symmetry

MDPI

Article

Some Interval Neutrosophic Linguistic Maclaurin Symmetric Mean Operators and Their Application in Multiple Attribute Decision Making

Yushui Geng [1,*], Xingang Wang [2], Xuemei Li [2], Kun Yu [2] and Peide Liu [3,*]

1. Graduate School, Qilu University of Technology, Jinan 250353, China
2. School of Information, Qilu University of Technology, Jinan 250353, China; Wxg@qlu.edu.cn (X.W.); leexm0911@gmail.com (X.L.); ytyukun@gmail.com (K.Y.)
3. School of Management Science and Engineering, Shandong University of Finance and Economics, Jinan 250353, China

* Correspondence: gys@qlu.edu.cn (Y.G.); liupd@sdufe.edu.cn or peide.liu@gmail.com (P.L.)

Received: 10 April 2018; Accepted: 17 April 2018; Published: 22 April 2018

check for updates

Abstract: There are many practical decision-making problems in people's lives, but the information given by decision makers (DMs) is often unclear and how to describe this information is of critical importance. Therefore, we introduce interval neutrosophic linguistic numbers (INLNs) to represent the less clear and uncertain information and give their operational rules and comparison methods. In addition, since the Maclaurin symmetric mean (*MSM*) operator has the special characteristic of capturing the interrelationships among multi-input arguments, we further propose an *MSM* operator for INLNs (*INLMSM*). Furthermore, considering the weights of attributes are the important parameters and they can influence the decision results, we also propose a weighted *INLMSM* (*WINLMSM*) operator. Based on the *WINLMSM* operator, we develop a multiple attribute decision making (MADM) method with INLNs and some examples are used to show the procedure and effectiveness of the proposed method. Compared with the existing methods, the proposed method is more convenient to express the complex and unclear information. At the same time, it is more scientific and flexible in solving the MADM problems by considering the interrelationships among multi-attributes.

Keywords: multiple attribute decision making (MADM); neutrosophic number; Maclaurin symmetric mean; linguistic variables

1. Introduction

The unclear set (FS) theory was put forward by Zadeh [1] in 1965. In this theory, the membership degree (MD) $T(x)$ is used to describe fuzzy information and it has also been widely used in practice. However, the inadequacies of FS are evident. For example, it is difficult to express the non-membership degree (NMD) $F(x)$. In order to fix this problem, Intuitionistic FS (IFS) was proposed by Atanassov [2] in 1986. It is made up of two parts: MD and NMD. IFS is an extension and development of Zadeh' FS and Zadeh' FS is a special case of IFS [3]. IFS needs to meet two conditions: (1) $T(x), F(x) \in [0,1]$; (2) $0 \leq T(x) + F(x) \leq 1$ [2]. Subsequently, the IFS theory was further extended such as Zadeh [4] proposed interval IFS (IIFS). Zwick et al. [5] put forward the triangular IFS while Zeng and Li [6] defined trapezoidal IFS. However, under some circumstances due to the limited cognitive ability of the DMs, they may hesitate in the two choices for accuracy and uncertainty. Since they choose both of them at the same time, this can produce an imprecise or contradictory evaluation result. Therefore, Smarandache [7,8] introduced a concept called neutrosophic set (NS), which included MD, NMD, and indeterminacy membership degree (IMD) in a non-standard unit interval [9]. Clearly, the NS is the generalization of FS and IFS. Furthermore, Wang [10] proposed the definition of interval NS (INS) which uses the standard interval to express the function of MD, IMD, and NMD. Broumi and Smarandache [11] presented the correlation coefficient of INS.

When dealing with the MADM problems with qualitative information, it is difficult for DMs to describe their own ideas with precise values. Generally, DMs ordinarily uses some linguistic terms (LTs) like “excellent”, “good”, “bad”, “very bad”, or “general” to indicate their evaluations. For example, when we look at a person’s height, we usually describe him as “high” or “very high” by visual inspection, but we will not give the exact value. In order to easily process the qualitative information, Herrera and Herrera-Viedma [12] proposed the LTs to deal with this kind of information instead of numerical computation. However, because LT such as “high” is not with MD, or we can think its MD is 1, which means LTs cannot describe the MD and NMD. Therefore, in order to facilitate DMs to describe the MD and NMD for one LT, Liu and Chen [13] defined the linguistic intuitionistic fuzzy number (LIFN), which combined the advantages of intuitionistic fuzzy numbers (IFNs) and linguistic variables (LVs). Therefore, LIFN can fully express the complex fuzzy information and there is a good prospect in MADM. After that, Ye [14] came up with the single-valued neutrosophic linguistic number (SVNLN). The most striking feature of the SVNLN is that it used LTs to describe the MD, IMD, and NMD. Sometimes, the three degrees are not expressed in a single real number, but is expressed in intervals [15]. And then, Ye [16] defined an interval neutrosophic linguistic set (INLS) and INLNs. INLNs is used to represent three values of MD, IMD, and NMD in the form of intervals. Clearly, INLS is a generalization of FS, IFS, NS, INS, LIFN, and SVNLN. It is general and beneficial for describing practical problems.

The aggregation operators (AOs) are an efficient way to handle MADM problems [17,18]. Many AOs are proposed for achieving some special functions. Yager [19] employed the ordered weighted average (OWA) operator for MADM. Bonferroni [20] proposed the Bonferroni mean (BM) operator, which can capture the correlation between input variables very well. Then BM operators have been extended to process different uncertain information such as IFS [21,22], interval-valued IFS [23], q-Rung Orthopai Fuzzy set [24], and Multi-valued Ns [25]. In addition, Beliakov [26] presented the Heronian mean (HM) operators, which have the same feature as the BM (i.e., they can capture the interrelationship between input parameters). Some HM operators have been proposed [27–30]. Furthermore, Yu [31] gave the comparison of BM with HM. However, since the BM operator and the HM operator can only reflect the relationship between any two parameters, they cannot process the MADM problems, which require the relationship for multiple inputs. In order to solve this shortcoming, Maclaurin [32] proposed the *MSM* operator, which has prominent features of capturing the relationship among multiple input parameters. Afterward, Qin and Liu [33] developed some *MSM* operators for uncertain LVs. Liu and Qin [34] developed some *MSM* for LIFNs. Liu and Zhang [35] proposed some *MSM* operators for single valued trapezoidal neutrosophic numbers.

Since the INLNs are superior to other ways of expressing complex uncertain information [16] and the *MSM* has good flexibility and adaptability, it can capture the relationship among multiple input parameters. However, now the *MSM* cannot deal with INLNs. Therefore, the objectives of this paper are to extend the *MSM* and weighted *MSM* (*WMSM*) operators to INLNs and to propose the *INLMSM* operator and the *WINLMSM* operator, to prove some properties of them and discuss some special cases, to propose a MADM approach with INLNs, and show the advantages of the proposed approach by comparing with other studies.

In Section 2 of this paper, we introduce some basic concepts about NS, INS, INLS, and *MSM*. In Section 3, we introduce the INLN and its operations including a new scoring function and a comparison method of INLN. In Section 4, we introduce an operator of *INLMSM*. Additionally, in order to improve flexibility, we propose the *INLGMSM* operator based on the *GMSM* operator. Furthermore, we develop the *WINLMSM* operator and the *WINLGMSM* operator to compare with operators that lack weight. Afterwards, we use examples to prove our theories. In Section 5, we give a MADM method for INLNs. In Section 6, we provide an example to demonstrate the effectiveness of the proposed method. Lastly, we provide the conclusions.

2. Preliminaries

In this section, we will introduce some existing definitions and basic concepts in order to understand this study.

2.1. The NS and INS

Definition 1 [7–9]. *Let X be a space of points (objects) with a generic element in X denoted by x. A NS A in X is expressed by a MD $T_A(x)$, an IMD $I(x)$, and a NMD $F_A(x)$.*

Then a NS A is denoted below.

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\} \tag{1}$$

$T_A(x)$, $I(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]^{-}0, 1^{+}[$. That is

$$T_A : X \to]^{-}0, 1^{+}[;\ \ I_A : X \to]^{-}0, 1^{+}[;\ \ F_A : X \to]^{-}0, 1^{+}[$$

With the condition $^{-}0 \le T_A(x) + I_A(x) + F_A(x) \le 3^{+}$.

Definition 2 [10,11]. *Let X be a space of points (objects) with a generic element in X denoted by x. For convenience, the lower and upper ends of T, I, F are expressed as $T_A^L(x)$, $T_A^U(x)$, $I_A^L(x)$, $I_A^U(x)$, $F_A^L(x)$, and $F_A^U(x)$. An INS A in X is defined below.*

$$A = \left\{x, \left\langle \left[T_A^L(x), T_A^U(x)\right], \left[I_A^L(x), I_A^U(x)\right], \left[F_A^L(x), F_A^U(x)\right]\right\rangle \middle| x \in X\right\} \tag{2}$$

For each point x in X, we have that $[T_A^L(x), T_A^U(x)] \subseteq [0,1]$, $[I_A^L(x), I_A^U(x)] \subseteq [0,1]$, $[F_A^L(x), F_A^U(x)] \subseteq [0,1]$, and $0 \le T_A^U(x) + I_A^U(x) + F_A^U(x) \le 3$.

Definition 3 [10,11]. *An INS A is contained in the INS B, $A \subseteq B$, if and only if $T_A^L(x) \le T_B^L(x)$, $T_A^U(x) \le T_B^U(x)$, $I_A^L(x) \ge I_B^L(x)$, $I_A^U(x) \ge I_B^U(x)$, $F_A^L(x) \ge F_B^L(x)$, and $F_A^U(x) \ge F_B^U(x)$. If $A \subseteq B$ and $A \supseteq B$, then $A = B$.*

2.2. LVs

Definition 4 [36,37]. *Let $S = \{s_i | i = 0, 1, \ldots, l, l \in N^*\}$ be a LT set (LTS) where N^* is a set of positive integers and s_i represents LV.*

Because the LTS is convenient and efficient, it is widely used by DMs in decision making. For instance, when we evaluate the production quality, we can set l = 9, then S is given below.

$$S = \{s_0 = extremely\ bad, s_1 = very\ bad, s_2 = bad, s_3 = slightly\ bad, s_4 = fair, s_5 = slightly\ good,$$
$$s_6 = good, s_7 = very\ good, s_8 = extremely\ good\}$$

To relieve the loss of linguistic information in operations, Xu [38,39] extended LTS S to continuous LTS $\overline{S} = \{s_\theta | 0 \le \theta \le l\}$. About the characteristics of LTS, please refer to References [38–40].

Definition 5 [13]. *Let s_α and s_β be any two LVs in $\overline{S}$. The related operations can be defined below.*

$$s_\alpha \oplus s_\beta = s_{\alpha + \beta - \frac{\alpha \cdot \beta}{l}} \tag{3}$$

$$\lambda s_\alpha = s_{l - l \cdot (1 - \frac{\alpha}{l})^\lambda}, \lambda > 0 \tag{4}$$

$$s_\alpha \otimes s_\beta = s_{\frac{\alpha \cdot \beta}{l}} \tag{5}$$

$$(s_\alpha)^\lambda = s_{l \cdot (\frac{\alpha}{l})^\lambda}, \lambda > 0 \tag{6}$$

2.3. MSM Operator

Definition 6 [15,32]. *Let $x_i (i = 1, 2, \ldots, n)$ be the set of the non-negative real number. An MSM operator of dimension n is a mapping $MSM^{(m)} : (R^+)^n \to R^+$ and it can be defined below.*

$$MSM^{(m)}(x_1, \ldots, x_n) = \left(\frac{\sum_{1 \le i_1 < \ldots < i_m \le n} \prod_{j=1}^{m} x_{i_j}}{C_n^m} \right)^{\frac{1}{m}} \tag{7}$$

where $(i_1, i_2, \ldots, i_m)$ traverses all the m-tuple combination of $(1, 2, \ldots, n)$ and $C_n^m = \frac{n!}{m!(n-m)!}$ is the binomial coefficient. In addition, x_{i_j} refers to i_jth element in a particular arrangement.

There are some properties of the $MSM^{(m)}$ operator, which are defined below.

(1) Idempotency. If $x_i = x$ for each i, and then $MSM^{(m)}(x, x, \ldots, x) = x$;
(2) Monotonicity. If $x_i <= y_i$ for all i, $MSM^{(m)}(x_1, x_2, \ldots, x_n) \le MSM^{(m)}(y_1, y_2, \ldots, y_n)$;
(3) Boundedness. $\min\{x_1, x_2, \ldots x_n\} \le MSM^{(m)}\{x_1, x_2, \ldots x_n\} \le \max\{x_1, x_2, \ldots x_n\}$.

Furthermore, the $MSM^{(m)}$ operator would degrade some particular forms when m takes some special values, which are shown as follows.

