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Models for Green Supplier Selection with Some 2-Tuple Linguistic Neutrosophic Number Bonferroni Mean Operators

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Abstract: In this paper, we extend the Bonferroni mean (BM) operator, generalized Bonferroni mean (GBM) operator, dual generalized Bonferroni mean (DGBM) operator and dual generalized geometric Bonferroni mean (DGGBM) operator with 2-tuple linguistic neutrosophic numbers (2TLNNs) to propose 2-tuple linguistic neutrosophic numbers weighted Bonferroni mean (2TLNNWBM) operator, 2-tuple linguistic neutrosophic numbers weighted geometric Bonferroni mean (2TLNNWGBM) operator, generalized 2-tuple linguistic neutrosophic numbers weighted Bonferroni mean (G2TLNNWBM) operator, generalized 2-tuple linguistic neutrosophic numbers weighted geometric Bonferroni mean (G2TLNNWGBM) operator, dual generalized 2-tuple linguistic neutrosophic numbers weighted Bonferroni mean (DG2TLNNWBM) operator, and dual generalized 2-tuple linguistic neutrosophic numbers weighted geometric Bonferroni mean (DG2TLNNWGBM) operator. Then, the MADM methods are proposed with these operators. In the end, we utilize an applicable example for green supplier selection in green supply chain management to prove the proposed methods.

Keywords: multiple attribute decision making (MADM); neutrosophic numbers; 2-tuple linguistic neutrosophic numbers set (2TLNNSs); Bonferroni mean (BM) operator; generalized Bonferroni mean (GBM) operator; dual generalized Bonferroni mean (DGBM) operator; dual generalized geometric Bonferroni mean (DGGBM) operator; green supplier selection; green supply chain management

1. Introduction

Zadeh [1] introduced a membership function between 0 and 1 instead of traditional crisp value of 0 and 1, and defined the fuzzy set (FS). In order to overcome the insufficiency of FS, Atanassov [2] proposed the concept of an intuitionistic fuzzy set (IFS), which is characterized by its membership function and non-membership function between 0 and 1. Furthermore, Atanassov and Gargov [3] introduced the concept of an interval-valued intuitionistic fuzzy set (IVIFS), which is characterized by its interval membership function and interval non-membership function in the unit interval [0,1]. Because IFSs and IVIFSs cannot depict indeterminate and inconsistent information, Smarandache [4] introduced a neutrosophic set (NS) from a philosophical point of view to express indeterminate and inconsistent information. A NS has more potential power than other modeling mathematical tools, such as fuzzy set [1], IFS [2], and IVIFS [3]. But, it is difficult to apply NSs in solving of real life problems. Therefore, Smarandache [4] and Wang et al. [5,6] defined a single valued neutrosophic set (SVNS) and an interval neutrosophic set (INS), which are characterized by a truth-membership, an indeterminacy membership, and a falsity membership. Ye [7] introduced a simplified neutrosophic set (SNS), including the concepts of SVNS and INS, which are the extension of IFS and IVIFS. Obviously, SNS is a subclass of NS, while SVNS and INS are subclasses of SNS. Ye [8] proposed the correlation and correlation coefficient of single-valued neutrosophic sets (SVNSs) that are based on the extension of the correlation of intuitionistic fuzzy sets and demonstrates that the cosine similarity measure is a special case of the correlation coefficient in SVNS. Broumi and Smarandache [9] extended the correlation coefficient to INSs. Biswas et al. [10] developed a new approach for multi-attribute group decision-making problems by extending the technique for order preference by similarity to ideal solution to single-valued neutrosophic environment. Liu et al. [11] combined Hamacher operations and generalized aggregation operators to NSs, and proposed the generalized neutrosophic number Hamacher weighted averaging (GNNHWA) operator, generalized neutrosophic number Hamacher ordered weighted averaging (GNNHOWA) operator, and generalized neutrosophic number Hamacher hybrid averaging (GNNHHA) operator, and explored some properties of these operators and analyzed some special cases of them. Sahin and Liu [12] developed a maximizing deviation method for solving the multiple attribute decision-making problems with the single-valued neutrosophic information or interval neutrosophic information. Ye [13] defined the Hamming and Euclidean distances between the interval neutrosophic sets (INSs) and proposed the similarity measures between INSs based on the relationship between similarity measures and distances. Zhang et al. [14] defined the operations for INSs and put forward a comparison approach that was based on the related research of interval valued intuitionistic fuzzy sets (IVIFSs) and developed two interval neutrosophic number aggregation operators. Peng et al. [15] developed a new outranking approach for multi-criteria decision-making (MCDM) problems in the context of a simplified neutrosophic environment, where the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree for each element are singleton subsets in [0,1] and defined some outranking relations for simplified neutrosophic number (SNNs) based on ELECTRE (ELimination and Choice Expressing REality), and the properties within the outranking relations are further discussed in detail. Zhang et al. [16] proposed a novel outranking approach for multi-criteria decision-making (MCDM) problems to address situations where there is a set of numbers in the real unit interval and not just a specific number with a neutrosophic set. Liu and Liu [17] proposed the neutrosophic number weighted power averaging (NNWPA) operator, the neutrosophic number weighted geometric power averaging (NNWGPA) operator, the generalized neutrosophic number weighted power averaging (GNNWPA) operator, and studied the properties of above operators are studied, such as idempotency, monotonicity, boundedness, and so on. Peng et al. [18] introduced the multi-valued neutrosophic sets (MVNSs), which allow for the truth-membership, indeterminacy- membership, and falsity-membership degree to have a set of crisp values between zero and one, respectively, and the multi-valued neutrosophic power weighted average (MVNPWA) operator and proposed the multi-valued neutrosophic power weighted geometric (MVNPWG) operator. Zhang et al. [19] presented a new correlation coefficient measure, which satisfies the requirement of this measure equaling one if and only if two interval neutrosophic sets (INSs) are the same and used the proposed weighted correlation coefficient measure of INSs to solve decision-making problems, which take into account the influence of the evaluations' uncertainty and both the objective and subjective weights. Chen and Ye [20] presented the Dombi operations of single-valued neutrosophic numbers (SVNNs) based on the operations of the Dombi T-norm and T-conorm, and then proposed the single-valued neutrosophic Dombi weighted arithmetic average (SVNDWAA) operator and the single-valued neutrosophic Dombi weighted geometric average (SVNDWGA) operator to deal with the aggregation of SVNNs and investigated their properties. Liu and Wang [21] proposed the single-valued neutrosophic normalized weighted Bonferroni mean (SVNNWBM) operator on the basis of Bonferroni mean, the weighted Bonferroni mean (WBM), and the normalized WBM, and developed an approach to solve the multiple attribute decision-making problems with SVNNs that were based on the SVNNWBM operator. Wu et al. [22] defined the prioritized weighted average operator and the prioritized weighted geometric operator for simplified neutrosophic numbers (SNNs) and proposed two novel effective cross-entropy measures for SNSs. Li et al. [23] proposed the improved generalized weighted Heronian mean (IGWHM) operator and the improved generalized weighted geometric Heronian mean (IGWGHM) operator, the single valued neutrosophic number improved generalized weighted Heronian mean (NNIGWHM) operator, and single valued the neutrosophic number improved generalized weighted geometric Heronian mean (NNIGWGHM) operator for multiple attribute group decision making (MAGDM) problems, in which attribute values take the form of SVNNs. Wang et al. [24] combined the generalized weighted BM (GWBM) operator and generalized weighted geometric Bonferroni mean (GWGBM) operator with single valued neutrosophic numbers (SVNNs) to propose the generalized single-valued neutrosophic number weight BM (GSVNNWBM) operator and the generalized single-valued neutrosophic numbers weighted GBM (GSVNNWGBM) operator and developed the MADM methods based on these operators. Wei & Zhang [25] utilized power aggregation operators and the Bonferroni mean to develop some single-valued neutrosophic Bonferroni power aggregation operators and single-valued neutrosophic geometric Bonferroni power aggregation operators. Peng & Dai [26] initiated a new axiomatic definition of single-valued neutrosophic distance measure and similarity measure, which is expressed by a single-valued neutrosophic number that will reduce the information loss and retain more original information.

Although SVNS theory has been successfully applied in some areas, the SVNS is also characterized by the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree information. However, all of the above approaches are unsuitable to describe the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree information of an element to a set by linguistic variables on the basis of the given linguistic term sets, which can reflect the decision maker's confidence level when they are making an evaluation. In order to overcome this limit, we shall propose the concept of 2-tuple linguistic neutrosophic numbers set (2TLNNSs) to solve this problem based on the SVNS [4–6] and the 2-tuple linguistic information processing model [27–42]. Thus, how to aggregate these 2-tuple linguistic neutrosophic numbers is an interesting topic. To solve this issue, in this paper, we shall develop some 2-tuple linguistic neutrosophic information aggregation operators that are based on the traditional Bonferroni mean (BM) operations [43–50]. In order to do so, the remainder of this paper is set out as follows. In the next section, we shall propose the concept of 2TLNNSs. In Section 3, we shall propose some Bonferroni mean (BM) operators with 2TLNNs. In Section 4, we shall propose some generalized Bonferroni mean (GBM) operators with 2TLNNs. In Section 5, we shall propose some dual generalized Bonferroni mean (DGBM) operators with 2TLNNs. In Section 6, we shall present a numerical example for green supplier selection in order to illustrate the method that is proposed in this paper. Section 7 concludes the paper with some remarks.

2. Preliminaries

In this section, we shall propose the concept of 2-tuple linguistic neutrosophic number sets (2TLNNSs) to solve this problem based on the SVNSs [6,7] and 2-tuple linguistic sets (2TLSs) [27,28].

2.1. 2-Tuple Fuzzy Linguistic Representation Model

Definition 1 ([27,28]). *Let $S = \{s_i | i = 0, 1, \ldots, t\}$ be a linguistic term set with odd cardinality. Any label, s_i represents a possible value for a linguistic variable, and S can be defined as:*

$$S = \left\{ \begin{array}{l} s_0 = extremely\ poor, s_1 = very\ poor, s_2 = poor, s_3 = medium, \\ s_4 = good, s_5 = very\ good, s_6 = extremely\ good. \end{array} \right\}$$

Herrera and Martinez [27,28] developed the 2-tuple fuzzy linguistic representation model based on the concept of symbolic translation. It is used for representing the linguistic assessment information by means of a 2-tuple (s_i, ρ_i), where s_i is a linguistic label for predefined linguistic term set S and ρ_i is the value of symbolic translation, and $\rho_i \in [-0.5, 0.5)$.

2.2. SVNSs

Let X be a space of points (objects) with a generic element in fix set X, denoted by x. A single-valued neutrosophic sets (SVNSs) A in X is characterized as following [4–6]:

$$A = \{(x, T_A(x), I_A(x), F_A(x))|x \in X\} \tag{1}$$

where the truth-membership function $T_A(x)$, indeterminacy-membership $I_A(x)$, and falsity-membership function $F_A(x)$ are single subintervals/subsets in the real standard $[0,1]$, that is, $T_A(x) : X \to [0,1], I_A(x) : X \to [0,1]$ and $F_A(x) : X \to [0,1]$. The sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$ satisfies the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. Then, a simplification of A is denoted by $A = \{(x, T_A(x), I_A(x), F_A(x))|x \in X\}$, which is a SVNS.

For a SVNS $\{(x, T_A(x), I_A(x), F_A(x))|x \in X\}$, the ordered triple components $(T_A(x), I_A(x), F_A(x))$, are described as a single-valued neutrosophic number (SVNN), and each SVNN can be expressed as $A = (T_A, I_A, F_A)$, where $T_A \in [0,1], I_A \in [0,1], F_A \in [0,1]$ $T_A \in [0,1], I_A \in [0,1], F_A \in [0,1]$, and $0 \leq T_A + I_A + F_A \leq 3$.

2.3. 2TLNNSs

Definition 2. *Assume that* $L = \{l_0, l_1, \ldots, l_t\}$ *is a 2TLSs with odd cardinality* $t + 1$. *If* $l = \langle (s_T, a), (s_I, b), (s_F, c) \rangle$ *is defined for* $(s_T, a), (s_I, b), (s_F, c) \in L$ *and* $a, b, c \in [0, t]$, *where* $(s_T, a), (s_I, b)$ *and* (s_F, c) *express independently the truth degree, indeterminacy degree, and falsity degree by 2TLSs, then 2TLNNSs is defined as follows:*

$$l_j = \left\{ \left(s_{T_j}, a_j\right), \left(s_{I_j}, b_j\right), \left(s_{F_j}, c_j\right) \right\} \tag{2}$$

where $0 \leq \Delta^{-1}\left(s_{T_j}, a_j\right) \leq t$, $0 \leq \Delta^{-1}\left(s_{I_j}, b_j\right) \leq t$, $0 \leq \Delta^{-1}\left(s_{F_j}, c_j\right) \leq t$, *and* $0 \leq \Delta^{-1}\left(s_{T_j}, a_j\right) + \Delta^{-1}\left(s_{I_j}, b_j\right) + \Delta^{-1}\left(s_{F_j}, c_j\right) \leq 3t$.

Definition 3. *Let* $l_1 = \langle (s_{T_1}, a_1), (s_{I_1}, b_1), (s_{F_1}, c_1) \rangle$ *be a 2TLNN in* L. *Then the score and accuracy functions of* l_1 *are defined as follows:*

$$S(l_1) = \Delta \left\{ \frac{(2t + \Delta^{-1}(s_{T_1}, a_1) - \Delta^{-1}(s_{I_1}, b_1) - \Delta^{-1}(s_{F_1}, c_1))}{3} \right\}, \Delta^{-1}(S(l_1)) \in [0, t] \tag{3}$$

$$H(l_1) = \Delta \left\{ \frac{t + \Delta^{-1}(s_{T_1}, a_1) - \Delta^{-1}(s_{F_1}, c_1)}{2} \right\}, \Delta^{-1}(H(l_1)) \in [0, t]. \tag{4}$$

Definition 4. *Let* $l_1 = \langle (s_{T_1}, a_1), (s_{I_1}, b_1), (s_{F_1}, c_1) \rangle$ *and* $l_2 = \langle (s_{T_2}, a_2), (s_{I_2}, b_2), (s_{F_2}, c_2) \rangle$ *be two 2TLNNs, then*

(1) if $S(l_1) < S(l_2)$, then $l_1 < l_2$;
(2) if $S(l_1) > S(l_2)$, then $l_1 > l_2$;
(3) if $S(l_1) = S(l_2), H(l_1) < H(l_2)$, then $l_1 < l_2$;
(4) if $S(l_1) = S(l_2), H(l_1) > H(l_2)$, then $l_1 > l_2$;
(5) if $S(l_1) = S(l_2), H(l_1) = H(l_2)$, then $l_1 = l_2$.

