International J.Math. Combin. Vol.4(2017), 103-109

On Terminal Hosoya Polynomial of Some Thorn Graphs

Harishchandra S.Ramane, Gouramma A.Gudodagi and Raju B.Jummannaver

(Department of Mathematics, Karnatak University, Dharwad - 580003, India)

E-mail: hsramane@yahoo.com, gouri.gudodagi@gmail.com, rajesh.rbj065@gmail.com

Abstract: The terminal Hosoya polynomial of a graph G is defined as $TH(G,\lambda) = \sum\limits_{k\geq 1} d_T(G,k)\lambda^k$ is the number of pairs of pendant vertices of G that are at distance k. In this paper we obtain the terminal Hosoya polynomial for caterpillers, thorn stars and thorn rings. These results generalizes the existing results.

Key Words: Terminal Hosoya polynomial, thorn graphs, thorn trees, thorn stars, thorn rings.

AMS(2010): 05C12.

§1. Introduction

Let G be a connected graph with a vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$, where $|V(G)| = n$ and $|E(G)| = m$. The *degree* of a vertex v in G is the number of edges incident to it and denoted by $deg_G(v)$. If $deg_G(v) = 1$, then v is called a *pendent vertex* or a *terminal vertex*. The *distance* between the vertices v_i and v_j in G is equal to the length of the shortest path joining them and is denoted by $d(v_i, v_j|G)$.

The *Wiener index* $W = W(G)$ of a graph G is defined as the sum of the distances between all pairs of vertices of G, that is

$$W = W(G) = \sum_{1\leq i<j\leq n} d(u_i, v_j|G).$$

This molecular structure descriptor was put forward by Harold Wiener [29] in 1947. Details on its chemical applications and mathematical properties can be found in [5, 12, 21, 28].

The Hosoya polynomial of a graph was introduced in Hosoya's seminal paper [16] in 1988 and received a lot of attention afterwards. The polynomial was later independently introduced and considered by Sagan et al. [22] under the name Wiener polynomial of a graph. Both names are still used for the polynomial but the term Hosoya polynomial is nowadays used by the majority of researchers. The main advantage of the Hosoya polynomial is that it contains a wealth of information about distance based graph invariants. For instance, knowing the Hosoya polynomial of a graph, it is straight forward to determine the Wiener index of a graph as the first derivative of the polynomial at the point $\lambda = 1$. Cash [2] noticed that the hyper-Wiener

[1]Received December 8, 2016, Accepted November 21, 2017.

index can be obtained from the Hosoya polynomial in a similar simple manner.

Estrada et al. [6] studied the chemical applications of Hosoya polynomial. The *Hosoya polynomial* of a graph is a distance based polynomial introduced by Hosoya [15] in 1988 under the name Wiener polynomial. However today it is called the Hosoya polynomial [8, 11, 17, 18, 23, 27]. For a connected graph G, the *Hosoya polynomial* denoted by $H(G,\lambda)$ is defined as

$$H(G,\lambda) = \sum_{k\geq 1} d(G,k)\lambda^k = \sum_{1\leq i<j\leq n} \lambda^{d(v_i,v_j|G)}. \tag{1.1}$$

where $d(G,k)$ is the number of pairs of vertices of G that are at distance k and λ is the parameter.

The Hosoya polynomial has been obtained for trees, composite graphs, benzenoid graphs, tori, zig-zag open-ended nano-tubes, certain graph decorations, armchair open-ended nanotubes, zigzag polyhex nanotorus, nanotubes, pentachains, polyphenyl chains, the circumcoronene series, Fibonacci and Lucas cubes, Hanoi graphs, and so forth. These can be found in [4].