1. When $m = 1$, the $MSM^{(m)}$ operator would become the average operator.

$$MSM^{(1)}(x_1, x_2, \ldots x_n) = \left(\frac{\sum_{1 \le i_1 \le n} x i_1}{C_n^1} \right) = \frac{\sum_{i=1}^{n} xi}{n} \tag{8}$$

2. When $m = 2$, the $MSM^{(m)}$ operator would become the following BM operator ($p = q = 1$).

$$\begin{aligned} MSM^{(2)}(x_1, \ldots, x_n) &= \left(\frac{\sum_{1 \le i_1 < i_2 \le n} \prod_{j=1}^{2} x_{i_j}}{C_n^2} \right)^{\frac{1}{2}} = \left(\frac{2 \sum_{1 \le i_1 < i_2 \le n} x i_1 x i_2}{n(n-1)} \right)^{\frac{1}{2}} \\ &= \left(\frac{\sum_{i.j=1, i \ne j}^{n} x i x j}{n(n-1)} \right)^{\frac{1}{2}} = BM^{1,1}(x_1, \ldots, x_n) \end{aligned} \tag{9}$$

3. When $m = n$, the $MSM^{(m)}$ operator would become the geometric mean.

$$MSM^{(n)}(x_1, \ldots, x_n) = \left(\prod_{j=1}^{n} x_j \right)^{\frac{1}{n}} \tag{10}$$

Definition 7 [15]. *Let $x_i (i = 1, 2, \ldots, n)$ be the set of non-negative real numbers and $p_1, p_2, \ldots, p_m \ge 0$. A generalized MSM operator of dimension n is a mapping $GMSM^{(m, p_1, p_2, \ldots, p_m)} : (R^+)^n \to R^+$ and it is defined below.*

$$GMSM^{(m, p_1, p_2, \ldots, p_m)}(x_1, \ldots, x_n) = \left(\frac{\sum_{1 \le i_1 < \ldots < i_m \le n} \prod_{j=1}^{m} x_{i_j}^{p_j}}{C_n^m} \right)^{\frac{1}{p_1 + p_2 + \ldots p_m}} \tag{11}$$

where $(i_1, i_2, \ldots, i_m)$ *traverses all the m-tuple combination of* $(1, 2, \ldots, n)$ *and* $C_n^m = \frac{n!}{m!(n-m)!}$ *is the binomial coefficient.*

There are some properties of the $GMSM^{(m,P_1,P_2,\ldots,P_m)}$ operator below.

(1) Idempotency. If $x_i = x$ for each i, and then $GMSM^{(m,P_1,P_2,\ldots,P_m)}(x, x, \ldots, x) = x$;
(2) Monotonicity. If $x_i \leq y_i$ for all i, $GMSM^{(m,p_1,p_2,\ldots,p_m)}(x_1, x_2, \ldots, x_n) \leq GMSM^{(m,P_1,P_2,\ldots,P_m)}(y_1, y_2, \ldots, y_n)$;
(3) Boundedness. $\min\{x_1, x_2, \ldots x_n\} \leq GMSM^{(m,p_1,p_2,\ldots,p_m)}\{x_1, x_2, \ldots x_n\} \leq \max\{x_1, x_2, \ldots x_n\}$.

In addition, the $GMSM^{(m,P_1,P_2,\ldots,P_m)}$ operator would degrade to some particular forms when m takes some special values, which are shown below.

1. When $m = 1$, we have the formula below.

$$GMSM^{(1,P_1)}(x_1, x_2, \ldots x_n) = \left(\frac{\sum_{1\leq i_1\leq n} {x_{i_1}}^{p_1}}{C_n^1}\right)^{\frac{1}{p_1}} = \left(\frac{\sum_{i=1}^{n} {x_i}^{p_1}}{n}\right)^{\frac{1}{p_1}} \tag{12}$$

2. When $m = 2$, the $GMSM^{(m,P_1,P_2,\ldots,P_m)}$ operator would become the following BM operator.

$$\begin{aligned} GMSM^{(2,p_1,p_2)}(x_1, \ldots, x_n) &= \left(\frac{\sum_{1\leq i_1<i_2\leq n} x_{i_1}^{p_1} x_{i_2}^{p_2}}{C_n^2}\right)^{\frac{1}{p_1+p_2}} = \left(\frac{2\sum_{1\leq i<j\leq n} x_i^{p_1} x_j^{p_2}}{n(n-1)}\right)^{\frac{1}{p_1+p_2}} \\ &= \left(\frac{\sum_{i,j=1,i\neq j}^{n} x_i^{p_1} x_j^{p_2}}{n(n-1)}\right)^{\frac{1}{p_1+p_2}} = BM^{p_1,p_2} \end{aligned} \tag{13}$$

3. When $m = n$, the $MSM^{(m)}$ operator would become the following formula.

$$GMSM^{(n,p_1,p_2,\ldots,p_n)}(x_1, \ldots, x_n) = \left(\prod_{j=1}^{n} x_j^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots p_n}} \tag{14}$$

4. When $p_1 = p_2 = \ldots = p_m = 1$, the $GMSM^{(m,P_1,P_2,\ldots,P_m)}$ operator would degenerate to the MSM operator and the parameter is m below.

$$GMSM^{(m,1,1,\ldots,1)}(x_1, \ldots, x_n) = \left(\frac{\sum_{1\leq i_1<\ldots<i_m\leq n} \prod_{j=1}^{m} x_{ij}^{1}}{C_n^m}\right)^{\frac{1}{m}} = MSM^{(m)}(x_1, \ldots, x_n). \tag{15}$$

3. INLNs and Operations

Definition 8 [16,41]. *Let X be a finite universal set. An INLS in X is defined by the equation below.*

$$A = \left\{x, \left\langle s_{\theta(x)}, [T_A(x), I_A(x), F_A(x)]\right\rangle | x \in X\right\} \tag{16}$$

where $s_{\theta(x)} \in \overline{S}$, $T_A(x) = [T_A^L(x), T_A^U(x)] \subseteq [0,1]$, $I_A(x) = [I_A^L(x), I_A^U(x)] \subseteq [0,1]$, $F_A(x) = [F_A^L(x), F_A^U(x)] \subseteq [0,1]$ *represent the MD, the IMD, and the NMD of the element x in X to the LV* $s_{\theta(x)}$, *respectively, with the condition* $0 \leq T_A^U(x) + I_A^U(x) + F_A^U(x) \leq 3$ *for any* $x \in X$.

Then the seven tuple $\left\langle s_{\theta(x)}, ([T_A^L(x), T_A^U(x)], [I_A^L(x), I_A^U(x)], [F_A^L(x), F_A^U(x)])\right\rangle$ in A is called an INLN. For convenience, an INLN can be represented as $a = \left\langle s_{\theta(a)}, ([T^L(a), T^U(a)], [I^L(a), I^U(a)], [F^L(a), F^U(a)])\right\rangle$.

Then we introduced the operational rules of operators of INLNs.

Definition 9 [16,37,42]. *Let* $a_1 = \left\langle s_{\theta(a_1)}, \left(\left[T^L(a_1), T^U(a_1)\right], \left[I^L(a_1), I^U(a_1)\right], \left[F^L(a_1), F^U(a_1)\right]\right)\right\rangle$ *and* $a_2 = \left\langle s_{\theta(a_2)}, \left(\left[T^L(a_2), T^U(a_2)\right], \left[I^L(a_2), I^U(a_2)\right], \left[F^L(a_2), F^U(a_2)\right]\right)\right\rangle$ *be two INLNs and* $\lambda \geq 0$. *Then the operation of the INLNs can be expressed by the equation below.*

$$\begin{aligned} a_1 \oplus a_2 = \Big\langle s_{\theta(a_1)+\theta(a_2)}, \big(&\left[T^L(a_1) + T^L(a_2) - T^L(a_1) \times T^L(a_2), T^U(a_1) + T^U(a_2) - T^U(a_1) \times T^U(a_2)\right], \\ &\left[I^L(a_1) \times I^L(a_2), I^U(a_1) \times I^U(a_2)\right], \left[F^L(a_1) \times F^L(a_2), F^U(a_1) \times F^U(a_2)\right]\big)\Big\rangle \end{aligned} \quad (17)$$

$$\begin{aligned} &I^U(a_1) + I^U(a_2) - I^U(a_1) \times I^U(a_2)], [F^L(a_1) + F^L(a_2) - F^L(a_1) \times F^L(a_2), \\ &a_1 \otimes a_2 = \Big\langle s_{\theta(a_1)\times\theta(a_2)}, \big(\left[T^L(a_1) \times T^L(a_2), T^U(a_1) \times T^U(a_2)\right], [I^L(a_1) + I^L(a_2) - I^L(a_1) \times I^L(a_2), \\ &F^U(a_1) + F^U(a_2) - F^U(a_1) \times F^U(a_2)]\big)\Big\rangle \end{aligned} \quad (18)$$

$$\begin{aligned} \lambda a_1 = \Big\langle s_{\lambda\times\theta(a_1)}, \Big(&\left[1 - (1 - T^L(a_1))^\lambda, 1 - (1 - T^U(a_1))^\lambda\right], \left[(I^L(a_1))^\lambda, (I^U(a_1))^\lambda\right], \\ &\left[(F^L(a_1))^\lambda, (F^U(a_1))^\lambda\right]\Big)\Big\rangle (\lambda > 0) \end{aligned} \quad (19)$$

$$\begin{aligned} a_1^\lambda = s_{\theta^\lambda(a_1)}, \Big(&\left[(T^L(a_1))^\lambda, (T^U(a_1))^\lambda\right], \left[1 - (1 - I^L(a_1))^\lambda, 1 - (1 - I^U(a_1))^\lambda\right], \\ &\left[1 - (1 - F^L(a_1))^\lambda, 1 - (1 - F^U(a_1))^\lambda\right]\Big)\Big\rangle, (\lambda > 0) \end{aligned} \quad (20)$$

Example 1. *Let* $a_1 = \langle s_3, ([0.1, 0.2], [0.2, 0.3], [0.4, 0.5])\rangle$ *and* $a_2 = \langle s_4, ([0.3, 0.5], [0.3, 0.4], [0.5, 0.6])\rangle$ *be two INLNs and* $S = \{s_0 = very\ bad, s_1 = bad, s_2 = slightly\ bad, s_3 = fair, s_4 = slightly\ good, s_5 = good, s_6 = very\ good\}$, *then we have the equations below.*

$$\begin{aligned} a_1 \oplus a_2 &= \langle s_{3+4}, ([0.1 + 0.3 - 0.1 \times 0.3, 0.2 + 0.5 - 0.2 \times 0.5], [0.2 \times 0.3, 0.3 \times 0.4], [0.4 \times 0.5, 0.5 \times 0.6]\rangle \\ &= \langle s_7, ([0.37, 0.6], [0.06, 0.12], [0.2, 0.3]\rangle \end{aligned}$$

$$\begin{aligned} a_1 \otimes a_2 &= \langle s_{3\times4}, ([0.1 \times 0.3, 0.2 \times 0.5], [0.2 + 0.3 - 0.2 \times 0.3, 0.3 + 0.4 - 0.3 \times 0.4], \\ &[0.4 + 0.5 - 0.4 \times 0.5, 0.5 + 0.6 - 0.5 \times 0.6])\rangle \\ &= \langle s_{12}, ([0.03, 0.1], [0.44, 0.58], [0.7, 0.8])\rangle \end{aligned}$$

As seen from the above examples, these results are not reasonable because they exceed the range of LTS. In order to overcome these limitations, we will improve these operations by Definition 10.