Definition 5. *Let* $l_1 = \langle (s_{T_1}, a_1), (s_{I_1}, b_1), (s_{F_1}, c_1) \rangle$ *and* $l_2 = \langle (s_{T_2}, a_2), (s_{I_2}, b_2), (s_{F_2}, c_2) \rangle$ *be two 2TLNNs, then*

$$\text{(1)}\quad l_1 \oplus l_2 = \left\{ \begin{array}{l} \Delta\left(t\left(\frac{\Delta^{-1}(s_{T_1},a_1)}{t} + \frac{\Delta^{-1}(s_{T_2},a_2)}{t} - \frac{\Delta^{-1}(s_{T_1},a_1)}{t} \cdot \frac{\Delta^{-1}(s_{T_2},a_2)}{t}\right)\right), \\ \Delta\left(t\left(\frac{\Delta^{-1}(s_{I_1},b_1)}{t} \cdot \frac{\Delta^{-1}(s_{I_2},b_2)}{t}\right)\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{F_1},c_1)}{t} \cdot \frac{\Delta^{-1}(s_{F_2},c_2)}{t}\right)\right) \end{array} \right\};$$

$$\text{(2)}\quad l_1 \otimes l_2 = \left\{ \begin{array}{l} \Delta\left(t\left(\frac{\Delta^{-1}(s_{T_1},a_1)}{t} \cdot \frac{\Delta^{-1}(s_{T_2},a_2)}{t}\right)\right), \\ \Delta\left(t\left(\frac{\Delta^{-1}(s_{I_1},b_1)}{t} + \frac{\Delta^{-1}(s_{I_2},b_2)}{t} - \frac{\Delta^{-1}(s_{I_1},b_1)}{t} \cdot \frac{\Delta^{-1}(s_{I_2},b_2)}{t}\right)\right), \\ \Delta\left(t\left(\frac{\Delta^{-1}(s_{F_1},c_1)}{t} + \frac{\Delta^{-1}(s_{F_2},c_2)}{t} - \frac{\Delta^{-1}(s_{F_1},c_1)}{t} \cdot \frac{\Delta^{-1}(s_{F_2},c_2)}{t}\right)\right) \end{array} \right\};$$

$$\text{(3)}\quad \lambda l_1 = \left\{ \begin{array}{l} \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{T_1},a_1)}{t}\right)^{\lambda}\right)\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{I_1},b_1)}{t}\right)^{\lambda}\right), \\ \Delta\left(t\left(\frac{\Delta^{-1}(s_{F_1},c_1)}{t}\right)^{\lambda}\right) \end{array} \right\}, \lambda > 0;$$

$$\text{(4)}\quad (l_1)^{\lambda} = \left\{ \begin{array}{l} \Delta\left(t\left(\frac{\Delta^{-1}(s_{T_1},a_1)}{t}\right)^{\lambda}\right), \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{I_1},b_1)}{t}\right)^{\lambda}\right)\right), \\ \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{F_1},c_1)}{t}\right)^{\lambda}\right)\right) \end{array} \right\}, \lambda > 0.$$

2.4. BM Operators

Definition 6 ([44]). *Let $p, q > 0$ and $b_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative crisp numbers then the Bonferroni mean (BM) is defined as follows:*

$$\mathrm{BM}^{p,q}(b_1, b_2, \cdots, b_n) = \left(\sum_{i,j=1}^{n} b_i^p b_j^q\right)^{1/(p+q)} \tag{5}$$

Then $\mathrm{BM}^{p,q}$ *is called Bonferroni mean (BM) operator.*

3. 2TLNNWBM and 2TLNNWGBM Operators

3.1. 2TLNNWBM Operator

To consider the attribute weights, the weighted Bonferroni mean (WBM) is defined, as follows.

Definition 7 ([44]). *Let $p, q > 0$ and $b_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative crisp numbers with the weights vector being $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$, thereby satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. The weighted Bonferroni mean (WBM) is defined as follows:*

$$\mathrm{WBM}_{\omega}^{p,q}(b_1, b_2, \cdots, b_n) = \left(\sum_{i,j=1}^{n} \omega_i \omega_j b_i^p b_j^q\right)^{1/(p+q)} \tag{6}$$

Then we extend WBM to fuse the 2TLNNs and propose 2-tuple linguistic neutrosophic number weighted Bonferroni mean (2TLNNWBM) aggregation operator.

Definition 8. *Let $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle$ be a set of 2TLNNs. The 2-tuple linguistic neutrosophic number weighted Bonferroni mean (2TLNNWBM) operator is:*

$$\mathrm{2TLNNWBM}_{\omega}^{p,q}(l_1, l_2, \ldots, l_n) = \left(\bigoplus_{i,j=1}^{n} \left(\omega_i \omega_j \left(l_i^p \otimes l_j^q\right)\right)\right)^{1/(p+q)} \tag{7}$$

Theorem 1. *Let $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle$ be a set of 2TLNNs. The aggregated value by using 2TLNNWBM operators is also a 2TLNN where*

$$\mathrm{2TLNNWBM}_{\omega}^{p,q}(l_1, l_2, \ldots, l_n) = \left(\mathop{\oplus}_{i,j=1}^{n} \left(\omega_i \omega_j \left(l_i^p \otimes l_j^q \right) \right) \right)^{1/(p+q)}$$
$$= \left\{ \begin{array}{l} \Delta\left(t\left(1 - \prod_{i,j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}(s_{T_i}, a_i)}{t} \right)^p \cdot \left(\frac{\Delta^{-1}(s_{T_j}, a_j)}{t} \right)^q \right)^{\omega_i \omega_j} \right)^{1/(p+q)} \right), \\ \Delta\left(t\left(1 - \left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_i}, b_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{I_j}, b_j)}{t} \right)^q \right)^{\omega_i \omega_j} \right)^{1/(p+q)} \right) \right), \\ \Delta\left(t\left(1 - \left(1 - \prod_{i,j=1}^{n} \left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_i}, c_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{F_j}, c_j)}{t} \right)^q \right)^{\omega_i \omega_j} \right)^{1/(p+q)} \right) \right) \end{array} \right\} \tag{8}$$

Proof. According to Definition 5, we can obtain

$$l_i^p = \left\{ \Delta\left(t\left(\frac{\Delta^{-1}(s_{T_i}, a_i)}{t} \right)^p \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_i}, b_i)}{t} \right)^p \right) \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_i}, c_i)}{t} \right)^p \right) \right) \right\} \tag{9}$$

$$l_j^q = \left\{ \Delta\left(t\left(\frac{\Delta^{-1}(s_{T_j}, a_j)}{t} \right)^q \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_j}, b_j)}{t} \right)^q \right) \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_j}, c_j)}{t} \right)^q \right) \right) \right\} \tag{10}$$

Thus,

$$l_i^p \otimes l_j^q = \left\{ \begin{array}{l} \Delta\left(t\left(\left(\frac{\Delta^{-1}(s_{T_i}, a_i)}{t} \right)^p \cdot \left(\frac{\Delta^{-1}(s_{T_j}, a_j)}{t} \right)^q \right) \right), \\ \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_i}, b_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{I_j}, b_j)}{t} \right)^q \right) \right), \\ \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_i}, c_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{F_j}, c_j)}{t} \right)^q \right) \right) \end{array} \right\} \tag{11}$$

Thereafter,

$$\omega_i \omega_j \left(l_i^p \otimes l_j^q \right)$$
$$= \left\{ \begin{array}{l} \Delta\left(t\left(1 - \left(1 - \left(\frac{\Delta^{-1}(s_{T_i}, a_i)}{t} \right)^p \cdot \left(\frac{\Delta^{-1}(s_{T_j}, a_j)}{t} \right)^q \right)^{\omega_i \omega_j} \right) \right), \\ \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_i}, b_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{I_j}, b_j)}{t} \right)^q \right)^{\omega_i \omega_j} \right), \\ \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_i}, c_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{F_j}, c_j)}{t} \right)^q \right)^{\omega_i \omega_j} \right) \end{array} \right\} \tag{12}$$

Furthermore,

$$\mathop{\oplus}_{i,j=1}^{n} \left(\omega_i \omega_j \left(l_i^p \otimes l_j^q \right) \right)$$
$$= \left\{ \begin{array}{l} \Delta\left(t\left(1 - \prod_{i,j=1}^{n} \left(1 - \left(\frac{\Delta^{-1}(s_{T_i}, a_i)}{t} \right)^p \cdot \left(\frac{\Delta^{-1}(s_{T_j}, a_j)}{t} \right)^q \right)^{\omega_i \omega_j} \right) \right), \\ \Delta\left(t \prod_{i,j=1}^{n} \left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_i}, b_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{I_j}, b_j)}{t} \right)^q \right)^{\omega_i \omega_j} \right), \\ \Delta\left(t \prod_{i,j=1}^{n} \left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_i}, c_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{F_j}, c_j)}{t} \right)^q \right)^{\omega_i \omega_j} \right) \end{array} \right\} \tag{13}$$

Therefore,

$$2\text{TLNNWBM}_{\omega}^{p,q}(l_1,l_2,\ldots,l_n)=\left(\underset{i,j=1}{\overset{n}{\oplus}}\left(\omega_i\omega_j\left(l_i^p\otimes l_j^q\right)\right)\right)^{1/(p+q)}$$

$$=\left\{\begin{array}{l}\Delta\left(t\left(1-\prod_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{1/(p+q)}\right),\\ \Delta\left(t\left(1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{I_i},b_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{1/(p+q)}\right)\right),\\ \Delta\left(t\left(1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{F_i},c_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{1/(p+q)}\right)\right)\end{array}\right\} \quad (14)$$

Hence, (8) is kept. □

Then, we need to prove that (8) is a 2TLNN. We need to prove two conditions, as follows:

① $0\le\Delta^{-1}(s_T,a)\le t, 0\le\Delta^{-1}(s_I,b)\le t, 0\le\Delta^{-1}(s_F,c)\le t$
② $0\le\Delta^{-1}(s_T,a)+\Delta^{-1}(s_I,b)+\Delta^{-1}(s_F,c)\le 3t$

Let

$$\frac{\Delta^{-1}(s_T,a)}{t}=\left(1-\prod_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{1/(p+q)}$$

$$\frac{\Delta^{-1}(s_I,b)}{t}=1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{I_i},b_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{1/(p+q)}$$

$$\frac{\Delta^{-1}(s_F,c)}{t}=1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{F_i},c_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{1/(p+q)}$$

Proof. ① Since $0\le\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\le 1, 0\le\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\le 1$ we get

$$0\le 1-\left(\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\le 1 \quad (15)$$

Then,

$$0\le 1-\prod_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\le 1 \quad (16)$$

$$0\le\left(1-\prod_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{1/(p+q)}\le 1 \quad (17)$$

That means $0\le\Delta^{-1}(s_T,a)\le t$, so ① is maintained, similarly, we can get $0\le\Delta^{-1}(s_I,b)\le t$ $0\le\Delta^{-1}(s_F,c)\le t$. ② Since $0\le\Delta^{-1}(s_T,a)\le t, 0\le\Delta^{-1}(s_I,b)\le t, 0\le\Delta^{-1}(s_F,c)\le t$, we get $0\le\Delta^{-1}(s_T,a)+\Delta^{-1}(s_I,b)+\Delta^{-1}(s_F,c)\le 3t$. □

Example 1. *Let* $\langle(s_3,0.4),(s_2,-0.3),(s_4,0.1)\rangle,\langle(s_2,0.3),(s_1,0.2),(s_4,-0.1)\rangle$ *be two 2TLNNs,* $(p,q)=(2,3)$, $\omega=(0.4,0.6)$ *according to (8), we have*

$$
\begin{aligned}
&\mathrm{2TLNNWBM}^{(2,3)}_{(0.4,0.6)}\begin{pmatrix}\langle (s_3,0.4),(s_2,-0.3),(s_4,0.1)\rangle,\\ \langle (s_2,0.3),(s_1,0.2),(s_4,-0.1)\rangle\end{pmatrix}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(1-\prod\limits_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^{p}\cdot\left(\frac{\Delta^{-1}(s_{T_j},a_j)}{t}\right)^{q}\right)^{\omega_i\omega_j}\right)^{1/(p+q)}\right),\\
\Delta\left(t\left(1-\left(1-\prod\limits_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^{p}\cdot\left(1-\frac{\Delta^{-1}(s_{I_j},b_j)}{t}\right)^{q}\right)^{\omega_i\omega_j}\right)^{1/(p+q)}\right)\right),\\
\Delta\left(t\left(1-\left(1-\prod\limits_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^{p}\cdot\left(1-\frac{\Delta^{-1}(s_{F_j},c_j)}{t}\right)^{q}\right)^{\omega_i\omega_j}\right)^{1/(p+q)}\right)\right)
\end{array}\right\}\\
&=\left\{\begin{array}{l}
\Delta\left(6\times\left(1-\left(\begin{array}{l}\left(1-\left(\frac{3.4}{6}\right)^2\times\left(\frac{3.4}{6}\right)^3\right)^{0.4\times0.4}\times\left(1-\left(\frac{3.4}{6}\right)^2\times\left(\frac{2.3}{6}\right)^3\right)^{0.4\times0.6}\\ \times\left(1-\left(\frac{2.3}{6}\right)^2\times\left(\frac{3.4}{6}\right)^3\right)^{0.6\times0.4}\times\left(1-\left(\frac{2.3}{6}\right)^2\times\left(\frac{2.3}{6}\right)^3\right)^{0.6\times0.6}\end{array}\right)\right)^{\frac{1}{2+3}}\right),\\
\Delta\left(6\times\left(1-\left(\begin{array}{l}1-\left(1-\left(1-\frac{1.7}{6}\right)^2\times\left(1-\frac{1.7}{6}\right)^3\right)^{0.4\times0.4}\times\left(1-\left(1-\frac{1.7}{6}\right)^2\times\left(1-\frac{1.2}{6}\right)^3\right)^{0.4\times0.6}\\ \times\left(1-\left(1-\frac{1.2}{6}\right)^2\times\left(1-\frac{1.7}{6}\right)^3\right)^{0.6\times0.4}\times\left(1-\left(1-\frac{1.2}{6}\right)^2\times\left(1-\frac{1.2}{6}\right)^3\right)^{0.6\times0.6}\end{array}\right)^{\frac{1}{2+3}}\right)\right),\\
\Delta\left(6\times\left(1-\left(\begin{array}{l}1-\left(1-\left(1-\frac{4.1}{6}\right)^2\times\left(1-\frac{4.1}{6}\right)^3\right)^{0.4\times0.4}\times\left(1-\left(1-\frac{4.1}{6}\right)^2\times\left(1-\frac{3.9}{6}\right)^3\right)^{0.4\times0.6}\\ \times\left(1-\left(1-\frac{3.9}{6}\right)^2\times\left(1-\frac{4.1}{6}\right)^3\right)^{0.6\times0.4}\times\left(1-\left(1-\frac{3.9}{6}\right)^2\times\left(1-\frac{3.9}{6}\right)^3\right)^{0.6\times0.6}\end{array}\right)^{\frac{1}{2+3}}\right)\right)
\end{array}\right\}\\
&=\langle (s_3,-0.173),(s_1,0.384),(s_4,-0.024)\rangle
\end{aligned}
$$

Then, we will discuss some properties of 2TLNNWBM operator.