Recently the *terminal Wiener index* $TW(G)$ was put forward by Gutman et al. [10]. The *terminal Wiener index* $TW(G)$ of a connected graph G is defined as the sum of the distances between all pairs of its pendant vertices. Thus if $V_T(G) = v_1, v_2, \ldots, v_k$ is the number of pendant vertices of G, then

$$TW(G) = \sum_{1\leq i<j\leq k} d(v_i,v_j|G).$$

The recent work on terminal Wiener index can be found in [3, 9, 14, 20, 24]. In analogy of (1.1), the *terminal Hosoya polynomial* $TH(G,\lambda)$ was put forward by Narayankar et al. [19] and is defined as follows: if $v_1, v_2, \ldots, v_k$ are the pendant vertices of G, then

$$TH(G,\lambda) = \sum_{k\geq 1} d_T(G,k)\lambda^k = \sum_{1\leq i<j\leq n} \lambda^{d(v_i,v_j|G)},$$

where $d_T(G,k)$ is the number of pairs of pendant vertices of the graph G that are at distance k. It is easy to check that

$$TW(G) = \frac{d}{d\lambda}(TH(G,\lambda))|_{\lambda=1}.$$

In [19], the terminal Hosoya polynomial of thorn graph is obtained. In this paper we generalize the results obtained in [19].

§2. Terminal Hosoya Polynomial of Thorn Graphs

Definition 2.1 *Let G be a connected n-vertex graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. The thorn graph $G_P = G(p_1, p_2, \cdots, p_n : k)$ is the graph obtained by attaching p_i paths of length k to the vertex v_i for $i = 1, 2, \cdots, n$ of a graph G. The p_i paths of length k attached to the vertex v_i will be called the thorns of v_i.*

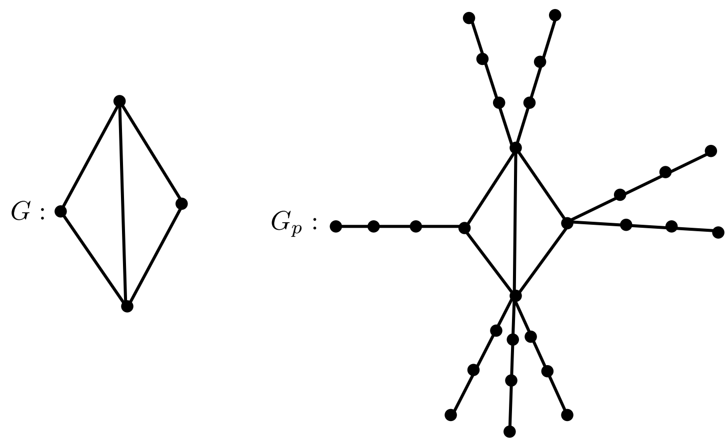

Fig. 1.

A thorn graph $Gp = G(2,1,3,2 : 3)$ obtained from G by attaching paths of length 3 is shown in Fig.1. Notice that the concept of thorny graph was introduced by Gutman [7] and eventually found a variety of applications [1, 25, 26, 27].

Theorem 2.2 *For a thorn graph $G_P = G(p_1, p_2, \ldots, p_n : k)$, the terminal Hosoya polynomial is*

$$TH(G_P, \lambda) = \sum_{i=1}^{n} \binom{p_i}{2} \lambda^{2k} + \sum_{1 \le i < j \le n} p_i p_j \lambda^{2k + d(v_i, v_j | G)}. \tag{2.1}$$

Proof Consider p_i path of length k attached to a vertex v_i, $i = 1, 2, \cdots, n$. Each of these are at distance $2k$. Thus for each v_i, there are $\binom{p_i}{2}$ pairs of vertices which are distance $2k$. This leads to the first term of (2.1).

For the second term of (2.1), consider p_i thorns $v_1^i, v_2^i, \cdots, v_{p_i}^i$ attached to the vertex v_i and p_j thorns $v_1^j, v_2^j, \cdots, v_{p_j}^j$ attached to the vertex v_j of G, $i \ne j$. In G_P,

$$d(v_m^i, v_l^j | G_P) = 2k + d(v_, v_j | G), \quad m = 1, 2, \cdots, p_i \;\text{ and }\; l = 1, 2, \cdots, p_j.$$

Since there are $p_i \times p_j$ pairs of paths of length k of such kind, their contribution to $TH(G_P, \lambda)$ is equal to $p_i p_j \lambda^{2k + d(v_i, v_j | G)}$, $i \ne j$. This leads to the second term of (2.1). $\square$

Corollary 2.3 *Let G be a connected graph with n vertices. If $p_i = p > 0$, $i = 1, 2, \cdots, n$. Then*

$$TH(G_P, \lambda) = \frac{np(p-1)}{2} \lambda^{2k} + p^2 \lambda^{2k} \sum_{1 \le i < j \le n} \lambda^{d(v, v_j | G)}. \tag{2.2}$$