Definition 10. *Let* $a_1 = \left\langle s_{\theta(a_1)}, \left(\left[T^L(a_1), T^U(a_1)\right], \left[I^L(a_1), I^U(a_1)\right], \left[F^L(a_1), F^U(a_1)\right]\right)\right\rangle$ *and* $a_2 = \left\langle s_{\theta(a_2)}, \left(\left[T^L(a_2), T^U(a_2)\right], \left[I^L(a_2), I^U(a_2)\right], \left[F^L(a_2), F^U(a_2)\right]\right)\right\rangle$ *be two INLNs and* $\lambda \geq 0$. *Then the operations of the INLNs can be defined by the equations below.*

$$\begin{aligned} a_1 \oplus a_2 = \Big\langle s_{\theta(a_1)+\theta(a_2)-\frac{\theta(a_1)\cdot\theta(a_2)}{l}}, \big(&\left[T^L(a_1) + T^L(a_2) - T^L(a_1) \times T^L(a_2), T^U(a_1) + T^U(a_2) - T^U(a_1) \times T^U(a_2)\right], \\ &\left[I^L(a_1) \times I^L(a_2), I^U(a_1) \times I^U(a_2)\right], \left[F^L(a_1) \times F^L(a_2), F^U(a_1) \times F^U(a_2)\right]\big)\Big\rangle \end{aligned} \quad (21)$$

$$\begin{aligned} a_1 \otimes a_2 = \Big\langle s_{\frac{\theta(a_1)\times\theta(a_2)}{l}}, \big(&\left[T^L(a_1) \times T^L(a_2), T^U(a_1) \times T^U(a_2)\right], [I^L(a_1) + I^L(a_2) - I^L(a_1) \times I^L(a_2), \\ &I^U(a_1) + I^U(a_2) - I^U(a_1) \times I^U(a_2)], [F^L(a_1) + F^L(a_2) - F^L(a_1) \times F^L(a_2), \\ &F^U(a_1) + F^U(a_2) - F^U(a_1) \times F^U(a_2)]\big)\Big\rangle \end{aligned} \quad (22)$$

$$\begin{aligned} \lambda a_1 = \Big\langle s_{l-l\cdot(1-\frac{\theta(a_1)}{l})^\lambda}, \Big(&\left[1 - (1 - T^L(a_1))^\lambda, 1 - (1 - T^U(a_1))^\lambda\right], \\ &\left[(I^L(a_1))^\lambda, (I^U(a_1))^\lambda\right], \left[(F^L(a_1))^\lambda, (F^U(a_1))^\lambda\right]\Big)\Big\rangle, (\lambda > 0) \end{aligned} \quad (23)$$

$$\begin{aligned} a_1^\lambda = s_{l\cdot(\frac{\theta(a_1)}{l})^\lambda}, \Big(&\left[(T^L(a_1))^\lambda, (T^U(a_1))^\lambda\right], \left[1 - (1 - I^L(a_1))^\lambda, 1 - (1 - I^U(a_1))^\lambda\right], \\ &\left[1 - (1 - F^L(a_1))^\lambda, 1 - (1 - F^U(a_1))^\lambda\right]\Big)\Big\rangle, (\lambda > 0). \end{aligned} \quad (24)$$

Based on the operational rules above, the above example is recalculated as follow.

Example 2. *Let* $a_1 = \langle s_3, ([0.1, 0.2], [0.2, 0.3], [0.4, 0.5])\rangle$ *and* $a_2 = \langle s_4, ([0.3, 0.5], [0.3, 0.4], [0.5, 0.6])\rangle$ *be two INLNs and* $S = \{s_0 = very\ bad, s_1 = bad, s_2 = slightly\ bad, s_3 = fair, s_4 = slightly\ good, s_5 = good, s_6 = very\ good\}$*, then we have the equations below.*

$$a_1 \oplus a_2 = \left\langle s_{3+4-\frac{3\times4}{6}}, ([0.1 + 0.3 - 0.1 \times 0.3, 0.2 + 0.5 - 0.2 \times 0.5], [0.2 \times 0.3, 0.3 \times 0.4], [0.4 \times 0.5, 0.5 \times 0.6]\right\rangle$$
$$= \langle s_5, ([0.37, 0.6], [0.06, 0.12], [0.2, 0.3]\rangle$$

$$a_1 \otimes a_2 = \left\langle s_{\frac{3\times4}{6}}, ([0.1 \times 0.3, 0.2 \times 0.5], [0.2 + 0.3 - 0.2 \times 0.3, 0.3 + 0.4 - 0.3 \times 0.4],\right.$$
$$[0.4 + 0.5 - 0.4 \times 0.5, 0.5 + 0.6 - 0.5 \times 0.6])\rangle$$
$$= \langle s_2, ([0.03, 0.1], [0.44, 0.58], [0.7, 0.8])\rangle$$

From the above example, the results are more reasonable than the previous ones.

In the following definitions, a new scoring function and a comparison method of INLN are described.

Definition 11. **[37].** *Let* $a = \left\langle s_{\theta(a)}, ([T^L(a), T^U(a)], [I^L(a), I^U(a)], [F^L(a), F^U(a)])\right\rangle$ *be an INLN. Then the score function of a can be expressed by the equation below.*

$$S(a) = \alpha \cdot \frac{\theta(a)}{6}\left[0.5(T^U(a) + 1 - F^L(a)) + \alpha I^U(a)\right] + (1-\alpha) \cdot \frac{\theta(a)}{6}\left[0.5(T^L(a) + 1 - F^U(a)) + \alpha I^L(a)\right] \quad (25)$$

where the values of $\alpha \in [0, 1]$ *reflect the attitudes of the decision makers.*

Definition 12. **[37].** *Let a and b be two INLNs. Then the INLN comparison method can be expressed by the statements below.*

$$\text{If } S(a) > S(b), \text{ then } a \succ b; \quad (26)$$

$$\text{If } S(a) = S(b), \text{ then } a \sim b; \quad (27)$$

$$\text{If } S(a) < (b), \text{ then } a \prec b; \quad (28)$$

4. Some Interval Neutrosophic Linguistic MSM Operators

In this section, we will propose *INLMSM* operators and *INLGMSM* operators.

4.1. The INLMSM Operators

Definition 13. *Let* $a_i = \langle s_{\theta_i}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)])\rangle$ $(i = 1, 2, ..., n)$ *be a set of INLNs. Then the INLMSM operator:* $\Omega^n \to \Omega$ *is shown below.*

$$INLMSM^{(m)}(a_1, \ldots, a_n) = \left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}} a_{i_j}\right)}{C_n^m}\right)^{\frac{1}{m}} \quad (29)$$

Ω *is a set of INLNs and* $m = 1, 2, ..., n$.

According to the operational laws of INLNs in Definition 10, we can get the expression of the INLMSM operator shown below.

Theorem 1. *Let* $a_i = \langle s_{\theta_i}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)]) \rangle$ $(i = 1, 2, ..., n)$ *be a set of INLNs and* $m = 1, 2, ..., n$. *Then the value aggregated from Definition 13 is still an INLN.*

$$INLMSM^{(m)}(a_1, \ldots, a_n) = \Big\langle s_{l \cdot (1 - \prod_{k=1}^{C_n^m} (1 - \prod_{j=1}^{m} (\frac{\theta_{i_j}(k)}{l}))^{\frac{1}{C_n^m}})^{\frac{1}{m}}}, \left(\left[\left(1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} T^L{}_{i_j(k)}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, \left(1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} T^U{}_{i_j(k)}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}} \right], \right.$$
$$\left[1 - \left(1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} \left(1 - I^L{}_{i_j(k)}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, 1 - \left(1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} \left(1 - I^U{}_{i_j(k)}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right], \tag{30}$$
$$\left.\left[1 - \left(1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} \left(1 - F^L{}_{i_j(k)}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, 1 - \left(1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} \left(1 - F^U{}_{i_j(k)}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right]\right) \Big\rangle$$

where $k = 1, 2, ...C_n^m$, $a_{i_j(k)}$ *is the* i_j*th element of* k *th permutation.*

Proof.

Because

$$a_{i_j(k)} = \Big\langle s_{\theta i_j(k)}, ((T^L i_j(k), T^U i_j(k)), (I^L i_j(k), I^U i_j(k)), (F^L i_j(k), F^U i_j(k))) \Big\rangle (j = 1, 2, ..., m)$$

$$\Rightarrow \bigotimes_{j=1}^{m} a_{i_j(k)} = \Big\langle s_{l \cdot \prod_{j=1}^{m} (\frac{\theta i_j(k)}{l})}, \left(\left[\prod_{j=1}^{m} T^L i_j(k), \prod_{j=1}^{m} T^U i_j(k) \right], \right.$$
$$\left.\left[1 - \prod_{j=1}^{m} \left(1 - I^L{}_{i_j(k)}\right), 1 - \prod_{j=1}^{m} \left(1 - I^U{}_{i_j(k)}\right)\right], \left[1 - \prod_{j=1}^{m} \left(1 - F^L{}_{i_j(k)}\right), 1 - \prod_{j=1}^{m} \left(1 - F^U{}_{i_j(k)}\right)\right] \right) \Big\rangle$$

$$\Rightarrow \bigoplus_{1 \le i_1 < \ldots < i_m \le n} \left(\bigotimes_{j=1}^{m} a i_j \right) = \Big\langle s_{l - l \cdot \prod_{k=1}^{C_n^m} (1 - \prod_{j=1}^{m} (\frac{\theta i_j(k)}{l}))},$$
$$\left(\left[1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} T^L{}_{i_j(k)}\right), 1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} T^U{}_{i_j(k)}\right) \right], \right.$$
$$\left[\prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} \left(1 - I^L{}_{i_j(k)}\right)\right), \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} \left(1 - I^U{}_{i_j(k)}\right)\right) \right],$$
$$\left.\left[\prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} \left(1 - F^L{}_{i_j(k)}\right)\right), \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} \left(1 - F^U{}_{i_j(k)}\right)\right) \right] \right) \Big\rangle$$

$$\Rightarrow \left(\frac{\bigoplus_{1 \le i_1 < \ldots < i_m \le n} \left(\bigotimes_{j=1}^{m} a_{i_j} \right)}{C_n^m} \right)^{\frac{1}{m}} = \Big\langle s_{l \cdot (1 - \prod_{k=1}^{C_n^m} (1 - \prod_{j=1}^{m} (\frac{\theta i_j(k)}{l}))^{\frac{1}{C_n^m}})^{\frac{1}{m}}},$$
$$\left(\left[\left(1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} T^L{}_{i_j(k)}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, \left(1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} T^U{}_{i_j(k)}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}} \right], \right.$$
$$\left[1 - \left(1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} \left(1 - I^L{}_{i_j(k)}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, 1 - \left(1 - \prod_{k=1}^{C_n^m} \left(1 - \prod_{j=1}^{m} \left(1 - I^U{}_{i_j(k)}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right],$$

$$\left[1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-F^L{}_{i_j(k)}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}},1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-F^U{}_{i_j(k)}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right]\Bigg)\Bigg\rangle$$

Therefore, Theorem 1 is kept. □

Property 1. *Let* $x_i = \langle s_{\alpha_i}, ([T^L(x_i), T^U(x_i)], [I^L(x_i), I^U(x_i)], [F^L(x_i), F^U(x_i)])\rangle$ $(i = 1,2,...,n)$ *and* $y_i = \langle s_{\beta_i}, ([T^L(y_i), T^U(y_i)], [I^L(y_i), I^U(y_i)], [F^L(y_i), F^U(y_i)])\rangle$ $(i = 1,2,...,n)$ *be sets of INLNs. There are four properties of* $INLMSM^{(m)}$ *operator, which is shown below.*

(1) *Idempotency. If the INLNs* $x_i = x = \langle s_{\theta_x}, ([T^L{}_x, T^U{}_x], [I^L{}_x, I^U{}_x], [F^L{}_x, F^U{}_x])\rangle$ *for each* $i(i = 1,2,...,n)$ *and then* $INLMSM^{(m)} = x = \langle s_{\theta_x}, (T_x, I_x, F_x)\rangle$.
(2) *Commutativity. If* x_i *is a permutation of* y_i *for all* i $(i = 1,2,...,n)$ *and then* $INLMSM^{(m)}(x_1, x_2, ..., x_n) = INLMSM^{(m)}(y_1, y_2, ..., y_n)$.
(3) *Monotonicity. If* $\alpha_i \le \beta_i$, $T^L(x_i) \le T^L(y_i)$, $T^U(x_i) \le T^U(y_i)$, $I^L(x_i) \ge I^L(y_i)$, $I^U(x_i) \ge I^U(y_i)$, $F^L(x_i) \ge F^L(y_i)$ *and* $F^U(x_i) \ge F^U(y_i)$ *for all* i $(i = 1,2,...,n)$, *then* $x_i \le y_i$ *and* $INLMSM^{(m)}(x_1, x_2, ..., x_n) \le INLMSM^{(m)}(y_1, y_2, ..., y_n)$.
(4) *Boundedness.* $\min\{x_1, x_2, ...x_n\} \le INLMSM^{(m)}\{x_1, x_2, ...x_n\} \le \max\{x_1, x_2, ...x_n\}$..

Proof.