Property 1. *(Idempotency) If* $l_i=\left\langle \left(s_{T_i},a_i\right),\left(s_{I_i},b_i\right),\left(s_{F_i},c_i\right)\right\rangle (i=1,2,\ldots,n)$ *are equal, then*

$$\mathrm{2TLNNWBM}^{p,q}_{\omega}(l_1,l_2,\cdots,l_n)=l \tag{18}$$

Proof. Since $l_i=l=\langle (s_T,a),(s_I,b),(s_F,c)\rangle$, then

$$
\begin{aligned}
&\mathrm{2TLNNWBM}^{p,q}_{\omega}(l_1,l_2,\ldots,l_n)=\left(\bigoplus_{i,j=1}^{n}\left(\omega_i\omega_j(l^p\otimes l^q)\right)\right)^{1/(p+q)}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(1-\prod\limits_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}(s_T,a)}{t}\right)^{p}\cdot\left(\frac{\Delta^{-1}(s_T,a)}{t}\right)^{q}\right)^{\omega_i\omega_j}\right)^{1/(p+q)}\right),\\
\Delta\left(t\left(1-\left(1-\prod\limits_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}(s_I,b)}{t}\right)^{p}\cdot\left(1-\frac{\Delta^{-1}(s_I,b)}{t}\right)^{q}\right)^{\omega_i\omega_j}\right)^{1/(p+q)}\right)\right),\\
\Delta\left(t\left(1-\left(1-\prod\limits_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}(s_F,c)}{t}\right)^{p}\cdot\left(1-\frac{\Delta^{-1}(s_F,c)}{t}\right)^{q}\right)^{\omega_i\omega_j}\right)^{1/(p+q)}\right)\right)
\end{array}\right\}\\
&=\left\{\begin{array}{l}
\Delta\left(t\left(1-\left(1-\left(\frac{\Delta^{-1}(s_T,a)}{t}\right)^{p+q}\right)^{\sum\limits_{i=1}^{n}\omega_i\sum\limits_{j=1}^{n}\omega_j}\right)^{1/(p+q)}\right),\\
\Delta\left(t\left(1-\left(1-\left(1-\left(1-\frac{\Delta^{-1}(s_I,b)}{t}\right)^{p+q}\right)^{\sum\limits_{i=1}^{n}\omega_i\sum\limits_{j=1}^{n}\omega_j}\right)^{1/(p+q)}\right)\right),\\
\Delta\left(t\left(1-\left(1-\left(1-\left(1-\frac{\Delta^{-1}(s_F,c)}{t}\right)^{p+q}\right)^{\sum\limits_{i=1}^{n}\omega_i\sum\limits_{j=1}^{n}\omega_j}\right)^{1/(p+q)}\right)\right)
\end{array}\right\}\\
&=\langle (s_T,a),(s_I,b),(s_F,c)\rangle=l
\end{aligned}
$$

□

Property 2. *(Monotonicity) Let* $l_{x_i} = \left\langle \left(s_{T_{x_i}}, a_{x_i}\right), \left(s_{I_{x_i}}, b_{x_i}\right), \left(s_{F_{x_i}}, c_{x_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *and* $l_{y_i} = \left\langle \left(s_{T_{y_i}}, a_{y_i}\right), \left(s_{I_{y_i}}, b_{y_i}\right), \left(s_{F_{y_i}}, c_{y_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *be two sets of 2TLNNs. If* $\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right) \leq \Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)$,$\Delta^{-1}\left(s_{I_{x_i}}, b_{x_i}\right) \geq \Delta^{-1}\left(s_{I_{y_i}}, b_{y_i}\right)$ *and* $\Delta^{-1}\left(s_{F_{x_i}}, c_{x_i}\right) \geq \Delta^{-1}\left(s_{F_{y_i}}, c_{y_i}\right)$ *hold for all i, then*

$$\text{2TLNNWBM}_{\omega}^{p,q}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) \leq \text{2TLNNWBM}_{\omega}^{p,q}(l_{y_1}, l_{y_2}, \cdots, l_{y_n}) \tag{19}$$

Proof. Let $\text{2TLNNWBM}_{\omega}^{p,q}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) = \left\langle \left(s_{T_{x_i}}, a_{x_i}\right), \left(s_{I_{x_i}}, b_{x_i}\right), \left(s_{F_{x_i}}, c_{x_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ and $\text{2TLNNWBM}_{\omega}^{p,q}(l_{y_1}, l_{y_2}, \cdots, l_{y_n}) = \left\langle \left(s_{T_{y_i}}, a_{y_i}\right), \left(s_{I_{y_i}}, b_{y_i}\right), \left(s_{F_{y_i}}, c_{y_i}\right) \right\rangle (i = 1, 2, \ldots, n)$, given that $\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right) \leq \Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)$, we can obtain

$$\left(\frac{\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_j}}, a_{x_j}\right)}{t}\right)^q \leq \left(\frac{\Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_j}}, a_{y_j}\right)}{t}\right)^q \tag{20}$$

$$\left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_j}}, a_{x_j}\right)}{t}\right)^q\right)^{\omega_i\omega_j} \geq \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_j}}, a_{y_j}\right)}{t}\right)^q\right)^{\omega_i\omega_j} \tag{21}$$

Thereafter,

$$1 - \prod_{i,j=1}^{n}\left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_j}}, a_{x_j}\right)}{t}\right)^q\right)^{\omega_i\omega_j} \leq 1 - \prod_{i,j=1}^{n}\left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_j}}, a_{y_j}\right)}{t}\right)^q\right)^{\omega_i\omega_j} \tag{22}$$

Furthermore,

$$\begin{aligned} &\left(1 - \prod_{i,j=1}^{n}\left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_j}}, a_{x_j}\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{1/(p+q)} \\ &\leq \left(1 - \prod_{i,j=1}^{n}\left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_j}}, a_{y_j}\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{1/(p+q)} \end{aligned} \tag{23}$$

That means $\Delta^{-1}(s_{T_x}, a_x) \leq \Delta^{-1}\left(s_{T_y}, a_y\right)$. Similarly, we can obtain $\Delta^{-1}(s_{I_x}, b_x) \geq \Delta^{-1}\left(s_{I_y}, b_y\right)$ and $\Delta^{-1}(s_{F_x}, c_x) \geq \Delta^{-1}\left(s_{F_y}, c_y\right)$.

If $\Delta^{-1}(s_{T_x}, a_x) < \Delta^{-1}\left(s_{T_y}, a_y\right)$ and $\Delta^{-1}(s_{I_x}, b_x) \geq \Delta^{-1}\left(s_{I_y}, b_y\right)$ and $\Delta^{-1}(s_{F_x}, c_x) \geq \Delta^{-1}\left(s_{F_y}, c_y\right)$

$$\text{2TLNNWBM}_{\omega}^{p,q}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) < \text{G2TLNNWBM}_{\omega}^{p,q}(l_{y_1}, l_{y_2}, \cdots, l_{y_n})$$

If $\Delta^{-1}(s_{T_x}, a_x) = \Delta^{-1}\left(s_{T_y}, a_y\right)$ and $\Delta^{-1}(s_{I_x}, b_x) > \Delta^{-1}\left(s_{I_y}, b_y\right)$ and $\Delta^{-1}(s_{F_x}, c_x) > \Delta^{-1}\left(s_{F_y}, c_y\right)$

$$\text{2TLNNWBM}_{\omega}^{p,q}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) < \text{G2TLNNWBM}_{\omega}^{p,q}(l_{y_1}, l_{y_2}, \cdots, l_{y_n})$$

If $\Delta^{-1}(s_{T_x}, a_x) = \Delta^{-1}\left(s_{T_y}, a_y\right)$ and $\Delta^{-1}(s_{I_x}, b_x) = \Delta^{-1}\left(s_{I_y}, b_y\right)$ and $\Delta^{-1}(s_{F_x}, c_x) = \Delta^{-1}\left(s_{F_y}, c_y\right)$

$$\text{2TLNNWBM}_{\omega}^{p,q}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) = \text{2TLNNWBM}_{\omega}^{p,q}(l_{y_1}, l_{y_2}, \cdots, l_{y_n})$$

So, Property 2 is right. □

Property 3. *(Boundedness) Let* $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle (i = 1, 2, \ldots, n)$ *be a set of 2TLNNs. If* $l^+ = (\max_i(S_{T_i}, a_i), \min_i(S_{I_i}, b_i), \min_i(S_{F_i}, c_i))$ *and* $l^- = (\min_i(S_{T_i}, a_i), \max_i(S_{I_i}, b_i), \max_i(S_{F_i}, c_i))$ *then*

$$l^- \leq \text{2TLNNWBM}_\omega^{p,q}(l_1, l_2, \cdots, l_n) \leq l^+ \tag{24}$$

From Property 1,

$$\text{2TLNNWBM}_\omega^{p,q}(l_1^-, l_2^-, \cdots, l_n^-) = l^-$$
$$\text{2TLNNWBM}_\omega^{p,q}(l_1^+, l_2^+, \cdots, l_n^+) = l^+$$

From Property 2,

$$l^- \leq \text{2TLNNWBM}_\omega^{p,q}(l_1, l_2, \cdots, l_n) \leq l^+$$

3.2. 2TLNNWGBM Operator

Similarly to WBM, to consider the attribute weights, the weighted geometric Bonferroni mean (WGBM) is defined, as follows:

Definition 9 ([51]). *Let* $p, q > 0$ *and* $b_i (i = 1, 2, \cdots, n)$ *be a collection of nonnegative crisp numbers with the weights vector being* $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$*, thereby satisfying* $\omega_i \in [0, 1]$ *and* $\sum\limits_{i=1}^{n} \omega_i = 1$*. If*

$$\text{WGBM}_\omega^{p,q}(b_1, b_2, \cdots, b_n) = \frac{1}{p+q} \prod_{i,j=1}^{n} (pb_i + qb_j)^{\omega_i \omega_j} \tag{25}$$

Then we extend WGBM to fuse the 2TLNNs and propose 2-tuple linguistic neutrosophic number weighted geometric Bonferroni mean (2TLNNWGBM) operator.

Definition 10. *Let* $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle (i = 1, 2, \ldots, n)$ *be a set of 2TLNNs with their weight vector be* $w_i = (w_1, w_2, \ldots, w_n)^T$*, thereby satisfying* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$*. If*

$$\text{2TLNNWGBM}_\omega^{p,q}(l_1, l_2, \ldots, l_n) = \frac{1}{p+q} \underset{i,j=1}{\overset{n}{\otimes}} (pl_i \oplus ql_j)^{\omega_i \omega_j} \tag{26}$$

Then we called $\text{2TLNNWGBM}_\omega^{p,q}$ *the 2-tuple linguistic neutrosophic number weighted geometric BM.*

Theorem 2. *Let* $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle (i = 1, 2, \ldots, n)$ *be a set of 2TLNNs. The aggregated value by using 2TLNNWGBM operators is also a 2TLNN where*

$$\text{2TLNNWGBM}_\omega^{p,q}(l_1, l_2, \ldots, l_n) = \frac{1}{p+q} \underset{i,j=1}{\overset{n}{\otimes}} (pl_i \oplus ql_j)^{\omega_i \omega_j}$$
$$= \left\{ \begin{array}{l} \Delta\left(t\left(1 - \left(1 - \prod\limits_{i,j=1}^{n}\left(1 - \left(1 - \frac{\Delta^{-1}(s_{T_i}, a_i)}{t}\right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{T_j}, a_j)}{t}\right)^q\right)^{\omega_i \omega_j}\right)^{\frac{1}{p+q}}\right)\right), \\ \Delta\left(t\left(1 - \prod\limits_{i,j=1}^{n}\left(1 - \left(\frac{\Delta^{-1}(s_{I_i}, b_i)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}(s_{I_j}, b_j)}{t}\right)^q\right)^{\omega_i \omega_j}\right)^{\frac{1}{p+q}}\right), \\ \Delta\left(t\left(1 - \prod\limits_{i,j=1}^{n}\left(1 - \left(\frac{\Delta^{-1}(s_{F_i}, c_i)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}(s_{F_j}, c_j)}{t}\right)^q\right)^{\omega_i \omega_j}\right)^{\frac{1}{p+q}}\right) \end{array} \right\} \tag{27}$$

Proof. From Definition 5, we can obtain,

$$pl_i = \left\{ \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p\right)\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^p\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^p\right) \right\} \quad (28)$$

$$ql_j = \left\{ \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{T_j},a_j)}{t}\right)^q\right)\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{I_j},b_j)}{t}\right)^q\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{F_j},c_j)}{t}\right)^q\right) \right\} \quad (29)$$

Thus,

$$pl_i \oplus ql_j = \left\{ \begin{array}{l} \Delta\left(t\left(1-(1-)^p\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\cdot\left(1-\frac{\Delta^{-1}(s_{T_j},a_j)}{t}\right)^q\right)\right), \\ \Delta\left(t\left(\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}(s_{I_j},b_j)}{t}\right)^q\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}(s_{F_j},c_j)}{t}\right)^q\right) \end{array} \right\} \quad (30)$$

Therefore,

$$\begin{array}{l} (pl_i \oplus ql_j)^{\omega_i\omega_j} \\ = \left\{ \begin{array}{l} \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}(s_{T_j},a_j)}{t}\right)^q\right)^{\omega_i\omega_j}\right), \\ \Delta\left(t\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}(s_{I_j},b_j)}{t}\right)^q\right)^{\omega_i\omega_j}\right)\right), \\ \Delta\left(t\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}(s_{F_j},c_j)}{t}\right)^q\right)^{\omega_i\omega_j}\right)\right) \end{array} \right\} \end{array} \quad (31)$$

Thereafter,

$$\begin{array}{l} \underset{i,j=1}{\overset{n}{\otimes}}(pl_i \oplus ql_j)^{\omega_i\omega_j} \\ = \left\{ \begin{array}{l} \Delta\left(t\prod_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}(s_{T_j},a_j)}{t}\right)^q\right)^{\omega_i\omega_j}\right), \\ \Delta\left(t\left(1-\prod_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}(s_{I_j},b_j)}{t}\right)^q\right)^{\omega_i\omega_j}\right)\right), \\ \Delta\left(t\left(1-\prod_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}(s_{F_j},c_j)}{t}\right)^q\right)^{\omega_i\omega_j}\right)\right) \end{array} \right\} \end{array} \quad (32)$$

Furthermore,

$$\begin{array}{l} \text{2TLNNWGBM}_{\omega}^{p,q}(l_1,l_2,\ldots,l_n) = \frac{1}{p+q}\underset{i,j=1}{\overset{n}{\otimes}}(pl_i \oplus ql_j)^{\omega_i\omega_j} \\ = \left\{ \begin{array}{l} \Delta\left(t\left(1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}(s_{T_j},a_j)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{\frac{1}{p+q}}\right)\right), \\ \Delta\left(t\left(1-\prod_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}(s_{I_j},b_j)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{\frac{1}{p+q}}\right), \\ \Delta\left(t\left(1-\prod_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}(s_{F_j},c_j)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{\frac{1}{p+q}}\right) \end{array} \right\} \end{array} \quad (33)$$

Hence, (27) is kept. □

Then, we need to prove that (27) is a 2TLNN. We need to prove two conditions, as follows:

① $0 \leq \Delta^{-1}(s_T, a) \leq t, 0 \leq \Delta^{-1}(s_I, b) \leq t, 0 \leq \Delta^{-1}(s_F, c) \leq t$
② $0 \leq \Delta^{-1}(s_T, a) + \Delta^{-1}(s_I, b) + \Delta^{-1}(s_F, c) \leq 3t$

Let

$$\frac{\Delta^{-1}(s_T,a)}{t} = 1 - \left(1 - \prod_{i,j=1}^{n}\left(1 - \left(1 - \frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p \cdot \left(1 - \frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{\frac{1}{p+q}}$$

$$\frac{\Delta^{-1}(s_I,b)}{t} = \left(1 - \prod_{i,j=1}^{n}\left(1 - \left(\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{\frac{1}{p+q}}$$

$$\frac{\Delta^{-1}(s_F,c)}{t} = \left(1 - \prod_{i,j=1}^{n}\left(1 - \left(\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{\frac{1}{p+q}}$$

Proof. ① Since $0 \leq \frac{\Delta^{-1}(s_{T_i},a_i)}{t} \leq 1, 0 \leq \frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t} \leq 1$ we get

$$0 \leq 1 - \left(1 - \frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p \cdot \left(1 - \frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q \leq 1 \quad (34)$$

Then,

$$0 \leq \prod_{i,j=1}^{n}\left(1 - \left(1 - \frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p \cdot \left(1 - \frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\right)^{\omega_i\omega_j} \leq 1 \quad (35)$$

$$0 \leq 1 - \left(1 - \prod_{i,j=1}^{n}\left(1 - \left(1 - \frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p \cdot \left(1 - \frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\right)^{\omega_i\omega_j}\right)^{\frac{1}{p+q}} \leq 1 \quad (36)$$

That means $0 \leq \Delta^{-1}(s_T, a) \leq t$, so ① is maintained, similarly, we can get $0 \leq \Delta^{-1}(s_I, b) \leq t, 0 \leq \Delta^{-1}(s_F, c) \leq t$. ② Since $0 \leq \Delta^{-1}(s_T, a) \leq t, 0 \leq \Delta^{-1}(s_I, b) \leq t, 0 \leq \Delta^{-1}(s_F, c) \leq t$, we get $0 \leq \Delta^{-1}(s_T, a) + \Delta^{-1}(s_I, b) + \Delta^{-1}(s_F, c) \leq 3t$. □

Example 2. *Let* $\langle (s_3, 0.4), (s_2, -0.3), (s_4, 0.1) \rangle, \langle (s_2, 0.3), (s_1, 0.2), (s_4, -0.1) \rangle$ *be two 2TLNNs,* $(p,q) = (2,3), \omega = (0.4, 0.6)$ *according to (27), we have*

$$\begin{aligned}
&2\text{TLNNWGBM}^{(2,3)}_{(0.4,0.6)}\begin{pmatrix} \langle (s_3,0.4),(s_2,-0.3),(s_4,0.1)\rangle, \\ \langle (s_2,0.3),(s_1,0.2),(s_4,-0.1)\rangle \end{pmatrix} \\
&= \left\{ \begin{array}{l}
\Delta\left(t\left(1-\left(1-\prod_{i,j=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^{p}\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^{q}\right)^{\omega_i\omega_j}\right)^{\frac{1}{p+q}}\right)\right), \\
\Delta\left(t\left(1-\prod_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{I_i},b_i\right)}{t}\right)^{p}\cdot\left(\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^{q}\right)^{\omega_i\omega_j}\right)^{\frac{1}{p+q}}\right), \\
\Delta\left(t\left(1-\prod_{i,j=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{F_i},c_i\right)}{t}\right)^{p}\cdot\left(\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^{q}\right)^{\omega_i\omega_j}\right)^{\frac{1}{p+q}}\right)
\end{array} \right\} \\
&= \left\{ \begin{array}{l}
\Delta\left(6\times\left(1-\left(\begin{array}{l} 1-\left(1-\left(1-\frac{3.4}{6}\right)^2\times\left(1-\frac{3.4}{6}\right)^3\right)^{0.4\times0.4}\times\left(1-\left(1-\frac{3.4}{6}\right)^2\times\left(1-\frac{2.3}{6}\right)^3\right)^{0.4\times0.6} \\ \times\left(1-\left(1-\frac{2.3}{6}\right)^2\times\left(1-\frac{3.4}{6}\right)^3\right)^{0.6\times0.4}\times\left(1-\left(1-\frac{2.3}{6}\right)^2\times\left(1-\frac{2.3}{6}\right)^3\right)^{0.6\times0.6} \end{array}\right)^{\frac{1}{2+3}}\right)\right), \\
\Delta\left(6\times\left(1-\left(\begin{array}{l} \left(1-\left(\frac{1.7}{6}\right)^2\times\left(\frac{1.7}{6}\right)^3\right)^{0.4\times0.4}\times\left(1-\left(\frac{1.7}{6}\right)^2\times\left(\frac{1.2}{6}\right)^3\right)^{0.4\times0.6} \\ \times\left(1-\left(\frac{1.2}{6}\right)^2\times\left(\frac{1.7}{6}\right)^3\right)^{0.6\times0.4}\times\left(1-\left(\frac{1.2}{6}\right)^2\times\left(\frac{1.2}{6}\right)^3\right)^{0.6\times0.6} \end{array}\right)\right)^{\frac{1}{2+3}}\right), \\
\Delta\left(6\times\left(1-\left(\begin{array}{l} \left(1-\left(\frac{4.1}{6}\right)^2\times\left(\frac{4.1}{6}\right)^3\right)^{0.4\times0.4}\times\left(1-\left(\frac{4.1}{6}\right)^2\times\left(\frac{3.9}{6}\right)^3\right)^{0.4\times0.6} \\ \times\left(1-\left(\frac{3.9}{6}\right)^2\times\left(\frac{4.1}{6}\right)^3\right)^{0.6\times0.4}\times\left(1-\left(\frac{3.9}{6}\right)^2\times\left(\frac{3.9}{6}\right)^3\right)^{0.6\times0.6} \end{array}\right)\right)^{\frac{1}{2+3}}\right)
\end{array} \right\} \\
&= \langle (s_3, -0.334), (s_1, 0.434), (s_4, -0.018) \rangle
\end{aligned}$$

Similar to 2TLNNWBM, the 2TLNNWGBM has the same properties, as follows. The proof are omitted here to save space.

Property 4. *(Idempotency) If* $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle (i = 1, 2, \ldots, n)$ *are equal, then*

$$2\text{TLNNWGBM}^{p,q}_{\omega}(l_1, l_2, \cdots, l_n) = l \tag{37}$$

Property 5. *(Monotonicity) Let* $l_{x_i} = \left\langle \left(s_{T_{x_i}}, a_{x_i}\right), \left(s_{I_{x_i}}, b_{x_i}\right), \left(s_{F_{x_i}}, c_{x_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *and* $l_{y_i} = \left\langle \left(s_{T_{y_i}}, a_{y_i}\right), \left(s_{I_{y_i}}, b_{y_i}\right), \left(s_{F_{y_i}}, c_{y_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *be two sets of 2TLNNs. If* $\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right) \leq \Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right), \Delta^{-1}\left(s_{I_{x_i}}, b_{x_i}\right) \geq \Delta^{-1}\left(s_{I_{y_i}}, b_{y_i}\right)$ *and* $\Delta^{-1}\left(s_{F_{x_i}}, c_{x_i}\right) \geq \Delta^{-1}\left(s_{F_{y_i}}, c_{y_i}\right)$ *hold for all i, then*

$$2\text{TLNNWGBM}^{p,q}_{\omega}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) \leq 2\text{TLNNWGBM}^{p,q}_{\omega}(l_{y_1}, l_{y_2}, \cdots, l_{y_n}) \tag{38}$$

Property 6. *(Boundedness) Let* $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle (i = 1, 2, \ldots, n)$ *be a set of 2TLNNs. If* $l^+ = (\max_i(S_{T_i}, a_i), \min_i(S_{I_i}, b_i), \min_i(S_{F_i}, c_i))$ *and* $l^- = (\min_i(S_{T_i}, a_i), \max_i(S_{I_i}, b_i), \max_i(S_{F_i}, c_i))$ *then*

$$l^- \leq 2\text{TLNNWGBM}^{p,q}_{\omega}(l_1, l_2, \cdots, l_n) \leq l^+ \tag{39}$$

4. G2TLNNWBM and G2TLNNWGBM Operators

4.1. G2TLNNWBM Operator

The primary advantage of BM is that it can determine the interrelationship between arguments. However, the traditional BM can only consider the correlations of any two aggregated arguments. Thereafter, Beliakov et al. [43] extended the BM and introduced the generalized BM(GBM) operator. Zhu et al. [51] introduced the generalized weighted BM(GWBM) operator, as follows.

Definition 11 ([51]). *Let $p, q, r > 0$ and $b_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative crisp numbers with the weights vector being $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$, thereby satisfying $\omega_i \in [0, 1]$ and $\sum_{i=1}^{n} \omega_i = 1$. The generalized weighted Bonferroni mean (GWBM) is defined as follows:*

$$\mathrm{GWBM}_{\omega}^{p,q,r}(b_1, b_2, \cdots, b_n) = \left(\sum_{i,j,k=1}^{n} \omega_i \omega_j \omega_k b_i^p b_j^q b_k^r \right)^{1/(p+q+r)} \tag{40}$$

Then we extend GWBM to fuse the 2TLNNs and propose generalized 2-tuple linguistic neutrosophic number weighted Bonferroni mean (G2TLNNWBM) aggregation operator.

Definition 12. *Let $l_i = \left\langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \right\rangle$ be a set of 2TLNNs. The generalized 2-tuple linguistic neutrosophic number weighted Bonferroni mean (G2TLNNWBM) operator is:*

$$\mathrm{G2TLNNWBM}_{\omega}^{p,q,r}(l_1, l_2, \ldots, l_n) = \left(\mathop{\oplus}_{i,j,k=1}^{n} \left(\omega_i \omega_j \omega_k \left(l_i^p \otimes l_j^q \otimes l_k^r \right) \right) \right)^{1/(p+q+r)} \tag{41}$$

Theorem 3. *Let $l_i = \left\langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \right\rangle$ be a set of 2TLNNs. The aggregated value by using G2TLNNWBM operators is also a 2TLNN where*

$$\begin{aligned} &\mathrm{G2TLNNWBM}_{\omega}^{p,q,r}(l_1, l_2, \ldots, l_n) = \left(\mathop{\oplus}_{i,j,k=1}^{n} \left(\omega_i \omega_j \omega_k \left(l_i^p \otimes l_j^q \otimes l_k^r \right) \right) \right)^{1/(p+q+r)} \\ &= \left\{ \begin{array}{l} \Delta\left(t\left(1 - \prod_{i,j,k=1}^{n} \left(1 - \left(\frac{\Delta^{-1}(s_{T_i}, a_i)}{t} \right)^p \cdot \left(\frac{\Delta^{-1}(s_{T_j}, a_j)}{t} \right)^q \cdot \left(\frac{\Delta^{-1}(s_{T_k}, a_k)}{t} \right)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(p+q+r)} \right), \\ \Delta\left(t\left(1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_i}, b_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{I_j}, b_j)}{t} \right) \cdot \left(1 - \frac{\Delta^{-1}(s_{I_k}, b_k)}{t} \right)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(p+q+r)} \right) \right), \\ \Delta\left(t\left(1 - \left(1 - \prod_{i,j,k=1}^{n} \left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_i}, c_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{F_j}, c_j)}{t} \right)^q \cdot \left(1 - \frac{\Delta^{-1}(s_{F_k}, c_k)}{t} \right)^r \right)^{\omega_i \omega_j \omega_k} \right)^{1/(p+q+r)} \right) \right) \end{array} \right\} \end{aligned} \tag{42}$$

Proof. According to Definition 5, we can obtain

$$l_i^p = \left\{ \Delta\left(t\left(\frac{\Delta^{-1}(s_{T_i}, a_i)}{t} \right)^p \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_i}, b_i)}{t} \right)^p \right) \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_i}, c_i)}{t} \right)^p \right) \right) \right\} \tag{43}$$

$$l_j^q = \left\{ \Delta\left(t\left(\frac{\Delta^{-1}(s_{T_j}, a_j)}{t} \right)^q \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_j}, b_j)}{t} \right)^q \right) \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_j}, c_j)}{t} \right)^q \right) \right) \right\} \tag{44}$$

$$l_k^r = \left\{ \Delta\left(t\left(\frac{\Delta^{-1}(s_{T_k}, a_k)}{t} \right)^r \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_k}, b_k)}{t} \right)^r \right) \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_k}, c_k)}{t} \right)^r \right) \right) \right\} \tag{45}$$

Thus,

$$l_i^p \otimes l_j^q \otimes l_k^r = \left\{ \begin{array}{l} \Delta\left(t\left(\left(\frac{\Delta^{-1}(s_{T_i}, a_i)}{t} \right)^p \cdot \left(\frac{\Delta^{-1}(s_{T_j}, a_j)}{t} \right)^q \cdot \left(\frac{\Delta^{-1}(s_{T_k}, a_k)}{t} \right)^r \right) \right), \\ \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_i}, b_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{I_j}, b_j)}{t} \right)^q \cdot \left(1 - \frac{\Delta^{-1}(s_{I_k}, b_k)}{t} \right)^r \right) \right), \\ \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_i}, c_i)}{t} \right)^p \cdot \left(1 - \frac{\Delta^{-1}(s_{F_j}, c_j)}{t} \right) \cdot \left(1 - \frac{\Delta^{-1}(s_{F_k}, c_k)}{t} \right)^r \right) \right) \end{array} \right\} \tag{46}$$

Thereafter,

$$\omega_i\omega_j\omega_k\left(l_i^p \otimes l_j^q \otimes l_k^r\right) = \left\{ \begin{array}{l} \Delta\left(t\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\cdot\left(\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)\right), \\ \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}\left(s_{I_i},b_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{I_k},b_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right), \\ \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}\left(s_{F_i},c_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{F_k},c_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right) \end{array} \right\} \tag{47}$$

Furthermore,

$$\bigoplus_{i,j,k=1}^{n}\left(\omega_i\omega_j\omega_k\left(l_i^p \otimes l_j^q \otimes l_k^r\right)\right) = \left\{ \begin{array}{l} \Delta\left(t\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\cdot\left(\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)\right), \\ \Delta\left(t\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{I_i},b_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{I_k},b_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right), \\ \Delta\left(t\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{F_i},c_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{F_k},c_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right) \end{array} \right\} \tag{48}$$

Therefore,

$$\mathrm{G2TLNNWBM}_{\omega}^{p,q,r}(l_1,l_2,\ldots,l_n) = \left(\bigoplus_{i,j,k=1}^{n}\left(\omega_i\omega_j\omega_k\left(l_i^p \otimes l_j^q \otimes l_k^r\right)\right)\right)^{1/(p+q+r)}$$
$$= \left\{ \begin{array}{l} \Delta\left(t\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\cdot\left(\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{1/(p+q+r)}\right), \\ \Delta\left(t\left(1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{I_i},b_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{I_k},b_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{1/(p+q+r)}\right)\right), \\ \Delta\left(t\left(1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{F_i},c_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{F_k},c_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{1/(p+q+r)}\right)\right) \end{array} \right\} \tag{49}$$