Corollary 2.4 *Let G be a complete graph on n vertices. If $p_i = p > 0$, $i = 1, 2, \cdots, n$. Then*

$$TH(G_P, \lambda) = \frac{np(p-1)}{2} \lambda^{2k} + \frac{p^2 n(n-1)}{2} \lambda^{2k+1}.$$

Proof If G is a complete graph then $d(v_, v_j | G) = 1$ for all $v_i, v_j \in V(G)$, $i \ne j$. Therefore from (2.2)

$$\begin{aligned} TH(G_P,\lambda) &= \frac{np(p-1)}{2}\lambda^{2k} + p^2\lambda^{2k}\sum_{1\le i<j\le n}\lambda \\ &= \frac{np(p-1)}{2}\lambda^{2k} + \frac{p^2 n(n-1)}{2}\lambda^{2k+1}. \end{aligned}$$

This completes the proof. □

Corollary 2.5 *Let G be a connected graph with n vertices and m edges. If $diam(G)\le 2$ and $p_i = p > 0$, $i = 1,2,\ldots,n$. Then*

$$TH(G_P,\lambda) = \frac{np(p-1)}{2}\lambda^{2k} + p^2\lambda^{2k+1}m + \left(\frac{n(n-1)}{2} - m\right)p^2\lambda^{2k+2}.$$

Proof Since $diam(G)\le 2$, there are m pairs of vertices at distance 1 and $\binom{n}{2} - m$ pairs of vertices are at distance 2 in G. Therefore from (2.2)

$$\begin{aligned} TH(G_P,\lambda) &= \frac{np(p-1)}{2}\lambda^{2k} + p^2\lambda^{2k}\left[\sum_{m}\lambda + \sum_{\binom{n}{2}-m}\lambda^2\right] \\ &= \frac{n(p-1)}{2}\lambda^{2k} + p^2\lambda^{2k}\left[m\lambda + \left(\frac{n(n-1)}{2} - m\right)\lambda^2\right] \\ &= \frac{np(p-1)}{2}\lambda^{2k} + p^2\lambda^{2k+1}m + \left(\frac{n(n-1)}{2} - m\right)p^2\lambda^{2k+2}. \end{aligned}$$

This completes the proof. □

Bonchev and Klein [1] proposed the terminology of thorn trees, where the parent graph is a tree. In a thorn tree if the parent graph is a path then it is a caterpiller [13].

Definition 2.6 *Let P_l be path on l vertices, $l\ge 3$ labeled as $u_1, u_2, \cdots, u_l$, where u_i is adjacent to u_{i+1}, $i = 1,2,\cdots,(l-1)$. Let $T_P = T(p_1,p_2,\cdots,p_l : k)$ be a thorn tree obtained from P_l by attaching $p_i\ge 0$ path of length k to u_i, $i = 1,2,\cdots,l$.*

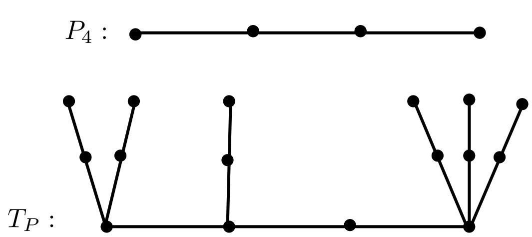

Fig. 2

A thorn graph $T_P = T(2,1,0,3 : 2)$ obtained from T by attaching paths of length 2 is shown in Fig.2.