1 If each $a_i = x$, then we get the equation below.

$$INLMSM^{(m)}(x, x, ..., x) =$$

$$\left\langle s_{l\cdot(1-\prod_{k=1}^{C_n^m}(1-\prod_{j=1}^{m}(\frac{\theta_x}{l}))^{\frac{1}{C_n^m}})^{\frac{1}{m}}}, \left(\left[\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}T^L{}_x\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, \left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}T^U{}_x\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right],\right.\right.$$
$$\left[1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}(1-I^L{}_x)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, 1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}(1-I^U{}_x)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right],$$
$$\left.\left[1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}(1-F^L{}_x)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, 1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}(1-F^U{}_x)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right]\right\rangle$$
$$= \langle s_{\theta_x}, (Tx, Ix, Fx)\rangle = x.$$

2 This property is clear and it is now omitted.

3 If $\alpha_i \le \beta_i$, $T^L(x_i) \le T^L(y_i)$, $T^U(x_i) \le T^U(y_i)$, $I^L(x_i) \ge I^L(y_i)$, $I^U(x_i) \ge I^U(y_i)$, $F^L(x_i) \ge F^L(y_i)$ and $F^U(x_i) \ge F^U(y_i)$ for all i, according to Theorem 1. Since

$$\prod_{j=1}^{m}\alpha_i \le \prod_{j=1}^{m}\beta_i, \prod_{j=1}^{m}T^L(x_i) \le \prod_{j=1}^{m}T^L(y_i), \prod_{j=1}^{m}T^U(x_i) \le \prod_{j=1}^{m}T^U(y_i), \prod_{j=1}^{m}I^L(x_i) \ge \prod_{j=1}^{m}I^L(y_i),$$
$$\prod_{j=1}^{m}I^U(x_i) \ge \prod_{j=1}^{m}I^U(y_i), \prod_{j=1}^{m}F^L(x_i) \ge \prod_{j=1}^{m}F^L(y_i), \prod_{j=1}^{m}F^U(x_i) \ge \prod_{j=1}^{m}F^U(y_i)$$

$$\text{then } l\cdot\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\frac{\alpha_i}{l}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}} \le l\cdot\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\frac{\beta_i}{l}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}},$$

$$\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}T^L(x_i)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}} \le \left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}T^L(y_i)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}},$$

$$\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}T^U(x_i)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}} \leq \left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}T^U(y_i)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}},$$

$$1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-I^L(x_i)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}} \geq 1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-I^L(y_i)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}},$$

$$1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-I^U(x_i)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}} \geq 1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-I^U(y_i)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}},$$

$$1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-F^L(x_i)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}} \geq 1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-F^L(y_i)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}},$$

$$1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-F^U(x_i)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}} \geq 1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-F^U(y_i)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}.$$

Therefore, we can get the following conclusion.

$$INLMSM^{(m)}(x_1, x_2, ..., x_n) \leq INLMSM^{(m)}(y_1, y_2, ..., y_n)$$

4 According to the idempotency, let $\min\{x_1, x_2, ...x_n\} = x_a = INLMSM^{(m)}(x_a, x_a, ..., x_a)$ and $\max\{x_1, x_2, ...x_n\} = x_b = INLMSM^{(m)}(x_b, x_b, ..., x_b)$. According to the monotonicity, if $x_a \leq x_i$ and $x_b \geq x_i$ for all i, then we have $x_a = INLMSM^{(m)}(x_a, x_a, ..., x_a) \leq INLMSM^{(m)}(x_1, x_2, ..., x_n)$ and

$$INLMSM^{(m)}(x_1, x_2, ..., x_n) \leq x_b = INLMSM^{(m)}(x_b, x_b, ..., x_b).$$

Therefore, we can get the conclusion below.

$$\min\{x_1, x_2, ...x_n\} \leq INLMSM^{(m)}\{x_1, x_2, ...x_n\} \leq \max\{x_1, x_2, ...x_n\}.$$

Furthermore, the $INLMSM^{(m)}$ operator would degrade to some particular forms when m takes some special values.

(1) When $m = 1$, we have the formula below.

$$\begin{gathered} INLMSM^{(1)}(x_1, x_2, ...x_n) = \left(\frac{\oplus_{i=1}^{n} x_i}{C_n^1}\right) = \\ \left\langle s_{l\cdot(1-\prod_{k=1}^{n}(1-\frac{k}{l})^{\frac{1}{n}})}, \left(\left[1-\prod_{k=1}^{n}(1-T^L{}_k)^{\frac{1}{n}}, 1-\prod_{k=1}^{n}(1-T^U{}_k)^{\frac{1}{n}}\right],\right.\right. \\ \left.\left.\left[\prod_{k=1}^{n}(I^L{}_k)^{\frac{1}{n}}, \prod_{k=1}^{n}(I^U{}_k)^{\frac{1}{n}}\right], \left[\prod_{k=1}^{n}(F^L{}_k)^{\frac{1}{n}}, \prod_{k=1}^{n}(F^U{}_k)^{\frac{1}{n}}\right]\right)\right\rangle \end{gathered} \tag{31}$$

(2) When $m = 2$, we have the formula below.

$$\begin{aligned}&INLMSM^{(2)}(x_1, x_2, \ldots x_n) = \\ &\Big\langle s_{l\cdot\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(\frac{\theta i_1(k)}{l}\right)\cdot\left(\frac{\theta i_2(k)}{l}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}}, \left(\left[\left(1-\prod_{k=1}^{C_n^2}\left(1-T^L{}_{i_1(k)}\cdot T^L{}_{i_2(k)}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}, \left(1-\prod_{k=1}^{C_n^2}\left(1-T^U{}_{i_1(k)}\cdot T^U{}_{i_2(k)}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}\right],\right. \\ &\left[1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-I^L i_1(k)\right)\cdot\left(1-I^L i_2(k)\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}, 1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-I^U i_1(k)\right)\cdot\left(1-I^U i_2(k)\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}\right], \\ &\left.\left[1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-F^L i_1(k)\right)\cdot\left(1-F^L i_2(k)\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}, 1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-F^U i_1(k)\right)\cdot\left(1-F^U i_2(k)\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}\right]\right)\Big\rangle\end{aligned} \tag{32}$$

(3) When $m = n$, the $INLMSM^{(m)}$ operator would reduce to the following form.

$$\begin{aligned}&INLMSM^{(n)}(x_1, \ldots, x_n) = \\ &\Big\langle s_{l\cdot\left(\prod_{j=1}^{n}\left(\frac{\theta_j}{l}\right)\right)^{\frac{1}{n}}}, \left(\left[\left(\prod_{j=1}^{n} T^L{}_j\right)^{\frac{1}{n}}, \left(\prod_{j=1}^{n} T^U{}_j\right)^{\frac{1}{n}}\right], \left[1-\left(\prod_{j=1}^{n}\left(1-I^L{}_j\right)\right)^{\frac{1}{n}}, 1-\left(\prod_{j=1}^{n}\left(1-I^U{}_j\right)\right)^{\frac{1}{n}}\right],\right. \\ &\left.\left[1-\left(\prod_{j=1}^{n}\left(1-F^L{}_j\right)\right)^{\frac{1}{n}}, 1-\left(\prod_{j=1}^{n}\left(1-F^U{}_j\right)\right)^{\frac{1}{n}}\right]\right)\Big\rangle\end{aligned} \tag{33}$$

□

Definition 14. *Let* $a_i = \langle s_{\theta_i}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)])\rangle$ $(i = 1, 2, \ldots, n)$ *be a set of INLNs. Then the INLGMSM operator:* $\Omega^n \to \Omega$ *is shown below.*

$$INLGMSM^{(m,p_1,p_2,\ldots,p_m)}(a_1, \ldots, a_n) = \left(\frac{\bigoplus_{1\le i_1<\ldots<i_m\le n}\left(\bigotimes_{j=1}^{m} a_{i_j}^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}, \tag{34}$$

Ω is a set of INLNs and $m = 1, 2, \ldots, n$.

According to the operational laws of INLNs in Definition 10, we can get the expression of the INLMSM operator shown below.

Theorem 2. *Let* $a_i = \langle s_{\theta_i}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)])\rangle$ $(i = 1, 2, \ldots, n)$ *be a set of INLNs and* $m = 1, 2, \ldots, n$. *Then the value aggregated from Definition 14 is still an INLN.*

$$\begin{aligned}&INLGMSM^{(m,p_1,p_2,\ldots,p_m)}(a_1, \ldots, a_n) = \Big\langle s_{l\cdot\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(\frac{\theta_{i_j(k)}}{l}\right)^{p_j}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}}, \\ &\left(\left[\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(T^L{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}},\right.\right. \\ &\left.\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(T^U{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}\right], \\ &\left[1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-I^L{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}},\right. \\ &\left.1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-I^U{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}\right], \\ &\left.\left[1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-F^L{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}, 1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-F^U{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}\right]\right)\Big\rangle\end{aligned} \tag{35}$$

where $k = 1, 2, \ldots C_n^m$, $a_{i_j(k)}$ is the i_jth element of k_jth permutation. Therefore, Theorem 2 is kept. The process of proof is similar to Theorem 1 and is now omitted.

Property 2. *Let $x_i = \langle s_{\alpha_i}, ([T^L(x_i), T^U(x_i)], [I^L(x_i), I^U(x_i)], [F^L(x_i), F^U(x_i)]) \rangle$ $(i = 1, 2, \ldots, n)$ and $y_i = \langle s_{\beta_i}, ([T^L(y_i), T^U(y_i)], [I^L(y_i), I^U(y_i)], [F^L(y_i), F^U(y_i)]) \rangle$ $(i = 1, 2, \ldots, n)$ be two sets of INLNs. There are four properties of $INLGMSM^{(m,p_1,p_2,\ldots,p_m)}$ operator shown as follows.*

1 Idempotency. If the INLNs $x_i = x = \langle s_{\theta_x}, ([T^L{}_x, T^U{}_x], [I^L{}_x, I^U{}_x], [F^L{}_x, F^U{}_x]) \rangle$ for each $i (i = 1, 2, \ldots, n)$ and then $INLGMSM^{(m,p_1,p_2,\ldots,p_m)} = x = \langle s_{\theta_x}, (T_x, I_x, F_x) \rangle$.
2 Commutativity. If x_i is a permutation of y_i for all I $(i = 1, 2, \ldots, n)$, and then $INLGMSM^{(m,p_1,p_2,\ldots,p_m)}(x_1, x_2, \ldots, x_n) = INLGMSM^{(m,p_1,p_2,\ldots,p_m)}(y_1, y_2, \ldots, y_n)$.
3 Monotonicity. If $\alpha_i \leq \beta_i$, $T^L(x_i) \leq T^L(y_i)$, $T^U(x_i) \leq T^U(y_i)$, $I^L(x_i) \geq I^L(y_i)$, $I^U(x_i) \geq I^U(y_i)$, $F^L(x_i) \geq F^L(y_i)$ and $F^U(x_i) \geq F^U(y_i)$ for all i $(i = 1, 2, \ldots, n)$, then $x_i \leq y_i$ and $INLGMSM^{(m,p_1,p_2,\ldots,p_m)}(x_1, x_2, \ldots, x_n) \leq INLGMSM^{(m,p_1,p_2,\ldots,p_m)}(y_1, y_2, \ldots, y_n)$.
4 Boundedness. $\min\{x_1, x_2, \ldots x_n\} \leq INLGMSM^{(m,p_1,p_2,\ldots,p_m)}\{x_1, x_2, \ldots x_n\} \leq \max\{x_1, x_2, \ldots x_n\}$.

The proofs are similar to Property 1, which are now omitted.

Furthermore, the $INLGMSM^{(m,p_1,p_2,\ldots,p_m)}$ operator would degrade to some particular forms when m takes some special values.

(1) When $m = 1$, we have the following formula.

$$\begin{aligned} & INLGMSM^{(1)}(x_1, x_2, \ldots, x_n) = \left(\frac{\oplus_{i=1}^n x i^{P_1}}{C_n^1} \right)^{\frac{1}{P_1}} = \\ & \Bigg\langle s_{l \cdot \left(1 - \prod_{k=1}^n \left(1 - \left(\frac{k}{l}\right)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{P_1}}}, \left(\left[\left(1 - \prod_{k=1}^n \left(1 - (T^L{}_k)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{P_1}}, \left(1 - \prod_{k=1}^n \left(1 - (T^U{}_k)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{P_1}} \right], \right. \\ & \left[1 - \left(1 - \prod_{k=1}^n \left(1 - (1 - I^L i_1(k))^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}, 1 - \left(1 - \prod_{k=1}^n \left(1 - (1 - I^U i_1(k))^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}} \right], \\ & \left. \left[1 - \left(1 - \prod_{k=1}^n \left(1 - (1 - F^L i_1(k))^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}, 1 - \left(1 - \prod_{k=1}^n \left(1 - (1 - F^U i_1(k))^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}} \right] \right) \Bigg\rangle \end{aligned} \tag{36}$$

(2) When $m = 2$, we have the following formula.

$$\begin{aligned} & INLMSM^{(2)}(x_1, x_2, \ldots, x_n) = \\ & \Bigg\langle s_{l \cdot \left(1 - \prod_{k=1}^{C_n^2} \left(1 - \left(\frac{\theta i_1(k)}{l}\right)^{p_1} \cdot \left(\frac{\theta i_2(k)}{l}\right)^{p_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}}, \\ & \left(\left[\left(1 - \prod_{k=1}^{C_n^2} \left(1 - \left(T^L{}_{i_1(k)}\right)^{P_1} \cdot \left(T^L{}_{i_2(k)}\right)^{P_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}, \left(1 - \prod_{k=1}^{C_n^2} \left(1 - \left(T^U{}_{i_1(k)}\right)^{P_1} \cdot \left(T^U{}_{i_2(k)}\right)^{P_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}} \right] \right. \\ & \left[1 - \left(1 - \prod_{k=1}^{C_n^2} \left(1 - (1 - I^L i_1(k))^{P_1} \cdot (1 - I^L i_2(k))^{P_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}, \right. \\ & \left. 1 - \left(1 - \prod_{k=1}^{C_n^2} \left(1 - (1 - I^U i_1(k))^{P_1} \cdot (1 - I^U i_2(k))^{P_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}} \right], \\ & \left[1 - \left(1 - \prod_{k=1}^{C_n^2} \left(1 - (1 - F^L i_1(k))^{P_1} \cdot (1 - F^L i_2(k))^{P_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}, \right. \\ & \left. \left. 1 - \left(1 - \prod_{k=1}^{C_n^2} \left(1 - (1 - F^U i_1(k))^{P_1} \cdot (1 - F^U i_2(k))^{P_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}} \right] \right) \Bigg\rangle \end{aligned} \tag{37}$$

When $m = 2$, the $INLGMSM^{(m,p_1,p_2,\ldots,p_m)}$ operator would reduce to the BM for INLNs (INLGBM) operator.