Hence, (42) is kept. □

Then, we need to prove that (42) is a 2TLNN. We need to prove two conditions as follows:

① $0 \le \Delta^{-1}(s_T,a) \le t, 0 \le \Delta^{-1}(s_I,b) \le t, 0 \le \Delta^{-1}(s_F,c) \le t$
② $0 \le \Delta^{-1}(s_T,a) + \Delta^{-1}(s_I,b) + \Delta^{-1}(s_F,c) \le 3t$

Let

$$\frac{\Delta^{-1}(s_T,a)}{t} = \left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\cdot\left(\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{1/(p+q+r)}$$
$$\frac{\Delta^{-1}(s_I,b)}{t} = 1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{I_i},b_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{I_k},b_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{1/(p+q+r)}$$
$$\frac{\Delta^{-1}(s_F,c)}{t} = 1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{F_i},c_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{F_k},c_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{1/(p+q+r)}$$

Proof. ① Since $0 \leq \frac{\Delta^{-1}(s_{T_i},a_i)}{t} \leq 1, 0 \leq \frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t} \leq 1, 0 \leq \frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t} \leq 1$ we get

$$0 \leq 1 - \left(\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}(s_{T_k},a_k)}{t}\right)^r \leq 1 \tag{50}$$

Then,

$$0 \leq 1 - \prod_{i,j,k=1}^{n}\left(1 - \left(\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}(s_{T_k},a_k)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k} \leq 1 \tag{51}$$

$$0 \leq \left(1 - \prod_{i,j,k=1}^{n}\left(1 - \left(\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{1/(p+q+r)} \leq 1 \tag{52}$$

That means $0 \leq \Delta^{-1}(s_T,a) \leq t$, so ① is maintained, similarly, we can get $0 \leq \Delta^{-1}(s_I,b) \leq t, 0 \leq \Delta^{-1}(s_F,c) \leq t$. ② Since $0 \leq \Delta^{-1}(s_T,a) \leq t, 0 \leq \Delta^{-1}(s_I,b) \leq t, 0 \leq \Delta^{-1}(s_F,c) \leq t$, we get $0 \leq \Delta^{-1}(s_T,a) + \Delta^{-1}(s_I,b) + \Delta^{-1}(s_F,c) \leq 3t$. □

Then we will discuss some properties of G2TLNNWBM operator.

Property 7. *(Idempotency) If* $l_i = \left\langle (s_{T_i},a_i), (s_{I_i},b_i), (s_{F_i},c_i)\right\rangle (i = 1,2,\ldots,n)$ *are equal, then*

$$\text{G2TLNNWBM}_{\omega}^{p,q,r}(l_1,l_2,\cdots,l_n) = l \tag{53}$$

Proof. Since $l_i = l = \langle (s_T,a),(s_I,b),(s_F,c)\rangle$, then

$$\begin{aligned}
&\text{G2TLNNWBM}_{\omega}^{p,q,r}(l_1,l_2,\ldots,l_n) = \left(\bigoplus_{i,j,k=1}^{n}\left(\omega_i\omega_j\omega_k\left(l_i^p \otimes l_j^q \otimes l_k^r\right)\right)\right)^{1/(p+q+r)} \\
&= \left\{\begin{array}{l}
\Delta\left(t\left(1 - \prod_{i,j,k=1}^{n}\left(1 - \left(\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{1/(p+q+r)}\right), \\
\Delta\left(t\left(1 - \left(1 - \prod_{i,j,k=1}^{n}\left(1 - \left(1 - \frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^p \cdot \left(1 - \frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q \cdot \left(1 - \frac{\Delta^{-1}(s_{I_k},b_k)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{1/(p+q+r)}\right)\right), \\
\Delta\left(t\left(1 - \left(1 - \prod_{i,j,k=1}^{n}\left(1 - \left(1 - \frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^p \cdot \left(1 - \frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q \cdot \left(1 - \frac{\Delta^{-1}(s_{F_k},c_k)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{1/(p+q+r)}\right)\right)
\end{array}\right\} \\
&= \left\{\begin{array}{l}
\Delta\left(t\left(1 - \left(1 - \left(\frac{\Delta^{-1}(s_T,a)}{t}\right)^{p+q+r}\right)^{\sum_{i=1}^{n}\omega_i\sum_{j=1}^{n}\omega_j\sum_{k=1}^{n}\omega_k}\right)^{1/(p+q+r)}\right), \\
\Delta\left(t\left(1 - \left(1 - \left(1 - \left(1 - \frac{\Delta^{-1}(s_I,b)}{t}\right)^{p+q+r}\right)^{\sum_{i=1}^{n}\omega_i\sum_{j=1}^{n}\omega_j\sum_{k=1}^{n}\omega_k}\right)^{1/(p+q+r)}\right)\right), \\
\Delta\left(t\left(1 - \left(1 - \left(1 - \left(1 - \frac{\Delta^{-1}(s_F,c)}{t}\right)^{p+q+r}\right)^{\sum_{i=1}^{n}\omega_i\sum_{j=1}^{n}\omega_j\sum_{k=1}^{n}\omega_k}\right)^{1/(p+q+r)}\right)\right)
\end{array}\right\} \\
&= \langle (s_T,a),(s_I,b),(s_F,c)\rangle = l
\end{aligned}$$

□

Property 8. *(Monotonicity) Let* $l_{x_i} = \left\langle \left(s_{T_{x_i}}, a_{x_i}\right), \left(s_{I_{x_i}}, b_{x_i}\right), \left(s_{F_{x_i}}, c_{x_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *and* $l_{y_i} = \left\langle \left(s_{T_{y_i}}, a_{y_i}\right), \left(s_{I_{y_i}}, b_{y_i}\right), \left(s_{F_{y_i}}, c_{y_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *be two sets of 2TLNNs. If* $\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right) \leq \Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)$,$\Delta^{-1}\left(s_{I_{x_i}}, b_{x_i}\right) \geq \Delta^{-1}\left(s_{I_{y_i}}, b_{y_i}\right)$ *and* $\Delta^{-1}\left(s_{F_{x_i}}, c_{x_i}\right) \geq \Delta^{-1}\left(s_{F_{y_i}}, c_{y_i}\right)$ *hold for all i, then*

$$\text{G2TLNNWBM}_{\omega}^{p,q,r}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) \leq \text{G2TLNNWBM}_{\omega}^{p,q,r}(l_{y_1}, l_{y_2}, \cdots, l_{y_n}) \tag{54}$$

Proof. Let $\text{G2TLNNWBM}_{\omega}^{p,q,r}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) = \left\langle \left(s_{T_{x_i}}, a_{x_i}\right), \left(s_{I_{x_i}}, b_{x_i}\right), \left(s_{F_{x_i}}, c_{x_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ and $\text{G2TLNNWBM}_{\omega}^{p,q,r}(l_{y_1}, l_{y_2}, \cdots, l_{y_n}) = \left\langle \left(s_{T_{y_i}}, a_{y_i}\right), \left(s_{I_{y_i}}, b_{y_i}\right), \left(s_{F_{y_i}}, c_{y_i}\right) \right\rangle (i = 1, 2, \ldots, n)$, given that $\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right) \leq \Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)$, we can obtain

$$\begin{aligned} &\left(\frac{\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_j}}, a_{x_j}\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_k}}, a_{x_k}\right)}{t}\right)^r \\ &\leq \left(\frac{\Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_j}}, a_{y_j}\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_k}}, a_{y_k}\right)}{t}\right)^r \end{aligned} \tag{55}$$

$$\begin{aligned} &\left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_j}}, a_{x_j}\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_k}}, a_{x_k}\right)}{t}\right)^r\right)^{\omega_i \omega_j \omega_k} \\ &\geq \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_j}}, a_{y_j}\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_k}}, a_{y_k}\right)}{t}\right)^r\right)^{\omega_i \omega_j \omega_k} \end{aligned} \tag{56}$$

Thereafter,

$$\begin{aligned} &1 - \prod_{i,j,k=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_j}}, a_{x_j}\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_k}}, a_{x_k}\right)}{t}\right)^r\right)^{\omega_i \omega_j \omega_k} \\ &\leq 1 - \prod_{i,j,k=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_j}}, a_{y_j}\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_k}}, a_{y_k}\right)}{t}\right)^r\right)^{\omega_i \omega_j \omega_k} \end{aligned} \tag{57}$$

Furthermore,

$$\begin{aligned} &t\left(1 - \prod_{i,j,k=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_j}}, a_{x_j}\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}\left(s_{T_{x_k}}, a_{x_k}\right)}{t}\right)^r\right)^{\omega_i \omega_j \omega_k}\right)^{1/(p+q+r)} \\ &\leq t\left(1 - \prod_{i,j,k=1}^{n} \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)}{t}\right)^p \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_j}}, a_{y_j}\right)}{t}\right)^q \cdot \left(\frac{\Delta^{-1}\left(s_{T_{y_k}}, a_{y_k}\right)}{t}\right)^r\right)^{\omega_i \omega_j \omega_k}\right)^{1/(p+q+r)} \end{aligned} \tag{58}$$

That means $\Delta^{-1}(s_{T_x}, a_x) \leq \Delta^{-1}\left(s_{T_y}, a_y\right)$. Similarly, we can obtain $\Delta^{-1}(s_{I_x}, b_x) \geq \Delta^{-1}\left(s_{I_y}, b_y\right)$ and $\Delta^{-1}(s_{F_x}, c_x) \geq \Delta^{-1}\left(s_{F_y}, c_y\right)$.

If $\Delta^{-1}(s_{T_x}, a_x) < \Delta^{-1}\left(s_{T_y}, a_y\right)$ and $\Delta^{-1}(s_{I_x}, b_x) \geq \Delta^{-1}\left(s_{I_y}, b_y\right)$ and $\Delta^{-1}(s_{F_x}, c_x) \geq \Delta^{-1}\left(s_{F_y}, c_y\right)$

$$\text{G2TLNNWBM}_{\omega}^{p,q,r}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) < \text{G2LNNWBM}_{\omega}^{p,q,r}(l_{y_1}, l_{y_2}, \cdots, l_{y_n})$$

If $\Delta^{-1}(s_{T_x}, a_x) = \Delta^{-1}\left(s_{T_y}, a_y\right)$ and $\Delta^{-1}(s_{I_x}, b_x) > \Delta^{-1}\left(s_{I_y}, b_y\right)$ and $\Delta^{-1}(s_{F_x}, c_x) > \Delta^{-1}\left(s_{F_y}, c_y\right)$

$$\text{G2TLNNWBM}_{\omega}^{p,q,r}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) < \text{DG2LNNWBM}_{\omega}^{p,q,r}(l_{y_1}, l_{y_2}, \cdots, l_{y_n})$$

If $\Delta^{-1}(s_{T_x}, a_x) = \Delta^{-1}\left(s_{T_y}, a_y\right)$ and $\Delta^{-1}(s_{I_x}, b_x) = \Delta^{-1}\left(s_{I_y}, b_y\right)$ and $\Delta^{-1}(s_{F_x}, c_x) = \Delta^{-1}\left(s_{F_y}, c_y\right)$

$$\text{G2TLNNWBM}_{\omega}^{p,q,r}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) = \text{G2TLNNWBM}_{\omega}^{p,q,r}(l_{y_1}, l_{y_2}, \cdots, l_{y_n})$$

So Property 8 is right. □

Property 9. *(Boundedness) Let* $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle (i = 1, 2, \ldots, n)$ *be a set of 2TLNNs. If* $l^+ = (\max_i(S_{T_i}, a_i), \min_i(S_{I_i}, b_i), \min_i(S_{F_i}, c_i))$ *and* $l^- = (\min_i(S_{T_i}, a_i), \max_i(S_{I_i}, b_i), \max_i(S_{F_i}, c_i))$ *then*

$$l^- \leq \text{G2TLNNWBM}_{\omega}^{p,q,r}(l_1, l_2, \cdots, l_n) \leq l^+ \tag{59}$$

From Property 7,

$$\text{G2TLNNWBM}_{\omega}^{p,q,r}(l_1^-, l_2^-, \cdots, l_n^-) = l^-$$
$$\text{G2TLNNWBM}_{\omega}^{p,q,r}(l_1^+, l_2^+, \cdots, l_n^+) = l^+$$

From Property 8,

$$l^- \leq \text{G2TLNNWBM}_{\omega}^{p,q,r}(l_1, l_2, \cdots, l_n) \leq l^+$$

4.2. G2TLNNWGBM Operator

Similarly to GWBM, to consider the attribute weights, the generalized weighted geometric Bonferroni mean (GWGBM) is defined, as follows.

Definition 13 ([51])**.** *Let* $p, q, r > 0$ *and* $b_i (i = 1, 2, \cdots, n)$ *be a collection of nonnegative crisp numbers with the weights vector being* $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$*, thereby satisfying* $\omega_i \in [0, 1]$ *and* $\sum_{i=1}^{n} \omega_i = 1$*. If*

$$\text{GWGBM}_{\omega}^{p,q,r}(b_1, b_2, \cdots, b_n) = \frac{1}{p+q+r} \prod_{i,j,k=1}^{n} \left(pb_i + qb_j + rb_k\right)^{\omega_i \omega_j \omega_k} \tag{60}$$

Then we extend GWGBM to fuse the 2TLNNs and propose generalized 2-tuple linguistic neutrosophic number weighted geometric Bonferroni mean (G2TLNNWGBM) aggregation operator.

