and through B draw a parallel to YF meeting OP in T. By Th. 31, $EA \parallel PX$ and $FB \parallel PY$, so that if OP is the number line, OS is the square of OE and OT is the square of OF, while the square of OP is OP. We wish to prove that OP is the sum of OS and OT, that is, that C, which is the mid-point of OP, is also the mid-point of TS.

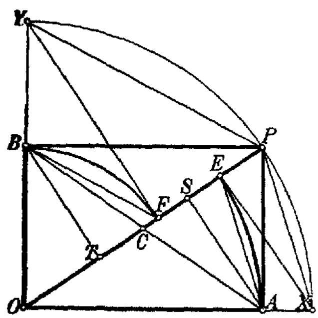

Fig. 28.

By Th. 31, $XE \equiv AP$, so that the triangles OAP and OEX are congruent (in the sense that the three sides of one are congruent respectively to the three sides of the other). But $PA \parallel OA$; therefore $XE \perp OE$. Similarly, $YF \equiv BP$, so that the triangles OBP and OFY are congruent; therefore $YF \perp OF$. Then by Th. 27, $XE \parallel YF$; that is, $BT \parallel AS$. Therefore, since $C = \text{mid } AB$, $ASBT$ is a parallelogram, and $C = \text{mid } TS$.

The extension to the case of any unit other than OP is effected immediately by simple algebra.

Proof of Theorem 46. (If two planes have a point in common, then they have a line in common.) Let the two planes be ABC and CDX, and consider the point X, which must lie in one of the fifteen regions of the space $ABCD$.

If X is in $[ABCD]$, then DX meets ABC, by Postulate 18.
If X is in $[AB'C'D']$, A is in $[BCDX]$, and BA meets CDX.
If X is in $[BA'C'D']$, B is in $[ACDX]$, and AB meets CDX.
If X is in $[DA'B'C']$, D is in $[ABCX]$, and DX meets ABC.
If X is in $[ABC'D']$, then AB meets CDX, by Def. 28.
If X is in $[ABCD']$, then DX meets ABC, by Def. 28.

If X is in $[ACB'D']$, AC meets BDX in E, and BE meets DX (Th. 13).
If X is in $[BCA'D']$, BC meets ADX in F, and AF meets DX.
If X is in $[CDA'B']$, CD meets ABX in G, and XG meets AB.
If X is in $[ABDC']$, CX meets ABD in H, and DH meets AB.
If X is in $[CA'B'D']$, then C is in $[ABDX]$, CX meets ABD in J, by Post. 18, and DJ meets AB by Th. 13.

There remain four cases to consider.
If X is in $[ADB'C']$, AD meets BCX in K, and XK meets BC in L.

Then DX and AL are coplanar, and will meet unless parallel.
If X is in $[BDA'C']$, BD meets ACX in M, and XM meets AC in N.
Then DX and BN are coplanar, and will meet unless parallel.
If X is in $[ACDB']$, ACD meets BX in P, and DP meets AC in Q.
Then DX and BQ are coplanar, and will meet unless parallel.
If X is in $[BCDA']$, BCD meets AX in R, and DR meets BC in S.
Then DX and AS are coplanar, and will meet unless parallel.

Hence we see that the only cases in which the existence of a second point of intersection is not immediately established are the cases in which DX is parallel to a line through A or B in the plane ABC. In these cases, draw through C a line parallel to the line in question in the plane ABC; the line so drawn will be parallel to DX also, and hence will lie in the plane CDX.