(3) When $m = n$, the $INLMSM^{(m)}$ operator would reduce to the form below.

$$\begin{aligned} &INLGMSM^{(n,p_1,p_2,\ldots,p_m)}(a_1,\ldots,a_n) = \\ &\left\langle s_{l\cdot\left(\prod_{j=1}^{n}\left(\frac{\theta_j(k)}{l}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}}, \right. \\ &\left(\left[\left(\prod_{j=1}^{n}\left(T^L{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}, \left(\prod_{j=1}^{n}\left(T^U{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}\right],\right. \\ &\left[1-\left(\prod_{j=1}^{m}\left(1-I^L{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}, 1-\left(\prod_{j=1}^{m}\left(1-I^U{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}}\right], \\ &\left.\left.\left[1-\left(\prod_{j=1}^{m}\left(1-F^L{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}, 1-\left(\prod_{j=1}^{m}\left(1-F^U{}_{i_j(k)}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}\right]\right)\right\rangle \end{aligned} \quad (38)$$

4.2. Some Weighted INLMSM Operators

We will introduce two operators, which are the weighted forms of the *INLMSM* operator and *INLGMSM* operator.

Definition 15. *Let $a_i = \langle s_{\theta_i}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)])\rangle$ i $(i = 1,2,...,n)$ be a set of INLNs. Let $\omega = (\omega_1, \omega_2, ..., \omega_n)$ T is the weight vector and satisfies $\sum_{i=1}^{n}\omega_i = 1$ with $\omega_i > 0$ $(i = 1,2,...,n)$. Each ω_i represents the importance of a_i. Then the WINLMSM operator: $\Omega^n \to \Omega$ is defined below.*

$$WINLMSM^{(m)}(a_1,\ldots,a_n) = \left(\frac{\underset{1\le i_1<\ldots<i_m\le n}{\oplus}\left(\overset{m}{\underset{j=1}{\otimes}}\left(n\omega_{i_j}\right)a_{i_j}\right)}{C_n^m}\right)^{\frac{1}{m}}, \quad (39)$$

Ω is a set of INLNs and $m = 1,2,...,n$.

According to the operational laws of INLNs in Definition 10, we can get the expression of the WINLMSM operator, which is shown below.

Theorem 3. *Let $a_i = \langle s_{\theta_i}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)])\rangle$ i $(i = 1,2,...,n)$ be a set of INLNs and $m = 1,2,...,n$, then the value aggregated from Definition 15 is still a WINLMSM operator.*

$$\begin{aligned} &WINLMSM^{(m)}(a_1,\ldots,a_n) = \left\langle s_{l\cdot\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\frac{\theta i_j(k)}{l}\right)^{n\cdot\omega_{i_j}}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}}, \right. \\ &\left(\left[\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-T^L{}_{i_j(k)}\right)^{n\cdot\omega_{ij}}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}},\right.\right. \\ &\left.\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-T^U{}_{i_j(k)}\right)^{n\cdot\omega_{ij}}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right], \\ &\left[1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(I^L{}_{i_j(k)}\right)^{n\cdot\omega_{ij}}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}},\right. \\ &\left.1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(I^U{}_{i_j(k)}\right)^{n\cdot\omega_{ij}}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right], \\ &\left.\left.\left[1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(F^L{}_{i_j(k)}\right)^{n\cdot\omega_{ij}}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, 1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(F^U{}_{i_j(k)}\right)^{n\cdot\omega_{ij}}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\right]\right)\right\rangle \end{aligned} \quad (40)$$

where $k = 1,2,...,C_n^m$, $a_{i_j(k)}$ is the i_jth element of kth permutation. The process of proof is similar to Theorem 1. Now it is omitted.

Property 3. *Let* $x_i = \langle s_{\alpha_i}, ([T^L(x_i), T^U(x_i)], [I^L(x_i), I^U(x_i)], [F^L(x_i), F^U(x_i)]) \rangle$ $(i = 1, 2, ..., n)$ *and* $y_i = \langle s_{\beta_i}, ([T^L(y_i), T^U(y_i)], [I^L(y_i), I^U(y_i)], [F^L(y_i), F^U(y_i)]) \rangle$ $(i = 1, 2, ..., n)$ *be sets of INLNs. There are some properties of the* $WINLMSM^{(m)}$ *operator as shown below.*

1 *Reducibility. When* $\omega = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$ *, then* $WINLMSM^{(m)}(a_1, a_2, ..., a_n) = INLMSM^{(m)}(a_1, a_2, ..., a_n)$.
2 *Monotonicity. If* $\alpha_i \le \beta_i$, $T^L(x_i) \le T^L(y_i)$, $T^U(x_i) \le T^U(y_i)$, $I^L(x_i) \ge I^L(y_i)$, $I^U(x_i) \ge I^U(y_i)$, $F^L(x_i) \ge F^L(y_i)$ *and* $F^U(x_i) \ge F^U(y_i)$ *for all i* $(i = 1, 2, ..., n)$, *then* $x_i \le y_i$ *and* $WINLMSM^{(m)}(x_1, x_2, ..., x_n) \le WINLMSM^{(m)}(y_1, y_2, ..., y_n)$.
3 *Boundedness.* $\min\{x_1, x_2, ...x_n\} \le WINLMSM^{(m)}\{x_1, x_2, ...x_n\} \le \max\{x_1, x_2, ...x_n\}$.

Proof.

1 If $\omega = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$, then $WINLMSM^{(m)}(a_1, a_2, ..., a_n) =$

$$\Bigg\langle s_{l\cdot\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-\frac{\theta_{i_j(k)}}{l}\right)^{n\cdot\frac{1}{n}}\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}},$$
$$\Bigg(\Bigg[\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-T^L_{i_j(k)}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, \left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(1-T^U_{i_j(k)}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\Bigg],$$
$$\Bigg[1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(I^L_{i_j(k)}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, 1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(I^U_{i_j(k)}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\Bigg]$$
$$\Bigg[1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(F^L_{i_j(k)}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}, 1-\left(1-\prod_{k=1}^{C_n^m}\left(1-\prod_{j=1}^{m}\left(1-\left(F^U_{i_j(k)}\right)\right)\right)^{\frac{1}{C_n^m}}\right)^{\frac{1}{m}}\Bigg]\Bigg)\Bigg\rangle = INLMSM^{(m)}(a_1, a_2, ..., a_n).$$

2 The proofs of Monotonicity and Boundedness are similar to Property 1, which are now omitted.

Furthermore, the $WINLMSM^{(m)}$ operator would degrade a particular form when m takes some special values.

(1) When $m = 1$, we have the formula below.

$$\begin{gathered} WINLMSM^{(1)}(a_1, \ldots, a_n) = \\ \left\langle s_{l\cdot\left(1-\prod_{i=1}^{n}\left(1-\frac{\theta_i}{l}\right)^{\omega_i}\right)}, \left(\left[\left(1-\prod_{i=1}^{n}\left(1-T_i^L\right)^{\omega_i}\right), \left(1-\prod_{i=1}^{n}\left(1-T_i^U\right)^{\omega_i}\right)\right],\right.\right. \\ \left.\left.\left[\prod_{i=1}^{n}\left(I_i^L\right)^{\omega_i}, \prod_{i=1}^{n}\left(I_i^U\right)^{\omega_i}\right], \left[\prod_{i=1}^{n}\left(F_i^L\right)^{\omega_i}, \prod_{i=1}^{n}\left(F_i^U\right)^{\omega_i}\right]\right)\right\rangle \end{gathered} \tag{41}$$

(2) When $m = 2$, we have the formula below.

$$\begin{gathered} WINLMSM^{(2)}(a_1, \ldots, a_n) = \Bigg\langle s_{l\cdot\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(1-\frac{\theta_{i_1(k)}}{l}\right)^{n\cdot\omega_{i_1}}\right)\cdot\left(1-\left(1-\frac{\theta_{i_2(k)}}{l}\right)^{n\cdot\omega_{i_2}}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}}, \\ \Bigg(\Bigg[\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(1-T^L_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)\cdot\left(1-\left(1-T^L_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}, \\ \left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(1-T^U_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)\cdot\left(1-\left(1-T^U_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}\Bigg], \\ \Bigg[1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(I^L_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)\cdot\left(1-\left(I^L_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}, \\ 1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(I^U_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)\cdot\left(1-\left(I^U_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}\Bigg], \\ \Bigg[1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(F^L_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)\cdot\left(1-\left(F^L_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}, \\ 1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(F^U_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)\cdot\left(1-\left(F^U_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{2}}\Bigg]\Bigg)\Bigg\rangle \end{gathered} \tag{42}$$

(3) When $m = n$, we have the formula below.

$$\begin{aligned} WINLMSM^{(n)}(a_1,\ldots,a_n) = \Big\langle & s_{l\cdot(\prod_{j=1}^{n}(1-(1-\frac{\theta_j}{l})^{n\cdot\omega_j}))^{\frac{1}{n}}}, \\ & \Bigg(\Bigg[\Bigg(\prod_{j=1}^{n}\Big(1-\big(1-T_j^L\big)^{n\cdot\omega_j}\Big)\Bigg)^{\frac{1}{n}}, \Bigg(\prod_{j=1}^{n}\Big(1-\big(1-T_j^U\big)^{n\cdot\omega_j}\Big)\Bigg)^{\frac{1}{n}}\Bigg], \\ & \Bigg[1-\Bigg(\prod_{j=1}^{n}\Big(1-\big(I_j^L\big)^{n\cdot\omega_j}\Big)\Bigg)^{\frac{1}{n}}, 1-\Bigg(\prod_{j=1}^{n}\Big(1-\big(I_j^U\big)^{n\cdot\omega_j}\Big)\Bigg)^{\frac{1}{n}}\Bigg], \\ & \Bigg[1-\Bigg(\prod_{j=1}^{n}\Big(1-\big(F_j^L\big)^{n\cdot\omega_j}\Big)\Bigg)^{\frac{1}{n}}, 1-\Bigg(\prod_{j=1}^{n}\Big(1-\big(F_j^U\big)^{n\cdot\omega_j}\Big)\Bigg)^{\frac{1}{n}}\Bigg]\Bigg)\Big\rangle. \end{aligned} \tag{43}$$

□

Definition 16. *Let $a_i = \langle s_{\theta_i}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)])\rangle$ i $(i = 1, 2, \ldots, n)$ be a set of INLNs. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ T is the weight vector and it satisfies $\sum_{i=1}^{n}\omega_i = 1$ with $\omega_i > 0$ $(i = 1, 2, \ldots, n)$. Each ω_i represents the importance of a_i. Then the WINLGMSM operator: $\Omega^n \to \Omega$ is defined below.*

$$WINLGMSM^{(m,p_1,p_2,\ldots p_m)}(a_1,\ldots,a_n) = \left(\frac{\bigoplus_{1\le i_1<\ldots<i_m\le n}\left(\bigotimes_{j=1}^{m}\left(n\omega_{i_j}\cdot a_{i_j}\right)^{p_j}\right)}{C_n^m}\right)^{\frac{1}{p_1+p_2+\ldots+p_m}} \tag{44}$$

Ω is a set of INLNs and $m = 1, 2, \ldots, n$.

According to the operational laws of INLNs in Definition 10, we can get the expression of WINLMSM operator shown below.