Definition 14. *Let* $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle (i = 1, 2, \ldots, n)$ *be a set of 2TLNNs with their weight vector be* $w_i = (w_1, w_2, \ldots, w_n)^T$*, thereby satisfying* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$*. If*

$$\text{G2TLNNWGBM}_{\omega}^{p,q,r}(l_1, l_2, \ldots, l_n) = \frac{1}{p+q+r} \bigotimes_{i,j,k=1}^{n} \left(pl_i \oplus ql_j \oplus rl_k\right)^{\omega_i \omega_j \omega_k} \tag{61}$$

Then we called $\text{G2TLNNWGBM}_{\omega}^{p,q,r}$ *the generalized 2-tuple linguistic neutrosophic number weighted geometric BM.*

Theorem 4. *Let $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle (i = 1, 2, \ldots, n)$ be a set of 2TLNNs. The aggregated value by using G2TLNNWGBM operators is also a 2TLNN where*

$$\text{G2TLNNWGBM}_{\omega}^{p,q,r}(l_1, l_2, \ldots, l_n) = \frac{1}{p+q+r} \underset{i,j,k=1}{\overset{n}{\otimes}} (pl_i \oplus ql_j \oplus rl_k)^{\omega_i\omega_j\omega_k}$$
$$= \left\{ \begin{array}{l} \Delta\left(t\left(1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^{p}\cdot\left(1-\frac{\Delta^{-1}(s_{T_j},a_j)}{t}\right)^{q}\cdot\left(1-\frac{\Delta^{-1}(s_{T_k},a_k)}{t}\right)^{r}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{p+q+r}}\right)\right), \\ \Delta\left(t\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^{p}\cdot\left(\frac{\Delta^{-1}(s_{I_j},b_j)}{t}\right)^{q}\cdot\left(\frac{\Delta^{-1}(s_{I_k},b_k)}{t}\right)^{r}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{p+q+r}}\right), \\ \Delta\left(t\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^{p}\cdot\left(\frac{\Delta^{-1}(s_{F_j},c_j)}{t}\right)^{q}\cdot\left(\frac{\Delta^{-1}(s_{F_k},c_k)}{t}\right)^{r}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{p+q+r}}\right) \end{array} \right\} \tag{62}$$

Proof. From Definition 5, we can obtain,

$$pl_i = \left\{ \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^{p}\right)\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^{p}\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^{p}\right) \right\} \tag{63}$$

$$ql_j = \left\{ \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{T_j},a_j)}{t}\right)^{q}\right)\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{I_j},b_j)}{t}\right)^{q}\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{F_j},c_j)}{t}\right)^{q}\right) \right\} \tag{64}$$

$$rl_k = \left\{ \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{T_k},a_k)}{t}\right)^{r}\right)\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{I_k},b_k)}{t}\right)^{r}\right), \Delta\left(t\left(\frac{\Delta^{-1}(s_{F_k},c_k)}{t}\right)^{r}\right) \right\} \tag{65}$$

Thus,

$$pl_i \oplus ql_j \oplus rl_k$$
$$= \left\{ \begin{array}{l} \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^{p}\cdot\left(1-\frac{\Delta^{-1}(s_{T_j},a_j)}{t}\right)^{q}\cdot\left(1-\frac{\Delta^{-1}(s_{T_k},a_k)}{t}\right)^{r}\right)\right), \\ \Delta\left(t\left(\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^{p}\cdot\left(\frac{\Delta^{-1}(s_{I_j},b_j)}{t}\right)^{q}\cdot\left(\frac{\Delta^{-1}(s_{I_k},b_k)}{t}\right)^{r}\right), \\ \Delta\left(t\left(\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^{p}\cdot\left(\frac{\Delta^{-1}(s_{F_j},c_j)}{t}\right)^{q}\cdot\left(\frac{\Delta^{-1}(s_{F_k},c_k)}{t}\right)^{r}\right) \end{array} \right\} \tag{66}$$

Therefore,

$$(pl_i \oplus ql_j \oplus rl_k)^{\omega_i\omega_j\omega_k}$$
$$= \left\{ \begin{array}{l} \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}(s_{T_i},a_i)}{t}\right)^{p}\cdot\left(1-\frac{\Delta^{-1}(s_{T_j},a_j)}{t}\right)^{q}\cdot\left(1-\frac{\Delta^{-1}(s_{T_k},a_k)}{t}\right)^{r}\right)^{\omega_i\omega_j\omega_k}\right), \\ \Delta\left(t\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{I_i},b_i)}{t}\right)^{p}\cdot\left(\frac{\Delta^{-1}(s_{I_j},b_j)}{t}\right)^{q}\cdot\left(\frac{\Delta^{-1}(s_{I_k},b_k)}{t}\right)^{r}\right)^{\omega_i\omega_j\omega_k}\right)\right), \\ \Delta\left(t\left(1-\left(1-\left(\frac{\Delta^{-1}(s_{F_i},c_i)}{t}\right)^{p}\cdot\left(\frac{\Delta^{-1}(s_{F_j},c_j)}{t}\right)^{q}\cdot\left(\frac{\Delta^{-1}(s_{F_k},c_k)}{t}\right)^{r}\right)^{\omega_i\omega_j\omega_k}\right)\right) \end{array} \right\} \tag{67}$$

Thereafter,

$$\begin{aligned}&\overset{n}{\underset{i,j,k=1}{\otimes}}\left(pl_i\oplus ql_j\oplus rl_k\right)^{\omega_i\omega_j\omega_k}\\&=\left\{\begin{array}{l}\Delta\left(t\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right),\\\Delta\left(t\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{I_i},b_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q\cdot\left(\frac{\Delta^{-1}\left(s_{I_k},b_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)\right),\\\Delta\left(t\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{F_i},c_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q\cdot\left(\frac{\Delta^{-1}\left(s_{F_k},c_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)\right)\end{array}\right\}\end{aligned}\tag{68}$$

Furthermore,

$$\begin{aligned}&\text{G2TLNNWGBM}_{\omega}^{p,q,r}(l_1,l_2,\ldots,l_n)=\tfrac{1}{p+q+r}\overset{n}{\underset{i,j,k=1}{\otimes}}\left(pl_i\oplus ql_j\oplus rl_k\right)^{\omega_i\omega_j\omega_k}\\&=\left\{\begin{array}{l}\Delta\left(t\left(1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{p+q+r}}\right)\right),\\\Delta\left(t\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{I_i},b_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q\cdot\left(\frac{\Delta^{-1}\left(s_{I_k},b_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{p+q+r}}\right),\\\Delta\left(t\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{F_i},c_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q\cdot\left(\frac{\Delta^{-1}\left(s_{F_k},c_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{p+q+r}}\right)\end{array}\right\}\end{aligned}\tag{69}$$

Hence, (62) is kept. □

Then, we need to prove that (62) is a 2TLNN. We need to prove two conditions, as follows:

① $0\le\Delta^{-1}(s_T,a)\le t, 0\le\Delta^{-1}(s_I,b)\le t, 0\le\Delta^{-1}(s_F,c)\le t$
② $0\le\Delta^{-1}(s_T,a)+\Delta^{-1}(s_I,b)+\Delta^{-1}(s_F,c)\le 3t$

Let

$$\begin{aligned}\frac{\Delta^{-1}(s_T,a)}{t}&=1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{p+q+r}}\\\frac{\Delta^{-1}(s_I,b)}{t}&=\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{I_i},b_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{I_j},b_j\right)}{t}\right)^q\cdot\left(\frac{\Delta^{-1}\left(s_{I_k},b_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{p+q+r}}\\\frac{\Delta^{-1}(s_F,c)}{t}&=\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(\frac{\Delta^{-1}\left(s_{F_i},c_i\right)}{t}\right)^p\cdot\left(\frac{\Delta^{-1}\left(s_{F_j},c_j\right)}{t}\right)^q\cdot\left(\frac{\Delta^{-1}\left(s_{F_k},c_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{p+q+r}}\end{aligned}$$

Proof. ① Since $0\le\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\le 1, 0\le\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\le 1, 0\le\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\le 1$ we get

$$0\le 1-\left(1-\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\le 1\tag{70}$$

Then,

$$0\le\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^p\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^q\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^r\right)^{\omega_i\omega_j\omega_k}\le 1\tag{71}$$

$$0 \leq 1-\left(1-\prod_{i,j,k=1}^{n}\left(1-\left(1-\frac{\Delta^{-1}\left(s_{T_i},a_i\right)}{t}\right)^{p}\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_j},a_j\right)}{t}\right)^{q}\cdot\left(1-\frac{\Delta^{-1}\left(s_{T_k},a_k\right)}{t}\right)^{r}\right)^{\omega_i\omega_j\omega_k}\right)^{\frac{1}{p+q+r}} \leq 1 \quad (72)$$

That means $0 \leq \Delta^{-1}(s_T,a) \leq t$, so ① is maintained, similarly, we can get $0 \leq \Delta^{-1}(s_I,b) \leq t, 0 \leq \Delta^{-1}(s_F,c) \leq t$. ② Since $0 \leq \Delta^{-1}(s_T,a) \leq t, 0 \leq \Delta^{-1}(s_I,b) \leq t, 0 \leq \Delta^{-1}(s_F,c) \leq t$, we get $0 \leq \Delta^{-1}(s_T,a)+\Delta^{-1}(s_I,b)+\Delta^{-1}(s_F,c) \leq 3t$. □

Similar to G2TLNNWBM, the G2TLNNWGBM has the same properties as follows. The proof are omitted here to save space.

Property 10. *(Idempotency) If* $l_i = \left\langle (s_{T_i},a_i),(s_{I_i},b_i),(s_{F_i},c_i)\right\rangle (i=1,2,\ldots,n)$ *are equal, then*

$$\text{G2TLNNWGBM}_{\omega}^{p,q,r}(l_1,l_2,\cdots,l_n)=l \quad (73)$$

Property 11. *(Monotonicity) Let* $l_{x_i} = \left\langle \left(s_{T_{x_i}},a_{x_i}\right),\left(s_{I_{x_i}},b_{x_i}\right),\left(s_{F_{x_i}},c_{x_i}\right)\right\rangle (i=1,2,\ldots,n)$ *and* $l_{y_i} = \left\langle \left(s_{T_{y_i}},a_{y_i}\right),\left(s_{I_{y_i}},b_{y_i}\right),\left(s_{F_{y_i}},c_{y_i}\right)\right\rangle (i=1,2,\ldots,n)$ *be two sets of 2TLNNs. If* $\Delta^{-1}\left(s_{T_{x_i}},a_{x_i}\right) \leq \Delta^{-1}\left(s_{T_{y_i}},a_{y_i}\right), \Delta^{-1}\left(s_{I_{x_i}},b_{x_i}\right) \geq \Delta^{-1}\left(s_{I_{y_i}},b_{y_i}\right)$ *and* $\Delta^{-1}\left(s_{F_{x_i}},c_{x_i}\right) \geq \Delta^{-1}\left(s_{F_{y_i}},c_{y_i}\right)$ *hold for all i, then*

$$\text{G2TLNNWGBM}_{\omega}^{p,q,r}(l_{x_1},l_{x_2},\cdots,l_{x_n}) \leq \text{G2TLNNWGBM}_{\omega}^{p,q,r}(l_{y_1},l_{y_2},\cdots,l_{y_n}) \quad (74)$$

Property 12. *(Boundedness) Let* $l_i = \left\langle (s_{T_i},a_i),(s_{I_i},b_i),(s_{F_i},c_i)\right\rangle (i=1,2,\ldots,n)$ *be a set of 2TLNNs. If* $l^+ = \left(\max_i(S_{T_i},a_i),\min_i(S_{I_i},b_i),\min_i(S_{F_i},c_i)\right)$ *and* $l^- = \left(\min_i(S_{T_i},a_i),\max_i(S_{I_i},b_i),\max_i(S_{F_i},c_i)\right)$ *then*

$$l^- \leq \text{G2TLNNWGBM}_{\omega}^{p,q,r}(l_1,l_2,\cdots,l_n) \leq l^+ \quad (75)$$

5. DG2TLNNWBM and DG2TLNNWGBM Operators

5.1. DG2TLNNWBM Operator

However, the GBM still has some drawbacks, GBWM and GWGBM can only consider the interrelationship between any three aggregated arguments. So, Zhang et al. [52] introduced a new generalization of the traditional BM because the correlations are ubiquitous among all of the arguments. The new generalization of the traditional BM is called the dual GWBM (DGBM). The DGWBM is defined, as follows.

Definition 15 ([52])**.** *Let* $b_i (i=1,2,\cdots,n)$ *be a collection of nonnegative crisp numbers with the weights vector being* $\omega=(\omega_1,\omega_2,\cdots\omega_n)^T$*, thereby satisfying* $\omega_i \in [0,1]$ *and* $\sum_{i=1}^{n}\omega_i=1$*. The dual generalized weighted Bonferroni mean (DGWBM) is defined as follows:*

$$\text{DGWBM}_{w}^{R}(b_1,b_2,\ldots,b_n)=\left(\sum_{i_1,i_2,\ldots,i_n=1}^{n}\left(\prod_{j=1}^{n}w_{i_j}b_{i_j}^{r_j}\right)\right)^{1/\sum_{j=1}^{n}r_j} \quad (76)$$

where $R=(r_1,r_2,\ldots,r_n)^T$ *is the parameter vector with* $r_i \geq 0\ (i=1,2,\ldots,n)$.

Then we extend DGWBM to fuse the 2TLNNs and propose dual generalized 2-tuple linguistic neutrosophic number weighted Bonferroni mean (DG2TLNNWBM) operator.

Definition 16. *Let $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle$ be a set of 2TLNNs. The dual generalized 2-tuple linguistic neutrosophic number weighted Bonferroni mean (DG2TLNNWBM) operator is:*

$$\text{DG2TLNNWBM}_w^R(l_1, l_2, \ldots, l_n) = \left(\underset{i_1, i_2, \ldots, i_n = 1}{\overset{n}{\oplus}} \left(\underset{j=1}{\overset{n}{\otimes}} w_{i_j} l_{ij}^{r_j} \right) \right)^{1/\sum_{i=1}^n r_j} \tag{77}$$

Theorem 5. *Let $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle$ be a set of 2TLNNs. The aggregated value by using DG2TLNNWBM operators is also a 2TLNN where*

$$\begin{aligned} &\text{DG2TLNNWBM}_w^R(l_1, l_2, \ldots, l_n) = \left(\underset{i_1, i_2, \ldots, i_n = 1}{\overset{n}{\oplus}} \left(\underset{j=1}{\overset{n}{\otimes}} w_{i_j} l_{i_j}^{r_j} \right) \right)^{1/\sum_{i=1}^n r_j} \\ &= \left\{ \begin{array}{l} \Delta\left(t\left(1 - \prod\limits_{i_1, i_2, \ldots, i_n = 1}^{n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^n r_j} \right), \\ \Delta\left(t\left(1 - \left(1 - \prod\limits_{i_1, i_2, \ldots, i_n = 1}^{n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \left(1 - \left(1 - \frac{\Delta^{-1}\left(s_{I_{i_j}}, b_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^n r_j}\right)\right), \\ \Delta\left(t\left(1 - \left(1 - \prod\limits_{i_1, i_2, \ldots, i_n = 1}^{n} \left(1 - \prod\limits_{j=1}^{n} \left(1 - \left(1 - \left(1 - \frac{\Delta^{-1}\left(s_{F_{i_j}}, c_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^n r_j}\right)\right) \end{array} \right\} \end{aligned} \tag{78}$$

Proof. According to Definition 5, we can obtain

$$l_{i_j}^{r_j} = \left\{ \Delta\left(t\left(\frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j} \right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}\left(s_{I_{i_j}}, b_{i_j}\right)}{t}\right)^{r_j}\right)\right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}\left(s_{F_{i_j}}, c_{i_j}\right)}{t}\right)^{r_j}\right)\right) \right\} \tag{79}$$

Thus,

$$w_{i_j} l_{i_j}^{r_j} = \left\{ \begin{array}{l} \Delta\left(t\left(1 - \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right), \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}\left(s_{I_{i_j}}, b_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right), \\ \Delta\left(t\left(1 - \left(1 - \frac{\Delta^{-1}\left(s_{F_{i_j}}, c_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right) \end{array} \right\} \tag{80}$$