Theorem 2.7 *For a thorn tree $T_P = T(p_1, p_2, \cdots, p_l : k)$ of order $n \geq 3$, the terminal Hosoya polynomial is*

$$Th(T_P, \lambda) = a_1\lambda + a_2\lambda^2 + \ldots + a_{2k+1}\lambda^{2k+1},$$

where

$$\begin{aligned} a_1 &= 0 \\ a_{2k} &= \sum_{i=1}^{l} \binom{p_i}{2} \\ a_{2k+l-j} &= \sum_{i=1}^{j} p_i p_{i+l-j} \qquad j = 1, 2, \ldots, (l-1). \end{aligned}$$

Proof Notice that there is no pair of pendant vertices which are at distance 1 and there are $\binom{p_i}{2}$ pairs of pendant vertices of which are at distance $2k$ in T. Therefore $a_1 = 0$ and

$$a_{2k} = \sum_{i=1}^{l} \binom{p_i}{2}.$$

For a_k, $2 \leq k \leq l$, $d(u, v|T) = 2k + l - j$, where u and v are the vertices of T_P. There are $p_i \times p_{i+l-j}$ pairs of pendant vertices which are at distance $2k + l - j$, where $j = 1, 2, \cdots, n-1$. Therefore

$$a_{2k+l-j} = \sum_{i=1}^{j} p_i p_{i+l-j}. \qquad \square$$

Definition 2.8 *Let $S_n = K_{1,n-1}$ be the star on n-vertices and let $u_1, u_2, \cdots, u_{n-1}$ be the pendant vertices of the star S_n and u_n be the central vertex. Let $S_P = S(p_1, p_2, \cdots, p_{n-1} : k)$ be the thorn star obtained from S_n by attaching p_i paths of length k to the vertex u_i, $i = 1, 2, \cdots, (n-1)$ and $p_i \geq 0$.*

Theorem 2.9 *The terminal Hosoya polynomial of thorn star S_P defined in Definition 2.8 is*

$$TH(S_P, \lambda) = a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \ldots + a_{2k}\lambda^{2k} + a_{2k+2}\lambda^{2k+2},$$

where

$$\begin{aligned} a_1 &= 0 \\ a_{2k} &= \sum_{i=1}^{n} \binom{p_i}{2} \\ a_{2k+2} &= \sum_{1 \leq i < j \leq n} p_i p_j. \end{aligned}$$

Proof There are no pair of pendant vertices which are at odd distance. Therefore, $a_{2k+1} = 0$ and the further proof follows from Theorem 2.7. $\square$

Definition 2.10 *Let C_n be the n-vertex cycle labeled consecutively as $u_1, u_2, \cdots, u_n$, $n \geq 3$. and let $\mathbb{C}_P = C(p_1, p_2, \cdots, p_n : k)$ be the thorn ring obtained from C_n by attaching p_i paths of length k to the vertex u_i, $i = 1, 2, \cdots, n$.*

Theorem 2.11 *The terminal Hosoya polynomial of thorn ring $\mathbb{C}_P$ defined in Definition* 2.10 *is*

$$TH(\mathbb{C}, \lambda) = a_1\lambda + a_2\lambda^2 + \ldots + a_{2k}\lambda^{2k} + a_{2k+1}\lambda^{2k+1},$$

where

$$\begin{aligned} a_1 &= 0 \\ a_{2k} &= \sum_{i=1}^{n} \binom{p_i}{2} \\ a_{2k+1} &= \sum_{i=1}^{n} (2k + d(v_i, v_j|G))p_i p_j. \end{aligned}$$

Proof The proof is analogous to that of Theorem 2.7. □

References

[1] D. Bonchev, D. J. Klein, On the Wiener number of thorn trees, stars, rings, and rods, *Croat. Chem. Acta*, 75 (2002), 613-620.

[2] G. G. Cash, Relationship between the Hosoya polynomial and the hyper-Wiener index, *Applied Mathematics Letters*, 15(7) (2002), 893-895.

[3] X. Deng, J. Zhang, Equiseparability on terminal Wiener index, in: A. V. Goldberg, Y. Zhou (Eds.), *Algorithmic Aspects in Information and Management*, Springer, Berlin, (2009), 166-174.

[4] E. Deutschand S. Klavžar, Computing the Hosoya polynomial of graphs from primary subgraphs, *MATCH Communications in Mathematical and in Computer Chemistry*, 70(2) (2013), 627-644.

[5] A. A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, *Acta Appl. Math.*, 66 (2001), 211-249.

[6] E. Estrada, O. Ivanciuc, I. Gutman, A. Gutierrez, and L. Rodrguez, Extended Wiener indices – A new set of descriptors for quantitative structure-property studies, *New Journal of Chemistry*, 22(8) (1998), 819-822.