Theorem 4. *Let $a_i = \langle s_{\theta_i}, ([T^L(a_i), T^U(a_i)], [I^L(a_i), I^U(a_i)], [F^L(a_i), F^U(a_i)])\rangle$ i $(i = 1, 2, \ldots, n)$ be a set of INLNs and $m = 1, 2, \ldots, n$. Then the value aggregated from Definition 16 is still an WINLGMSM.*

$$\begin{aligned} WINLGMSM^{(m,p_1,p_2,\ldots,p_m)}(a_1,\ldots,a_n) = \Big\langle & s_{l\cdot(1-\prod_{k=1}^{C_n^m}(1-\prod_{j=1}^{m}(1-(1-\frac{\theta_{i_j(k)}}{l})^{n\cdot\omega_{i_j}})^{p_j})^{\frac{1}{C_n^m}})^{\frac{1}{p_1+p_2+\ldots+p_m}}}, \\ & \Bigg(\Bigg[\Bigg(1-\prod_{k=1}^{C_n^m}\Bigg(1-\prod_{j=1}^{m}\Big(1-\big(1-T^L{i_j}(k)\big)^{n\cdot\omega_{i_j}}\Big)^{p_j}\Bigg)^{\frac{1}{C_n^m}}\Bigg)^{\frac{1}{p_1+p_2+\ldots+p_m}}, \\ & \Bigg(1-\prod_{k=1}^{C_n^m}\Bigg(1-\prod_{j=1}^{m}\Big(1-\big(1-T^U{i_j}(k)\big)^{n\cdot\omega_{i_j}}\Big)^{p_j}\Bigg)^{\frac{1}{C_n^m}}\Bigg)^{\frac{1}{p_1+p_2+\ldots+p_m}}\Bigg], \\ & \Bigg[1-\Bigg(1-\prod_{k=1}^{C_n^m}\Bigg(1-\prod_{j=1}^{m}\Big(1-\big(I^L{i_j}(k)\big)^{n\cdot\omega_{i_j}}\Big)^{p_j}\Bigg)^{\frac{1}{C_n^m}}\Bigg)^{\frac{1}{p_1+p_2+\ldots+p_m}}, \\ & 1-\Bigg(1-\prod_{k=1}^{C_n^m}\Bigg(1-\prod_{j=1}^{m}\Big(1-\big(I^U{i_j}(k)\big)^{n\cdot\omega_{i_j}}\Big)^{p_j}\Bigg)^{\frac{1}{C_n^m}}\Bigg)^{\frac{1}{p_1+p_2+\ldots+p_m}}\Bigg], \\ & \Bigg[1-\Bigg(1-\prod_{k=1}^{C_n^m}\Bigg(1-\prod_{j=1}^{m}\Big(1-\big(F^L{i_j}(k)\big)^{n\cdot\omega_{i_j}}\Big)^{p_j}\Bigg)^{\frac{1}{C_n^m}}\Bigg)^{\frac{1}{p_1+p_2+\ldots+p_m}}, \\ & 1-\Bigg(1-\prod_{k=1}^{C_n^m}\Bigg(1-\prod_{j=1}^{m}\Big(1-\big(F^U{i_j}(k)\big)^{n\cdot\omega_{i_j}}\Big)^{p_j}\Bigg)^{\frac{1}{C_n^m}}\Bigg)^{\frac{1}{p_1+p_2+\ldots+p_m}}\Bigg]\Bigg)\Big\rangle \end{aligned} \tag{45}$$

where $k = 1, 2, \ldots, C_n^m$, $a_{i_j(k)}$ is the i_jth element of kth permutation. The process of proof is similar to Theorem 1. It is now omitted.

Property 4. *Let* $x_i = \langle s_{\alpha_i}, ([T^L(x_i), T^U(x_i)], [I^L(x_i), I^U(x_i)], [F^L(x_i), F^U(x_i)]) \rangle$ $(i = 1,2,...,n)$ *and* $y_i = \langle s_{\beta_i}, ([T^L(y_i), T^U(y_i)], [I^L(y_i), I^U(y_i)], [F^L(y_i), F^U(y_i)]) \rangle$ $(i = 1,2,...,n)$ *be two sets of INLNs. There are some properties of the* $WINLGMSM^{(m,p_1,p_2,...,p_m)}$ *operator shown below.*

1 Reducibility. When $\omega = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})$. Additionally, $WINLGMSM^{(m,p_1,p_2,...,p_m)}(a_1, a_2, ..., a_n) = INLGMSM^{(m,p_1,p_2,...,p_m)}(a_1, a_2, ..., a_n)$.
2 Monotonicity. If $\alpha_i \le \beta_i$, $T^L(x_i) \le T^L(y_i)$, $T^U(x_i) \le T^U(y_i)$, $I^L(x_i) \ge I^L(y_i)$, $I^U(x_i) \ge I^U(y_i)$, $F^L(x_i) \ge F^L(y_i)$ and $F^U(x_i) \ge F^U(y_i)$ for all i $(i = 1,2,...,n)$, then $x_i \le y_i$ and $WINLGMSM^{(m,p_1,p_2,...,p_m)}(x_1, x_2, ..., x_n) \le WINLGMSM^{(m,p_1,p_2,...,p_m)}(y_1, y_2, ..., y_n)$.
3 Boundedness. $\min\{x_1, x_2, ...x_n\} \le WINLGMSMSM^{(m,p_1,p_2,...,p_m)}\{x_1, x_2, ...x_n\} \le \max\{x_1, x_2, ...x_n\}$.

The process of proof is similar to Property 3 and is now omitted.

Furthermore, the $WINLGMSM^{(m,p_1,p_2,...,p_m)}$ operator would degrade some particular forms when m takes some special values.

(1) When $m = 1$, we have the following formula.

$$\begin{aligned} &WINLGMSM^{(1,p_1)}(a_1, \ldots, a_n) = \Big\langle s_{l\cdot\left(1-\prod_{k=1}^{n}\left(1-\left(1-\left(1-\frac{\theta_{1}(k)}{l}\right)^{n\cdot\omega_1}\right)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}}, \\ &\Bigg(\Bigg[\left(1-\prod_{k=1}^{n}\left(1-\left(1-\left(1-T^L_{i_j(k)}\right)^{n\cdot\omega_1}\right)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}, \left(1-\prod_{k=1}^{n}\left(1-\left(1-\left(1-T^U_{i_j(k)}\right)^{n\cdot\omega_1}\right)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}\Bigg], \\ &\Bigg[1-\left(1-\prod_{k=1}^{n}\left(1-\left(1-\left(I^L_{i_j(k)}\right)^{n\cdot\omega_1}\right)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}, 1-\left(1-\prod_{k=1}^{n}\left(1-\left(1-\left(I^U_{i_j(k)}\right)^{n\cdot\omega_1}\right)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}\Bigg], \\ &\Bigg[1-\left(1-\prod_{k=1}^{n}\left(1-\left(1-\left(F^L_{i_j(k)}\right)^{n\cdot\omega_1}\right)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}, 1-\left(1-\prod_{k=1}^{n}\left(1-\left(1-\left(F^U_{i_j(k)}\right)^{n\cdot\omega_1}\right)^{p_1}\right)^{\frac{1}{n}}\right)^{\frac{1}{p_1}}\Bigg]\Bigg)\Big\rangle \end{aligned} \tag{46}$$

(2) When $m = 2$, we have the formula below.

$$\begin{aligned} &WINLGMSM^{(2,p_1,p_2)}(a_1, \ldots, a_n) = \Big\langle s_{l\cdot\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(1-\frac{\theta_{i_1}(k)}{l}\right)^{n\cdot\omega_{i_1}}\right)^{p_1}\cdot\left(1-\left(1-\frac{\theta_{i_2}(k)}{l}\right)^{n\cdot\omega_{i_2}}\right)^{p_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}}, \\ &\Bigg(\Bigg[\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(1-T^L_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)^{p_1}\cdot\left(1-\left(1-T^L_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)^{p_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}, \\ &\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(1-T^U_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)^{p_1}\cdot\left(1-\left(1-T^U_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)^{p_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}\Bigg], \\ &\Bigg[1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(I^L_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)^{p_1}\cdot\left(1-\left(I^L_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)^{p_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}, \\ &1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(I^U_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)^{p_1}\cdot\left(1-\left(I^U_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)^{p_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}\Bigg], \\ &\Bigg[1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(F^L_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)^{p_1}\cdot\left(1-\left(F^L_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)^{p_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}, \\ &1-\left(1-\prod_{k=1}^{C_n^2}\left(1-\left(1-\left(F^U_{i_1(k)}\right)^{n\cdot\omega_{i_1}}\right)^{p_1}\cdot\left(1-\left(F^U_{i_2(k)}\right)^{n\cdot\omega_{i_2}}\right)^{p_2}\right)^{\frac{1}{C_n^2}}\right)^{\frac{1}{p_1+p_2}}\Bigg]\Bigg)\Big\rangle \end{aligned} \tag{47}$$

(3) When $m = n$, we have the formula below.

$$
\begin{aligned}
&WINLGMSM^{(n,p_1,p_2,\ldots,p_n)}(a_1,\ldots,a_n) = \Bigg\langle s_{l\cdot\left(\prod_{j=1}^{n}\left(1-\left(1-\frac{\theta_j}{l}\right)^{n\cdot\omega_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}}, \\
&\Bigg(\Bigg[\left(\prod_{j=1}^{n}\left(1-\left(1-T_j^L\right)^{n\cdot\omega_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}, \left(\prod_{j=1}^{n}\left(1-\left(1-T_j^U\right)^{n\cdot\omega_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}\Bigg], \\
&\Bigg[1-\left(\prod_{j=1}^{n}\left(1-\left(I_j^L\right)^{n\cdot\omega_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}, 1-\left(\prod_{j=1}^{n}\left(1-\left(I_j^U\right)^{n\cdot\omega_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}\Bigg], \\
&\Bigg[1-\left(\prod_{j=1}^{n}\left(1-\left(F_j^L\right)^{n\cdot\omega_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}, 1-\left(\prod_{j=1}^{n}\left(1-\left(F_j^U\right)^{n\cdot\omega_j}\right)^{p_j}\right)^{\frac{1}{p_1+p_2+\ldots+p_n}}\Bigg]\Bigg)\Bigg\rangle
\end{aligned} \tag{48}
$$

5. MADM Method Based on INLMSM Operator

In this section, we introduce the MADM method based on the *WINLMSM* and *WINLGMSM* operators. Let $d = \{d_1, d_2, \ldots, d_m\}$ be a collection of alternatives and $c = \{c_1, c_2, \ldots, c_n\}$ is a collection of n criteria. The weight vector is $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with satisfying $\sum_{i=1}^{n} \omega_i = 1(\omega_i \geq 0,\ i = 1, 2, \ldots, n)$, and each ω_i represents the importance of c_j. The performance of alternative d_j in criteria c_j is surveyed by INLNs and the decision matrix is $A = (a_{ij})_{m\times n}$, where $a_{\text{ij}} = \langle s_{\theta_{ij}}, ([T^L(r_{\text{ij}}), T^U(r_{\text{ij}})], [I^L(r_{\text{ij}}), I^U(r_{\text{ij}})], [F^L(r_{\text{ij}}), F^U(r_{\text{ij}})])\rangle$. The objective is to rank the alternatives.

The detailed steps are shown below.

Step 1 Normalize the decision matrix.

We should normalize the decision-making information in the matrix. The benefit (the bigger the better) and the cost (the smaller the better) are the two possible types. In order to keep the consistency of the types, it is necessary to convert the decision matrix A into a standardized matrix $R = (r_{\text{ij}})m \times n$.

If c_j is cost type, then $r_{\text{ij}} = \langle s_{\theta_{ij}}, ([F^L(r_{\text{ij}}), F^U(r_{\text{ij}})], [1 - I^U(r_{\text{ij}}), 1 - I^L(r_{\text{ij}})][T^L(r_{\text{ij}}), T^U(r_{\text{ij}})])\rangle$

else $r_{\text{ij}} = \langle s_{\theta_{ij}}, ([T^L(r_{\text{ij}}), T^U(r_{\text{ij}})], [I^L(r_{\text{ij}}), I^U(r_{\text{ij}})], [F^L(r_{\text{ij}}), F^U(r_{\text{ij}})])\rangle$.

Step 2 Aggregate the criterion values of each alternative. We would use Definition 15 and Definition 16 to aggregate $r_{ij}(j = 1, 2, \ldots, n)$ of the ith alternative and get the overall value r_i.

Step 3 Calculate the score values of $r_i(i = 1, 2, \ldots, m)$ according to Definition 11. If two score values are equal, then calculate the accuracy values and certainty values.

Step 4 According to Step 3 and Definition 12, rank the alternatives.

6. Illustrative Example

There are many decision-making problems to be solved in the current society, which requires some decision-making methods.

In this section, we investigate an example (adapted from Ref [43]) about the MADM. In a MADM problem, there are four possible alternatives for an investment company including a car company (A_1), a food company (A_2), a computer company (A_3), and an arms company (A_4). The following three attributes can be used to evaluate alternatives by the investment company: the risk (C_1), the growth (C_2), and the environmental impact (C_3) where C_1 and C_2 are benefit types and C_3 is cost type. Then the evaluation values of alternatives are shown in Table 1 where the LTS is $S = \{s_0 = extremely\ poor(EP),\ s_1 = very\ poor(VP),\ s_2 = poor(P),\ s_3 = medium(M),\ s_4 = good(G),\ s_5 = very\ good(VG),\ s_6 = extremely\ good(EG)\}$, and the weight vector of criteria is $\omega = (0.35, 0.25, 0.4)^T$. Now we will use the method proposed in this paper, according to the above LTs and three criteria. Then we evaluate and sort the four options in Table 1.