Thereafter,

$$\underset{j=1}{\overset{n}{\otimes}} w_{i_j} l_{i_j}^{r_j} = \left\{ \begin{array}{l} \Delta\left(t\prod\limits_{j=1}^{n} \left(1 - \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right), \\ \Delta\left(t\left(1 - \prod\limits_{j=1}^{n} \left(1 - \left(1 - \left(1 - \frac{\Delta^{-1}\left(s_{I_{i_j}}, b_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right), \\ \Delta\left(t\left(1 - \prod\limits_{j=1}^{n} \left(1 - \left(1 - \left(1 - \frac{\Delta^{-1}\left(s_{F_{i_j}}, c_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right) \end{array} \right\} \tag{81}$$

Furthermore,

$$\underset{i_1,i_2,\ldots,i_n=1}{\overset{n}{\oplus}}\left(\underset{j=1}{\overset{n}{\otimes}} w_{i_j} l_{i_j}^{r_j}\right) = \left\{\begin{array}{l} \Delta\left(t\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)\right), \\ \Delta\left(t\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\frac{\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right), \\ \Delta\left(t\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\frac{\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right) \end{array}\right\} \quad (82)$$

Therefore,

$$\mathrm{DG2TLNNWBM}_w^R(l_1,l_2,\ldots,l_n) = \left(\underset{i_1,i_2,\ldots,i_n=1}{\overset{n}{\oplus}}\left(\underset{j=1}{\overset{n}{\otimes}} w_{i_j} l_{i_j}^{r_j}\right)\right)^{1/\sum_{i=1}^{n} r_j}$$

$$= \left\{\begin{array}{l} \Delta\left(t\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}\right), \\ \Delta\left(t\left(1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\frac{\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}\right)\right), \\ \Delta\left(t\left(1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\frac{\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}\right)\right) \end{array}\right\} \quad (83)$$

Hence, (78) is kept. □

Then, we need to prove that (78) is a 2TLNN. We need to prove two conditions as follows:

① $0 \le \Delta^{-1}(s_T,a) \le t, 0 \le \Delta^{-1}(s_I,b) \le t, 0 \le \Delta^{-1}(s_F,c) \le t$
② $0 \le \Delta^{-1}(s_T,a) + \Delta^{-1}(s_I,b) + \Delta^{-1}(s_F,c) \le 3t$

Let

$$\frac{\Delta^{-1}(s_T,a)}{t} = \left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}$$

$$\frac{\Delta^{-1}(s_I,b)}{t} = 1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\frac{\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}$$

$$\frac{\Delta^{-1}(s_F,c)}{t} = 1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\frac{\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}$$

Proof. ① Since $0 \leq \frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t} \leq 1$, we get

$$0 \leq 1 - \left(\frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j} \leq 1 \tag{84}$$

Then,

$$0 \leq 1 - \prod_{j=1}^{n}\left(1 - \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right) \leq 1 \tag{85}$$

Furthermore,

$$0 \leq 1 - \prod_{i_1, i_2, \ldots, i_n = 1}^{n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right) \leq 1 \tag{86}$$

Therefore,

$$0 \leq \left(1 - \prod_{i_1, i_2, \ldots, i_n = 1}^{n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j} \leq 1 \tag{87}$$

That means $0 \leq \Delta^{-1}(s_T, a) \leq t$, so ① is maintained, similarly, we can get $0 \leq \Delta^{-1}(s_I, b) \leq t, 0 \leq \Delta^{-1}(s_F, c) \leq t$. ② Since $0 \leq \Delta^{-1}(s_T, a) \leq t, 0 \leq \Delta^{-1}(s_I, b) \leq t, 0 \leq \Delta^{-1}(s_F, c) \leq t$, we get $0 \leq \Delta^{-1}(s_T, a) + \Delta^{-1}(s_I, b) + \Delta^{-1}(s_F, c) \leq 3t$. □

Then, we will discuss some properties of the DG2TLNNWBM operator.

Property 13. *(Monotonicity) Let* $l_{x_i} = \left\langle \left(s_{T_{x_i}}, a_{x_i}\right), \left(s_{I_{x_i}}, b_{x_i}\right), \left(s_{F_{x_i}}, c_{x_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *and* $l_{y_i} = \left\langle \left(s_{T_{y_i}}, a_{y_i}\right), \left(s_{I_{y_i}}, b_{y_i}\right), \left(s_{F_{y_i}}, c_{y_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *be two sets of 2TLNNs. If* $\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right) \leq \Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right), \Delta^{-1}\left(s_{I_{x_i}}, b_{x_i}\right) \geq \Delta^{-1}\left(s_{I_{y_i}}, b_{y_i}\right)$ *and* $\Delta^{-1}\left(s_{F_{x_i}}, c_{x_i}\right) \geq \Delta^{-1}\left(s_{F_{y_i}}, c_{y_i}\right)$ *hold for all i, then*

$$\text{DG2TLNNWBM}_{w}^{R}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) \leq \text{DG2TLNNWBM}_{w}^{R}(l_{y_1}, l_{y_2}, \cdots, l_{y_n}) \tag{88}$$

Proof. Let $\text{DG2TLNNWBM}_{w}^{R}(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) = \left\langle \left(s_{T_{x_i}}, a_{x_i}\right), \left(s_{I_{x_i}}, b_{x_i}\right), \left(s_{F_{x_i}}, c_{x_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ and $\text{DG2TLNNWBM}_{w}^{R}(l_{y_1}, l_{y_2}, \cdots, l_{y_n}) = \left\langle \left(s_{T_{y_i}}, a_{y_i}\right), \left(s_{I_{y_i}}, b_{y_i}\right), \left(s_{F_{y_i}}, c_{y_i}\right) \right\rangle (i = 1, 2, \ldots, n)$, given that $\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right) \leq \Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)$, we can obtain

$$\left(\frac{\Delta^{-1}\left(s_{T_{x_{i_j}}}, a_{x_{i_j}}\right)}{t}\right)^{r_j} \leq \left(\frac{\Delta^{-1}\left(s_{T_{y_{i_j}}}, a_{y_{i_j}}\right)}{t}\right)^{r_j} \tag{89}$$

$$\left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{x_{i_j}}}, a_{x_{i_j}}\right)}{t}\right)^{r_j}\right)^{w_{i_j}} \geq \left(1 - \left(\frac{\Delta^{-1}\left(s_{T_{y_{i_j}}}, a_{y_{i_j}}\right)}{t}\right)^{r_j}\right)^{w_{i_j}} \tag{90}$$

Thereafter,

$$\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{T_{x_{i_j}}},a_{x_{i_j}}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\leq\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{T_{y_{i_j}}},a_{y_{i_j}}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right) \tag{91}$$

Furthermore,

$$\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{T_{x_{i_j}}},a_{x_{i_j}}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\geq\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{T_{y_{i_j}}},a_{y_{i_j}}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right) \tag{92}$$

$$\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{T_{x_{i_j}}},a_{x_{i_j}}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n}r_j}$$
$$\leq\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\Delta^{-1}\left(s_{T_{y_{i_j}}},a_{y_{i_j}}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n}r_j}$$

That means $\Delta^{-1}(s_{T_x},a_x)\leq\Delta^{-1}\left(s_{T_y},a_y\right)$. Similarly, we can obtain $\Delta^{-1}(s_{I_x},b_x)\geq\Delta^{-1}\left(s_{I_y},b_y\right)$ and $\Delta^{-1}(s_{F_x},c_x)\geq\Delta^{-1}\left(s_{F_y},c_y\right)$.

If $\Delta^{-1}(s_{T_x},a_x)<\Delta^{-1}\left(s_{T_y},a_y\right)$ and $\Delta^{-1}(s_{I_x},b_x)\geq\Delta^{-1}\left(s_{I_y},b_y\right)$ and $\Delta^{-1}(s_{F_x},c_x)\geq\Delta^{-1}\left(s_{F_y},c_y\right)$

$$\text{DG2TLNNWBM}_w^R(l_{x_1},l_{x_2},\cdots,l_{x_n})<\text{DG2TLNNWBM}_w^R(l_{y_1},l_{y_2},\cdots,l_{y_n})$$

If $\Delta^{-1}(s_{T_x},a_x)=\Delta^{-1}\left(s_{T_y},a_y\right)$ and $\Delta^{-1}(s_{I_x},b_x)>\Delta^{-1}\left(s_{I_y},b_y\right)$ and $\Delta^{-1}(s_{F_x},c_x)>\Delta^{-1}\left(s_{F_y},c_y\right)$

$$\text{DG2TLNNWBM}_w^R(l_{x_1},l_{x_2},\cdots,l_{x_n})<\text{DG2TLNNWBM}_w^R(l_{y_1},l_{y_2},\cdots,l_{y_n})$$

If $\Delta^{-1}(s_{T_x},a_x)=\Delta^{-1}\left(s_{T_y},a_y\right)$ and $\Delta^{-1}(s_{I_x},b_x)=\Delta^{-1}\left(s_{I_y},b_y\right)$ and $\Delta^{-1}(s_{F_x},c_x)=\Delta^{-1}\left(s_{F_y},c_y\right)$

$$\text{DG2TLNNWBM}_w^R(l_{x_1},l_{x_2},\cdots,l_{x_n})=\text{DG2TLNNWBM}_w^R(l_{y_1},l_{y_2},\cdots,l_{y_n})$$

So Property 13 is right. □

Property 14. *(Boundedness) Let*$l_i=\langle(s_{T_i},a_i),(s_{I_i},b_i),(s_{F_i},c_i)\rangle(i=1,2,\ldots,n)$ *be a set of 2TLNNs. If* $l^+=\left(\max_i\left(S_{T_i},a_i\right),\min_i\left(S_{I_i},b_i\right),\min_i\left(S_{F_i},c_i\right)\right)$ *and* $l^-=\left(\min_i\left(S_{T_i},a_i\right),\max_i\left(S_{I_i},b_i\right),\max_i\left(S_{F_i},c_i\right)\right)$ *then*

$$l^-\leq\text{DG2TLNNWBM}_w^R(l_1,l_2,\cdots,l_n)\leq l^+ \tag{93}$$

From Theorem 5, we can obtain

$$\mathrm{DG2TLNNWBM}_w^R(l_1^-, l_2^-, \cdots, l_n^-) = \left\{ \begin{array}{l} \Delta\left(t\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\min\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}\right), \\ \Delta\left(t\left(1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\frac{\max\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}\right)\right), \\ \Delta\left(t\left(1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\frac{\max\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}\right)\right) \end{array} \right\}$$

$$\mathrm{DG2TLNNWBM}_w^R(l_1^+, l_2^+, \cdots, l_n^+) = \left\{ \begin{array}{l} \Delta\left(t\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(\frac{\max\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}\right), \\ \Delta\left(t\left(1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\frac{\min\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}\right)\right), \\ \Delta\left(t\left(1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(1-\frac{\min\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)^{w_{i_j}}\right)\right)\right)^{1/\sum_{i=1}^{n} r_j}\right)\right) \end{array} \right\}$$

From Property 13,

$$l^- \leq \mathrm{DG2TLNNWBM}_w^R(l_1, l_2, \cdots, l_n) \leq l^+$$

5.2. DG2TLNNWGBM Operator

Similarly to DGWBM, to consider the attribute weights, the dual generalized weighted geometric Bonferroni mean (DGWGBM) is defined, as follows.

Definition 17 ([52]). *Let and $b_i (i = 1, 2, \cdots, n)$ be a collection of nonnegative crisp numbers with the weights vector being $\omega = (\omega_1, \omega_2, \cdots \omega_n)^T$, thereby satisfying $\omega_i \in [0, 1]$ and $\sum\limits_{i=1}^{n} \omega_i = 1$. If*

$$\mathrm{DGWGBM}_w^R(b_1, b_2, \cdots, b_n) = \frac{1}{\sum_{j=1}^{n} r_j}\left(\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(\sum_{j=1}^{n}\left(r_j l_{i_j}\right)\right)\right)^{\prod_{j=1}^{n} w_{i_j}} \tag{94}$$

where $R = (r_1, r_2, \ldots, r_n)^T$ is the parameter vector with $r_i \geq 0$ $(i = 1, 2, \ldots, n)$.

Then we extend DGWGBM to fuse the 2TLNNs and propose dual generalized 2-tuple linguistic neutrosophic number weighted geometric Bonferroni mean (DG2TLNNWGBM) aggregation operator.

Definition 18. *Let $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle (i = 1, 2, \ldots, n)$be a set of 2TLNNs with their weight vector be $w_i = (w_1, w_2, \ldots, w_n)^T$, thereby satisfying $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. If*

$$\mathrm{DG2TLNNWGBM}_w^R(l_1, l_2, \ldots, l_n) = \frac{1}{\sum_{j=1}^{n} r_j}\left(\bigotimes_{i_1,i_2,\ldots,i_n=1}^{n}\left(\bigoplus_{j=1}^{n}\left(r_j l_{i_j}\right)\right)^{\prod_{j=1}^{n} w_{i_j}}\right) \tag{95}$$

Then we called $\mathrm{DG2TLNNWGBM}_w^R$ *the dual generalized 2-tuple linguistic neutrosophic number weighted geometric BM.*

Theorem 6. *Let* $l_i = \left\langle \left(s_{T_i}, a_i\right), \left(s_{I_i}, b_i\right), \left(s_{F_i}, c_i\right) \right\rangle (i = 1, 2, \ldots, n)$ *be a set of 2TLNNs. The aggregated value by using DG2TLNNWGBM operators is also a 2TLNN where*

$$\begin{aligned}
&\mathrm{DG2TLNNWGBM}_w^R(l_1, l_2, \ldots, l_n) \\
&= \frac{1}{\sum_{j=1}^n r_j}\left(\underset{i_1,i_2,\ldots,i_n=1}{\overset{n}{\otimes}}\left(\overset{n}{\underset{j=1}{\oplus}}\left(r_j l_{i_j}\right)\right)^{\prod_{j=1}^n w_{i_j}}\right) \\
&= \left\{\begin{array}{l}
\Delta\left(t\left(1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^n\left(1-\prod_{j=1}^n\left(1-\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)^{\frac{1}{\sum_{j=1}^n r_j}}\right)\right), \\
\Delta\left(t\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^n\left(1-\prod_{j=1}^n\left(\frac{\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)^{\frac{1}{\sum_{j=1}^n r_j}}\right), \\
\Delta\left(t\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^n\left(1-\prod_{j=1}^n\left(\frac{\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)^{\frac{1}{\sum_{j=1}^n r_j}}\right)
\end{array}\right\}
\end{aligned} \tag{96}$$

Proof. From Definition 5, we can obtain,

$$r_j l_{i_j} = \left\{\begin{array}{l}
\Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)\right), \Delta\left(t\left(1-\left(1-\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)\right), \\
\Delta\left(t\left(\frac{\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)
\end{array}\right\} \tag{97}$$

Thus,

$$\overset{n}{\underset{j=1}{\oplus}}\left(r_j l_{i_j}\right) = \left\{\begin{array}{l}
\Delta\left(t\left(1-\prod_{j=1}^n\left(1-\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)\right), \Delta\left(t\prod_{j=1}^n\left(\frac{\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right), \\
\Delta\left(t\prod_{j=1}^n\left(\frac{\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)
\end{array}\right\} \tag{98}$$