[7] I. Gutman, Distance in thorny graphs, Publ. Inst. Math. (Beograd), 63 (1998), 31-36.

[8] I. Gutman, Hosoya polynomial and the distance of the total graph of a tree, *Publ. Elektrotehn. Fak. (Beograd)*, Ser. Mat., 10 (1999), 53-58.

[9] I. Gutman, B. Furtula, A survey on terminal Wiener index, in: I. Gutman, B. Furtula (Eds.), *Novel Molecular Structure Descriptors - Theory and Applications I*, Univ. Kragujevac, Kragujevac, (2010), 173-190.

[10] I. Gutman, B. Furtula, M. Petrovic, Terminal Wiener index, *J. Math. Chem.*, 46 (2009), 522-531.

[11] I. Gutman, S. Klavžar, M. Petkovšek, P. Žigert, On Hosoya polynomials of benzenoid graphs, *MATCH Commun. Math. Comput. Chem.*, 43 (2001), 49-66.

[12] I. Gutman, Y. N. Yeh, S. L. Lee, Y. L. Luo, Some recent results in the theory of the Wiener number, *Indian J. Chem.*, 32A (1993), 651-661.

[13] F. Harary, A. J. Schwenk, The number of caterpillars, *Discrete Math.*, 6 (1973), 359-365.

[14] A. Heydari, I. Gutman, On the terminal Wiener index of thorn graphs, *Kragujevac J. Sci.*, 32 (2010), 57-64.

[15] H. Hosoya, On some counting polynomials in chemistry, *Discrete Appl. Math.*, 19 (1988) 239-257.

[16] H. Hosoya, On some counting polynomials in chemistry, *Discrete Applied Mathematics*, 19 (13)(1988), 239-257.

[17] M. Lepović, I. Gutman, A collective property of trees and chemical trees, *J. Chem. Inf. Comput. Sci.*, 38 (1998), 823-826.

[18] K. P. Narayankar, S. B. Lokesh, V. Mathad, I. Gutman, Hosoya polynomial of Hanoi graphs, *Kragujevac J. Math.*, 36 (2012), 51-57.

[19] K. P. Narayankar, S. B. Lokesh, S. S. Shirkol, H. S. Ramane, Terminal Hosoya polynomial of thorn graphs, *Scientia Magna*, 9(3) (2013), 37–42.

[20] H. S. Ramane, K. P. Narayankar, S. S. Shirkol, A. B. Ganagi, Terminal Wiener index of line graphs, *MATCH Commun. Math. Comput. Chem.*, 69 (2013) 775-782.

[21] D. H. Rouvray, The rich legacy of half century of the Wiener index, in: D. H. Rouvray, R. B. King (Eds.), *Topology in Chemistry–Discrete Mathematics of Molecules*, Horwood, Chichester, (2002), 16-37.

[22] B. E. Sagan, Y. Yeh, and P. Zhang, The Wiener polynomial of a graph, *International Journal of Quantum Chemistry*, 60 (5) (1996), 959-969.

[23] D. Stevanović, Hosoya polynomial of composite graphs, *Discrete Math.*, 235 (2001), 237-244.

[24] L. A. Székely, H. Wang, T. Wu, The sum of distances between the leaves of a tree and the semi-regular property, *Discrete Math.*, 311 (2011), 1197-1203.

[25] D. Vukičevi c, A. Graovac, On modied Wiener indices of thorn graphs, *MATCH Commun. Math. Comput. Chem.*, 50 (2004), 93-108.

[26] D. Vukičević, B. Zhou, N. Trinajstić, Altered Wiener indices of thorn trees, *Croat. Chem. Acta*, 80 (2007) 283-285.

[27] H. B. Walikar, H. S. Ramane, L. Sindagi, S. S. Shirkol, I. Gutman, Hosoya polynomial of thorn trees, rods, rings and stars, *Kragujevac J. Sci.*, 28 (2006) 47-56.

[28] H. B. Walikar, V. S. Shigehalli, H. S. Ramane, Bounds on the Wiener number of a graph, *MATCH Commun. Math. Comput. Chem.*, 50 (2004) 117-132.

[29] H. Wiener, Structural determination of paran boiling points, *J. Am. Chem. Soc.*, 69 (1947) 17-20.