Table 1. Evaluation values of alternatives.

	C_1	C_2	C_3
A_1	$\langle s_5, ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4])\rangle$	$\langle s_6, ([0.4, 0.6], [0.1, 0.2], [0.2, 0.4])\rangle$	$\langle s_5, ([0.2, 0.3], [0.1, 0.2], [0.5, 0.6])\rangle$
A_2	$\langle s_6, ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle$	$\langle s_5, ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle$	$\langle s_5, ([0.5, 0.7], [0.2, 0.2], [0.1, 0.2])\rangle$
A_3	$\langle s_6, ([0.3, 0.5], [0.1, 0.2], [0.3, 0.4])\rangle$	$\langle s_5, ([0.5, 0.6], [0.1, 0.3], [0.3, 0.4])\rangle$	$\langle s_4([0.5, 0.6], [0.1, 0.3], [0.1, 0.3])\rangle$
A_4	$\langle s_4, ([0.7, 0.8], [0.0, 0.1], [0.1, 0.2])\rangle$	$\langle s_4, ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle$	$\langle s_6([0.3, 0.4], [0.1, 0.2], [0.1, 0.2])\rangle$

6.1. The Method Based on the WINLMSM Operator

Generally, we can give $m = \frac{n}{2}$, so $m = 1$ and $m = 2$. Then, according to Section 5, we have the statements below.

(1) When $m = 1$, the steps are shown below.

Step 1 Normalize the decision matrix.

From the example, the risk (C_1) and the growth (C_2) are benefit types while the environmental impact (C_3) is cost type. We set up the decision matrix as shown below.

$$R = \begin{bmatrix} \langle s_5, ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4])\rangle & \langle s_6, ([0.4, 0.6], [0.1, 0.2], [0.2, 0.4])\rangle & \langle s_5, ([0.2, 0.3], [0.1, 0.2], [0.5, 0.6])\rangle \\ \langle s_6, ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle & \langle s_5, ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle & \langle s_5, ([0.5, 0.7], [0.2, 0.2], [0.1, 0.2])\rangle \\ \langle s_6, ([0.3, 0.5], [0.1, 0.2], [0.3, 0.4])\rangle & \langle s_5, ([0.5, 0.6], [0.1, 0.3], [0.3, 0.4])\rangle & \langle s_4([0.5, 0.6], [0.1, 0.3], [0.1, 0.3])\rangle \\ \langle s_4, ([0.7, 0.8], [0.0, 0.1], [0.1, 0.2])\rangle & \langle s_4, ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle & \langle s_6([0.3, 0.4], [0.1, 0.2], [0.1, 0.2])\rangle \end{bmatrix}$$

Step 2 Aggregate all attribute values of each alternative and get the overall value of each alternative a_i denoted as $r_i (i = 1, 2, 3, 4)$.

$$\begin{aligned} r_1 &= \langle s_6, ([0.3268, 0.4590], [0.1275, 0.2305], [0.3325, 0.4704])\rangle, \\ r_2 &= \langle s_6, ([0.5271, 0.7000], [0.1320, 0.2000], [0.1516, 0.2551])\rangle, \\ r_3 &= \langle s_6, ([0.4375, 0.5675], [0.1000, 0.2603], [0.1933, 0.3565])\rangle, \\ r_4 &= \langle s_6, ([0.5216, 0.6565], [0.0000, 0.1569], [0.1189, 0.2213])\rangle \end{aligned}$$

Step 3 According to Definition 11, we assume $\alpha = 0.7$ and calculate the score values of $r_i (i = 1, 2, 3, 4)$ below.

$$S_{(r_1)} = s_{0.6228}, S_{(r_2)} = s_{0.8306}, S_{(r_3)} = s_{0.7462}, S_{(r_4)} = s_{0.7778}$$

Step 4 According to Step 3 and Definition 12, we would get the ranking of the alternatives, which are $A_2 \succ A_4 \succ A_3 \succ A_1$.

(2) When $m = 2$, the steps are shown below.

Step 1 Normalize the decision matrix.

From the example, the risk (C_1) and the growth (C_2) are benefit types while the environmental impact (C_3) is cost type. We set up the decision matrix as shown below.

$$R = \begin{bmatrix} \langle s_5, ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4])\rangle & \langle s_6, ([0.4, 0.6], [0.1, 0.2], [0.2, 0.4])\rangle & \langle s_5, ([0.2, 0.3], [0.1, 0.2], [0.5, 0.6])\rangle \\ \langle s_6, ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle & \langle s_5, ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle & \langle s_5, ([0.5, 0.7], [0.2, 0.2], [0.1, 0.2])\rangle \\ \langle s_6, ([0.3, 0.5], [0.1, 0.2], [0.3, 0.4])\rangle & \langle s_5, ([0.5, 0.6], [0.1, 0.3], [0.3, 0.4])\rangle & \langle s_4([0.5, 0.6], [0.1, 0.3], [0.1, 0.3])\rangle \\ \langle s_4, ([0.7, 0.8], [0.0, 0.1], [0.1, 0.2])\rangle & \langle s_4, ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3])\rangle & \langle s_6([0.3, 0.4], [0.1, 0.2], [0.1, 0.2])\rangle \end{bmatrix}$$

Step 2 Aggregate all attribute values of each alternative and get the overall value of each alternative a_i denoted as $r_i(i = 1,2,3,4)$.

$$\begin{aligned} r_1 &= \langle s_{5.4841}, ([0.3190, 0.4520], [0.1406, 0.2420], [0.3391, 0.4771]) \rangle, \\ r_2 &= \langle s_{5.3016}, ([0.5260, 0.6922], [0.1366, 0.2083], [0.1791, 0.2772]) \rangle, \\ r_3 &= \langle s_{4.9567}, ([0.4224, 0.5587], [0.1077, 0.2741], [0.2494, 0.3754]) \rangle, \\ r_4 &= \langle s_{4.4896}, ([0.4794, 0.6190], [0.0711, 0.1739], [0.1415, 0.2416]) \rangle \end{aligned}$$

Step 3 According to Definition 11, we assume $\alpha = 0.7$ and calculate the score values of $r_i(i = 1,2,3,4)$. We get the values below.

$$S_{(r_1)} = s_{0.5695}, S_{(r_2)} = s_{0.7170}, S_{(r_3)} = s_{0.6004}, S_{(r_4)} = s_{0.5765}.$$

Step 4 According to Step 3 and Definition 12, we get the ranking of the alternatives below.

$$A_2 \succ A_3 \succ A_4 \succ A_1$$

6.2. The Method Based on the WINLGMSM Operator

When $m = 1, p = 1$, the $WINLGMSM^{(1)}$ operator is the same as the $WINLMSM^{(1)}$ operator. The steps are omitted here. When $m = 2$, the steps are below.

Step 1 Normalize the decision matrix.

From the example, the risk (C_1), the growth (C_2) are benefit types and the environmental impact (C_3) is cost type, so we set up the matrix as step 1 of Section 6.1.

Step 2 Aggregate all attribute values of each alternative by the $WINLMSM^{(2)}$ operator and get the overall value of each alternative a_i denoted as $r_i(i = 1,2,3,4)$

$$\begin{aligned} r_1 &= \langle s_{5.4988}, ([0.3221, 0.4549], [0.1387, 0.2401], [0.3374, 0.4752]) \rangle, \\ r_2 &= \langle s_{5.3735}, ([0.5264, 0.6938], [0.1358, 0.2070], [0.1772, 0.2745]) \rangle, \\ r_3 &= \langle s_{5.0083}, ([0.4296, 0.5610], [0.1069, 0.2702], [0.2449, 0.3721]) \rangle, \\ r_4 &= \langle s_{4.5371}, ([0.4892, 0.6244], [0.0634, 0.1698], [0.1390, 0.2381]) \rangle \end{aligned}$$

Step 3 According to Definition 11, we assume $\alpha = 0.7$, calculate the score values of $r_i(i = 1,2,3,4)$, and get the values shown below.

$$S_{(r_1)} = s_{0.5722}, S_{(r_2)} = s_{0.7276}, S_{(r_3)} = s_{0.6087}, S_{(r_4)} = s_{0.5839}$$

Step 4 According to Step 3 and Definition 12, we get the rankings of the alternatives, which are shown below.

$$A_2 \succ A_3 \succ A_4 \succ A_1$$

6.3. Comparative Analysis and Discussion

(1) Based on the results in Sections 6.1 and 6.2, we can show them by using Table 2. From Table 2, we know that there are the same ranking results in two methods when m = 1 or $m = 2$. However, the result when $m = 1$ is different from the one when $m = 2$. It can be explained that, when $m = 1$, the interrelationship between the attributes doesn't need to be considered when $m = 2$. We can consider the interrelationship between two attributes.

(2) Furthermore, we get the comparisons for different values of P_1 and P_2 when $m = 2$, which are shown in Table 3. From Table 3, we know when m = 2 and P_1 and P_2 are not equal to zero, we can get the same ranking results, i.e., $A_2 \succ A_4 \succ A_3 \succ A_1$. However, when $P_1 = 0$ or $P_2 = 0$, the ranking results are different from the ones when P_1 and P_2 are not equal to zero. When $P_1 = 0$

or $P_2 = 0$, the interrelationship between the attributes doesn't need to be considered, so it can get the same ranking results as the ones when $m = 1$.

Table 2. Comparison of different operator.

Operator	m	P_1	P_2	Ranking
$WINLMSM^{(m)}$	1	-	-	$A_2 \succ A_4 \succ A_3 \succ A_1$
	2	-	-	$A_2 \succ A_3 \succ A_4 \succ A_1$
$WINLGMSM^{(m,p_1,p_2)}$	1	1	-	$A_2 \succ A_4 \succ A_3 \succ A_1$
	2	1	2	$A_2 \succ A_3 \succ A_4 \succ A_1$

Table 3. Comparisons of different values of P_1 and P_2 when m = 2.

Operator	P_1	P_2	$S_{(r_i)}(i = 1, 2, 3, 4)$	Ranking
	0	1	$S_{(r_1)} = s_{0.6228}$ $S_{(r_2)} = s_{0.8306}$ $S_{(r_3)} = s_{0.7462}$ $S_{(r_4)} = s_{0.7778}$	$A_2 \succ A_4 \succ A_3 \succ A_1$
	1	0	$S_{(r_1)} = s_{0.6228}$ $S_{(r_2)} = s_{0.8306}$ $S_{(r_3)} = s_{0.7462}$ $S_{(r_4)} = s_{0.7778}$	$A_2 \succ A_4 \succ A_3 \succ A_1$
	1	1	$S_{(r_1)} = s_{0.5695}$ $S_{(r_2)} = s_{0.7170}$ $S_{(r_3)} = s_{0.6004}$ $S_{(r_4)} = s_{0.5765}$	$A_2 \succ A_3 \succ A_4 \succ A_1$
	1	2	$S_{(r_1)} = s_{0.5722}$ $S_{(r_2)} = s_{0.7276}$ $S_{(r_3)} = s_{0.6087}$ $S_{(r_4)} = s_{0.5839}$	$A_2 \succ A_3 \succ A_4 \succ A_1$
	1	3	$S_{(r_1)} = s_{0.5769}$ $S_{(r_2)} = s_{0.7387}$ $S_{(r_3)} = s_{0.6227}$ $S_{(r_4)} = s_{0.6027}$	$A_2 \succ A_3 \succ A_4 \succ A_1$
$WINLGMSM^{(m,p_1,p_2)}$	2	1	$S_{(r_1)} = s_{0.5745}$ $S_{(r_2)} = s_{0.7199}$ $S_{(r_3)} = s_{0.6116}$ $S_{(r_4)} = s_{0.6004}$	$A_2 \succ A_3 \succ A_4 \succ A_1$
	2	2	$S_{(r_1)} = s_{0.5717}$ $S_{(r_2)} = s_{0.7196}$ $S_{(r_3)} = s_{0.6037}$ $S_{(r_4)} = s_{0.5837}$	$A_2 \succ A_3 \succ A_4 \succ A_1$
	2	3	$S_{(r_1)} = s_{0.5733}$ $S_{(r_2)} = s_{0.7256}$ $S_{(r_3)} = s_{0.6079}$ $S_{(r_4)} = s_{0.5859}$	$A_2 \succ A_3 \succ A_4 \succ A_1$
	3	1	$S_{(r_1)} = s_{0.5806}$ $S_{(r_2)} = s_{0.7276}$ $S_{(r_3)} = s_{0.6269}$ $S_{(r_4)} = s_{0.6280}$	$A_2 \succ A_4 \succ A_3 \succ A_1$
	3	2	$S_{(r_1)} = s_{0.5751}$ $S_{(r_2)} = s_{0.7206}$ $S_{(r_3)} = s_{0.6101}$ $S_{(r_4)} = s_{0.5997}$	$A_2 \succ A_3 \succ A_4 \succ A_1$
	3	3	$S_{(r_1)} = s_{0.5741}$ $S_{(r_2)} = s_{0.7223}$ $S_{(r_3)} = s_{0.6071}$ $S_{(r_4)} = s_{0.5909}$	$A_2 \succ A_3 \succ A_4 \succ A_1$

Furthermore, in order to verify the validity of the methods proposed in this paper, we can compare them with methods from Ye [16] and the ranking results are shown in Table 4.

From Table 4, we know that the best choice is A_2 for all methods, which is the same as the results produced above. However, the ranking results are different. Compared with the approach proposed by Ye [16], when $m = 1$, our ranking results have the same values as that of Ye [16], but when $m = 2$, our ranking results are different from the Ye method [16]. When $m = 1$, all methods don't consider the interrelationship. They produce the same results, however, when $m = 2$. Our methods in this paper can take into account the interrelationship while the method by Ye [16] doesn't consider the interrelationship. Therefore, there are different ranking results. Therefore, our methods are more suitable for the different applications.

Table 4. Comparison of different methods.

Methods	Operator	Ranking
Methods in this paper	$WINLMSM^{(m)} m = 1$	$A_2 \succ A_4 \succ A_3 \succ A_1$
	$WINLMSM^{(m)} m = 2$	$A_2 \succ A_3 \succ A_4 \succ A_1$
	$WINLGMSM^{(m,p_1,p_2)}\ m = 1$	$A_2 \succ A_4 \succ A_3 \succ A_1$
	$WINLGMSM^{(m,p_1,p_2)}\ m = 2$	$A_2 \succ A_3 \succ A_4 \succ A_1$
Method in [16]	*INLWAA*	$A_2 \succ A_4 \succ A_3 \succ A_1$
	INLWGA	$A_2 \succ A_4 \succ A_3 \succ A_1$

From the above comparison results, we can obtain that the methods proposed by this paper are feasible and adaptable for the MADM problems. Additionally, they have better reliability and wider application space than other existing methods.

7. Conclusions

In this study, we propose the concept of *INLMSM*, which can not only adapt to the cognitive situation of decision maker, but also provide convenience for decision making. We introduce the basic concept of *INLMSM* and its generalized form, give some operators based on *INLMSM*, and introduce the theory of weight to investigate *WINLMSM* and *WINLGMSM*. Afterwards, we put forward the *INLMSM* operator, the *INLGMSM* operator, the *WINLMSM* operator, and the *WINLGMSM* operator. In addition, we proved these operators. In addition, we introduce the MADM methods with *INLMSM* in detail and illustrate their usefulness and effectiveness by showing examples. Finally, we compare other methods to demonstrate our approach. From this paper, we can see that *WINLGMSM* is more practical and flexible in application and *INLMSM* can express fuzzy information more conveniently. In further study, we can use the *INLMSM* operator to solve practical problems and pattern recognition. We should develop other aggregation operators for future research.

Acknowledgments: This paper is supported by the National Natural Science Foundation of China (Nos. 71771140 and 71471172), the Special Funds of Taishan Scholars Project of Shandong Province (No. ts201511045), Shandong Provincial Social Science Planning Project (Nos. 17BGLJ04,16CGLJ31 and 16CKJJ27), the Natural Science Foundation of Shandong Province (Nos. ZR2013FM017, ZR2017MG007), and the Teaching Reform Research Project of Undergraduate Colleges and Universities in Shandong Province (2015Z057).

Author Contributions: Yushui Geng proposed extended MSM operators for INLNs; Xingang Wang designed the experiments; Xuemei Li and Kun Yu analyzed the data, and wrote the paper; Peide Liu proposed the ideas of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Compliance with Ethical Standards: (1) Disclosure of potential conflicts of interest. We declare that we have no commercial or associative interests that represent a conflict of interest in connection with this manuscript. There are no professional or other personal interests that can inappropriately influence our submitted work. (2) Research involving human participants and/or animals. This article does not contain any studies with human participants or animals performed by any of the authors.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–356. [CrossRef]
2. Atanassov, T.K. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96. [CrossRef]
3. Abbas, S.E. On intuitionistic fuzzy compactness. *Inf. Sci.* **2005**, *173*, 75–91. [CrossRef]
4. Zadeh, L.A. Similarity relations and fuzzy ordering. *Inf. Sci.* **1971**, *3*, 177–200. [CrossRef]
5. Zwick, R.; Carlstein, E.; Budescu, D.V. Measures of similarity among fuzzy concepts: A comparative analysis. *Int. J. Approx. Reason.* **1987**, *1*, 221–242. [CrossRef]
6. Zeng, W.; Li, H. Relationship between similarity measure and entropy of interval-valued fuzzy sets. *Fuzzy Sets Syst.* **2006**, *157*, 1477–1484. [CrossRef]
7. Smarandache, F. *Neutrosophy: Neutrosophic Probability, Set, and Logic: Analytic Synthesis and Synthetic Analysis*; ProQuest Information & Learning: Ann Harbor, MI, USA, 1998; p. 105.
8. Smarandache, F. A unifying field in logic: Neutrsophic logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. *J. Mol. Biol.* **2001**, *332*, 489–503.
9. Smarandache, F. A geometric interpretation of the neutrosophic set—A generalization of the intuitionistic fuzzy set. *IEEE Int. Conf. Granul. Comput.* **2003**, *28*, 1496.
10. Wang, H.; Smarandache, F.; Zhang, Y.Q.; Sunderraman, R. *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*; ProQuest Information & Learning: Ann Harbor, MI, USA, 2005.
11. Broumi, S.; Smarandache, F. Correlation Coefficient of Interval Neutrosophic Set. *Appl. Mech. Mater.* **2013**, *436*, 511–517. [CrossRef]
12. Herrera, F.; Herrera-Viedma, E. Aggregation operators for linguistic weighted information. *IEEE Trans. Syst. Man Cybern. Part A Syst. Hum.* **1997**, *27*, 646–656. [CrossRef]
13. Chen, Z.C.; Liu, P.H. An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. *Int. J. Comput. Intell. Syst.* **2015**, *8*, 747–760. [CrossRef]
14. Ye, J. An extended TOPSIS method for multiple attribute group decision making based on single valued neutrosophic linguistic numbers. *J. Intell. Fuzzy Syst.* **2015**, *28*, 247–255.
15. Wang, J.Q.; Yang, Y.; Li, L. Multi-criteria decision-making method based on single-valued neutrosophic linguistic Maclaurin symmetric mean operators. *Neural Comput. Appl.* **2016**, *2016*, 1–19. [CrossRef]
16. Ye, J. Some aggregation operators of interval neutrosophic linguistic numbers for multiple attribute decision making. *J. Intell. Fuzzy Syst.* **2014**, *27*, 2231–2241.
17. Liu, P.; Chen, S.M. Multiattribute Group Decision Making Based on Intuitionistic 2-Tuple Linguistic Information. *Inf. Sci.* **2018**, *430–431*, 599–619. [CrossRef]
18. Liu, P.; Wang, P. Some q-Rung Orthopair Fuzzy Aggregation Operators and Their Applications to Multiple-Attribute Decision Making. *Int. J. Intell. Syst.* **2018**, *33*, 259–280. [CrossRef]
19. Yager, R.R. On ordered weighted averaging aggregation operators in multicriteria decisionmaking. *IEEE Trans. Syst. Man Cybern.* **1988**, *18*, 183–190. [CrossRef]
20. Bonferroni, C. Sulle medie multiple di potenze. *Boll. Mater. Ital.* **1950**, *5*, 267–270.
21. Liu, P.; Liu, J.; Chen, S.M. Some Intuitionistic Fuzzy Dombi Bonferroni Mean Operators and Their Application to Multi-Attribute Group Decision Making. *J. Oper. Res. Soc.* **2018**, *69*, 1–24. [CrossRef]
22. Liu, P.; Chen, S.M.; Liu, J. Some intuitionistic fuzzy interaction partitioned Bonferroni mean operators and their application to multi-attribute group decision making. *Inf. Sci.* **2017**, *411*, 98–121. [CrossRef]
23. Liu, P.; Li, H. Interval-valued intuitionistic fuzzy power Bonferroni aggregation operators and their application to group decision making. *Cognit. Comput.* **2017**, *9*, 494–512. [CrossRef]
24. Liu, P.; Liu, J. Some q-Rung Orthopai Fuzzy Bonferroni Mean Operators and Their Application to Multi-Attribute Group Decision Making. *Int. J. Intell. Syst.* **2018**, *33*, 315–347. [CrossRef]
25. Liu, P.; Zhang, L.; Liu, X.; Wang, P. Multi-valued Neutrosophic number Bonferroni mean operators and their application in multiple attribute group decision making. *Int. J. Inf. Technol. Decis. Mak.* **2016**, *15*, 1181–1210. [CrossRef]
26. Beliakov, G.; Pradera, A.; Calvo, T. Aggregation functions: A guide for practitioners. *Stud. Fuzziness Soft Comput.* **2010**, *2010*, 139–141.
27. Liu, P.; Liu, J.; Merigó, J.M. Partitioned Heronian means based on linguistic intuitionistic fuzzy numbers for dealing with multi-attribute group decision making. *Appl. Soft Comput.* **2018**, *62*, 395–422. [CrossRef]

28. Liu, P. Multiple attribute group decision making method based on interval-valued intuitionistic fuzzy power Heronian aggregation operators. *Comput. Ind. Eng.* **2017**, *108*, 199–212. [CrossRef]
29. Liu, P.; Chen, S.M. Group decision making based on Heronian aggregation operators of intuitionistic fuzzy numbers. *IEEE Trans. Cybern.* **2017**, *47*, 2514–2530. [CrossRef] [PubMed]
30. Liu, P.; Shi, L. Some Neutrosophic uncertain linguistic number Heronian mean operators and their application to multi-attribute group decision making. *Neural Comput. Appl.* **2017**, *28*, 1079–1093. [CrossRef]
31. Yu, D. Interval-valued intuitionistic fuzzy Heronian mean operators and their application in multi-criteria decision making. *Afr. J. Bus. Manag.* **2012**, *6*, 4158–4168. [CrossRef]
32. Maclaurin, C. A second letter to Martin Folkes, Esq., Concerning the roots of equations, with demonstration of other rules of algebra. *Philos. Trans. R. Soc. Lond. Ser. A* **1972**, *36*, 59–96.
33. Qin, J.; Liu, X. Approaches to uncertain linguistic multiple attribute decision making based on dual Maclaurin symmetric mean. *J. Intell. Fuzzy Syst.* **2015**, *29*, 171–186. [CrossRef]
34. Liu, P.; Qin, X. Maclaurin symmetric mean operators of linguistic intuitionistic fuzzy numbers and their application to multiple-attribute decision-making. *J. Exp. Theor. Artif. Intell.* **2017**, *29*, 1173–1202. [CrossRef]
35. Liu, P.; Zhang, X. Some Maclaurin symmetric mean operators for single valued trapezoidal neutrosophic numbers and their applications to group decision making. *Int. J. Fuzzy Syst.* **2017**. [CrossRef]
36. Delgado, M.; Verdegay, J.L.; Vila, M.A. Linguistic decision-making models. *Int. J. Intell. Syst.* **2010**, *7*, 479–492. [CrossRef]
37. Ma, Y.X.; Wang, J.Q.; Wang, J.; Wu, X.H. An interval neutrosophic linguistic multi-criteria group decision-making method and its application in selecting medical treatment options. *Neural Comput. Appl.* **2017**, *28*, 2745–2765. [CrossRef]
38. Xu, Z.S. A note on linguistic hybrid arithmetic averagingoperator in multiple attribute group decision making with linguistic information. *Group Decis. Negot.* **2006**, *15*, 593–604. [CrossRef]
39. Xu, Z.S. Fuzzy harmonic mean operators. *Int. J. Intell. Syst.* **2009**, *24*, 152–172. [CrossRef]
40. Xu, Z.S. Group decision making based on multiple types of linguistic preference relations. *Inf. Sci.* **2008**, *178*, 452–467. [CrossRef]
41. Broumi, S.; Ye, J.; Smarandache, F. An Extended TOPSIS Method for Multiple Attribute Decision Making based on Interval Neutrosophic Uncertain Linguistic Variables. *Neutrosophic Sets Syst.* **2015**, *8*, 22–31.
42. Wang, J.Q.; Wu, J.T.; Wang, J.; Zhang, H.Y.; Chen, X.H. Interval-valued hesitant fuzzy linguistic sets and their applications in multi-criteria decision-making problems. *Inf. Sci.* **2014**, *288*, 55–72. [CrossRef]
43. Ye, J. Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making. *J. Intell. Fuzzy Syst.* **2014**, *26*, 165–172.