Therefore,

$$\begin{aligned}
&\left(\overset{n}{\underset{j=1}{\oplus}}\left(r_j l_{i_j}\right)\right)^{\prod_{j=1}^n w_{i_j}} \\
&= \left\{\begin{array}{l}
\Delta\left(t\left(1-\prod_{j=1}^n\left(1-\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right), \Delta\left(t\left(1-\left(1-\prod_{j=1}^n\left(\frac{\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)\right), \\
\Delta\left(t\left(1-\left(1-\prod_{j=1}^n\left(\frac{\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)\right)
\end{array}\right\}
\end{aligned} \tag{99}$$

Thereafter,

$$\underset{i_1,i_2,\ldots,i_n=1}{\overset{n}{\otimes}}\left(\overset{n}{\underset{j=1}{\oplus}}\left(r_j l_{i_j}\right)\right)^{\prod_{j=1}^n w_{i_j}} = \left\{\begin{array}{l} \Delta\left(t\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right), \\ \Delta\left(t\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)\right), \\ \Delta\left(t\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)\right) \end{array}\right\} \quad (100)$$

Furthermore,

$$\begin{array}{l} \mathrm{DG2TLNNWGBM}_w^R(l_1,l_2,\ldots,l_n) \\ = \frac{1}{\sum_{j=1}^n r_j}\left(\underset{i_1,i_2,\ldots,i_n=1}{\overset{n}{\otimes}}\left(\overset{n}{\underset{j=1}{\oplus}}\left(r_j l_{i_j}\right)\right)^{\prod_{j=1}^n w_{i_j}}\right) \\ = \left\{\begin{array}{l} \Delta\left(t\left(1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)^{\frac{1}{\sum_{j=1}^n r_j}}\right)\right), \\ \Delta\left(t\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)^{\frac{1}{\sum_{j=1}^n r_j}}\right), \\ \Delta\left(t\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)^{\frac{1}{\sum_{j=1}^n r_j}}\right) \end{array}\right\} \end{array} \quad (101)$$

Hence, (96) is kept. □

Then, we need to prove that (96) is a 2TLNN. We need to prove two conditions, as follows:

① $0 \le \Delta^{-1}(s_T,a) \le t, 0 \le \Delta^{-1}(s_I,b) \le t, 0 \le \Delta^{-1}(s_F,c) \le t$
② $0 \le \Delta^{-1}(s_T,a) + \Delta^{-1}(s_I,b) + \Delta^{-1}(s_F,c) \le 3t$

Let

$$\frac{\Delta^{-1}(s_T,a)}{t} = 1-\left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(1-\frac{\Delta^{-1}\left(s_{T_{i_j}},a_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)^{\frac{1}{\sum_{j=1}^n r_j}}$$

$$\frac{\Delta^{-1}(s_I,b)}{t} = \left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{I_{i_j}},b_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)^{\frac{1}{\sum_{j=1}^n r_j}}$$

$$\frac{\Delta^{-1}(s_F,c)}{t} = \left(1-\prod_{i_1,i_2,\ldots,i_n=1}^{n}\left(1-\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{F_{i_j}},c_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^n w_{i_j}}\right)^{\frac{1}{\sum_{j=1}^n r_j}}$$

Proof. ① Since $0 \leq \frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t} \leq 1$, we get

$$0 \leq 1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j} \leq 1 \quad (102)$$

Then,

$$0 \leq \prod_{i_1, i_2, \ldots, i_n = 1}^{n}\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^{n} w_{i_j}} \leq 1 \quad (103)$$

$$0 \leq 1 - \left(1 - \prod_{i_1, i_2, \ldots, i_n = 1}^{n}\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}\left(s_{T_{i_j}}, a_{i_j}\right)}{t}\right)^{r_j}\right)^{\prod_{j=1}^{n} w_{i_j}}\right)^{\frac{1}{\sum_{j=1}^{n} r_j}} \leq 1 \quad (104)$$

That means $0 \leq \Delta^{-1}(s_T, a) \leq t$, so ① is maintained, similarly, we can get $0 \leq \Delta^{-1}(s_I, b) \leq t, 0 \leq \Delta^{-1}(s_F, c) \leq t$. ② Since $0 \leq \Delta^{-1}(s_T, a) \leq t, 0 \leq \Delta^{-1}(s_I, b) \leq t, 0 \leq \Delta^{-1}(s_F, c) \leq t$, we get $0 \leq \Delta^{-1}(s_T, a) + \Delta^{-1}(s_I, b) + \Delta^{-1}(s_F, c) \leq 3t$. □

Similar to DG2TLNNWBM, the DG2TLNNWGBM has the same properties, as follows. The proof are omitted here to save space.

Property 15. *(Monotonicity) Let* $l_{x_i} = \left\langle \left(s_{T_{x_i}}, a_{x_i}\right), \left(s_{I_{x_i}}, b_{x_i}\right), \left(s_{F_{x_i}}, c_{x_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *and* $l_{y_i} = \left\langle \left(s_{T_{y_i}}, a_{y_i}\right), \left(s_{I_{y_i}}, b_{y_i}\right), \left(s_{F_{y_i}}, c_{y_i}\right) \right\rangle (i = 1, 2, \ldots, n)$ *be two sets of 2TLNNs. If* $\Delta^{-1}\left(s_{T_{x_i}}, a_{x_i}\right) \leq \Delta^{-1}\left(s_{T_{y_i}}, a_{y_i}\right)$,$\Delta^{-1}\left(s_{I_{x_i}}, b_{x_i}\right) \geq \Delta^{-1}\left(s_{I_{y_i}}, b_{y_i}\right)$ *and* $\Delta^{-1}\left(s_{F_{x_i}}, c_{x_i}\right) \geq \Delta^{-1}\left(s_{F_{y_i}}, c_{y_i}\right)$ *hold for all i, then*

$$\mathrm{DG2TLNNWGBM}_w^R(l_{x_1}, l_{x_2}, \cdots, l_{x_n}) \leq \mathrm{DG2TLNNWGBM}_w^R(l_{y_1}, l_{y_2}, \cdots, l_{y_n}) \quad (105)$$

Property 16. *(Boundedness) Let* $l_i = \langle (s_{T_i}, a_i), (s_{I_i}, b_i), (s_{F_i}, c_i) \rangle (i = 1, 2, \ldots, n)$ *be a set of 2TLNNs. If* $l^+ = (\max_i(S_{T_i}, a_i), \min_i(S_{I_i}, b_i), \min_i(S_{F_i}, c_i))$ *and* $l^- = (\min_i(S_{T_i}, a_i), \max_i(S_{I_i}, b_i), \max_i(S_{F_i}, c_i))$ *then*

$$l^- \leq \mathrm{DG2TLNNWGBM}_w^R(l_1, l_2, \cdots, l_n) \leq l^+ \quad (106)$$

6. Numerical Example and Comparative Analysis

6.1. Numerical Example

Given the rise in environmental and resource conservation importance, green supply chain management has seen growth within industry. In addition, balancing economic development and environmental development is one of the critical issues faced by managers to help organizations maintain a strategically competitive position. Green supplier management as one of important part of green supply chain is also critical issue for effective green supply chain management. Thus, in this section, we shall present a numerical example to select green suppliers in green supply chain management with 2TLNNs in order to illustrate the method that is proposed in this paper. There is a panel with five possible green suppliers in green supply chain management A_i $(i = 1, 2, 3, 4, 5)$ to select. The experts selects four attribute to evaluate the five possible green suppliers: ① G_1 is the product quality factor; ② G_2 is the environmental factors; ③ G_3 is the delivery factor; and, ④ G_4 is the price factor. The five possible green suppliers A_i $(i = 1, 2, 3, 4, 5)$ are to be evaluated using the 2TLNNs by the three decision maker under the above four attributes (whose weighting vector $\omega = (0.35, 0.10, 0.25, 0.30)$, expert weighting vector, which are listed in Tables 1–3.

Table 1. 2-tuple linguistic neutrosophic number (2TLNN) decision matrix (R_1).

	G$_1$	**G$_2$**	**G$_3$**	**G$_4$**
A$_1$	<(s_3, 0), (s_2, 0) (s_1, 0)>	<(s_3, 0), (s_4, 0) (s_2, 0)>	<(s_4, 0), (s_4, 0) (s_2, 0)>	<(s_4, 0), (s_4, 0) (s_2, 0)>
A$_2$	<(s_4, 0), (s_3, 0) (s_2, 0)>	<(s_5, 0), (s_4, 0) (s_4, 0)>	<(s_4, 0), (s_4, 0) (s_2, 0)>	<(s_4, 0), (s_3, 0) (s_3, 0)>
A$_3$	<(s_4, 0), (s_3, 0) (s_4, 0)>	<(s_3, 0), (s_3, 0) (s_2, 0)>	<(s_5, 0), (s_3, 0) (s_2, 0)>	<(s_4, 0), (s_3, 0) (s_4, 0)>
A$_4$	<(s_5, 0), (s_5, 0) (s_4, 0)>	<(s_4, 0), (s_3, 0) (s_3, 0)>	<(s_5, 0), (s_4, 0) (s_5, 0)>	<(s_3, 0), (s_4, 0) (s_1, 0)>
A$_5$	<(s_3, 0), (s_4, 0) (s_2, 0)>	<(s_4, 0), (s_5, 0) (s_2, 0)>	<(s_3, 0), (s_4, 0) (s_1, 0)>	<(s_4, 0), (s_3, 0) (s_2, 0)>

Table 2. 2TLNN decision matrix (R_2).

	G$_1$	**G$_2$**	**G$_3$**	**G$_4$**
A$_1$	<(s_5, 0), (s_4, 0) (s_3, 0)>	<(s_3, 0), (s_5, 0) (s_2, 0)>	<(s_3, 0), (s_1, 0) (s_2, 0)>	<(s_4, 0), (s_1, 0) (s_3, 0)>
A$_2$	<(s_2, 0), (s_3, 0) (s_3, 0)>	<(s_3, 0), (s_3, 0) (s_3, 0)>	<(s_3, 0), (s_4, 0) (s_2, 0)>	<(s_5, 0), (s_4, 0) (s_3, 0)>
A$_3$	<(s_5, 0), (s_3, 0) (s_3, 0)>	<(s_3, 0), (s_2, 0) (s_2, 0)>	<(s_2, 0), (s_3, 0) (s_4, 0)>	<(s_3, 0), (s_2, 0) (s_4, 0)>
A$_4$	<(s_3, 0), (s_5, 0) (s_2, 0)>	<(s_4, 0), (s_2, 0) (s_3, 0)>	<(s_3, 0), (s_4, 0) (s_5, 0)>	<(s_5, 0), (s_1, 0) (s_4, 0)>
A$_5$	<(s_3, 0), (s_3, 0) (s_1, 0)>	<(s_3, 0), (s_4, 0) (s_5, 0)>	<(s_4, 0), (s_5, 0) (s_1, 0)>	<(s_5, 0), (s_3, 0) (s_2, 0)>

Table 3. 2TLNN decision matrix (R_3).

	G$_1$	**G$_2$**	**G$_3$**	**G$_4$**
A$_1$	<(s_5, 0), (s_3, 0) (s_1, 0)>	<(s_4, 0), (s_2, 0) (s_1, 0)>	<(s_4, 0), (s_4, 0) (s_3, 0)>	<(s_4, 0), (s_1, 0) (s_3, 0)>
A$_2$	<(s_4, 0), (s_2, 0) (s_2, 0)>	<(s_4, 0), (s_5, 0) (s_4, 0)>	<(s_3, 0), (s_2, 0) (s_3, 0)>	<(s_2, 0), (s_1, 0) (s_3, 0)>
A$_3$	<(s_2, 0), (s_1, 0) (s_4, 0)>	<(s_3, 0), (s_2, 0) (s_2, 0)>	<(s_4, 0), (s_5, 0) (s_2, 0)>	<(s_2, 0), (s_4, 0) (s_4, 0)>
A$_4$	<(s_5, 0), (s_4, 0) (s_4, 0)>	<(s_5, 0), (s_4, 0) (s_2, 0)>	<(s_3, 0), (s_4, 0) (s_5, 0)>	<(s_5, 0), (s_3, 0) (s_1, 0)>
A$_5$	<(s_3, 0), (s_3, 0) (s_2, 0)>	<(s_4, 0), (s_2, 0) (s_2, 0)>	<(s_4, 0), (s_2, 0) (s_3, 0)>	<(s_5, 0), (s_3, 0) (s_4, 0)>

In the following, we utilize the approach that was developed to select green suppliers in green supply chain management.

Definition 19. *Let* $l_j = \left\langle \left(s_{T_j}, a_j\right), \left(s_{I_j}, b_j\right), \left(s_{F_j}, c_j\right) \right\rangle (j = 1, 2, \ldots, n)$ *be a set of 2TLNNs with their weight vector be* $w_i = (w_1, w_2, \ldots, w_n)^T$*, thereby satisfying* $w_i \in [0, 1]$ *and* $\sum_{i=1}^{n} w_i = 1$*, then we can obtain*

$$\begin{aligned} \text{2TLNNWAA}(l_1, l_2, \ldots, l_n) &= \sum_{j=1}^{n} w_j l_j \\ &= \left\{ \begin{array}{l} \Delta\left(t\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}\left(s_{T_j}, a_j\right)}{t}\right)^{w_j}\right)\right), \Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{I_j}, b_j\right)}{t}\right)^{w_j}\right), \\ \Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{F_j}, c_j\right)}{t}\right)^{w_j}\right) \end{array} \right\} \end{aligned} \tag{107}$$

$$\begin{aligned} \text{2TLNNWGA}(l_1, l_2, \ldots, l_n) &= \sum_{j=1}^{n} (l_j)^{w_j} \\ &= \left\{ \begin{array}{l} \Delta\left(t\prod_{j=1}^{n}\left(\frac{\Delta^{-1}\left(s_{T_j}, a_j\right)}{t}\right)^{w_j}\right), \Delta\left(t\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}\left(s_{I_j}, b_j\right)}{t}\right)^{w_j}\right)\right), \\ \Delta\left(t\left(1 - \prod_{j=1}^{n}\left(1 - \frac{\Delta^{-1}\left(s_{F_j}, c_j\right)}{t}\right)^{w_j}\right)\right) \end{array} \right\} \end{aligned} \tag{108}$$

Step 1. According to 2TLNNs $r_{ij} (i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$, we can aggregate all of the 2TLNNs r_{ij} by using the 2TLNNWAA (2TLNNWGA) operator to get the overall 2TLNNs A_i $(i = 1, 2, 3, 4, 5)$ of the green suppliers A_i. Then the aggregating results are shown in Table 4.

Table 4. The aggregating results by the 2TLNNWAA operator.

	G_1	G_2
A_1	$<(s_5, 0.000), (s_3, 0.464), (s_2, -0.268)>$	$<(s_4, -0.449), (s_3, 0.162), (s_1, 0.414)>$
A_2	$<(s_3, 0.172), (s_2, 0.449), (s_2, 0.449)>$	$<(s_4, -0.449), (s_4, -0.127), (s_3, 0.464)>$
A_3	$<(s_4, 0.000), (s_2, -0.268), (s_3, 0.464)>$	$< (s_3, 0.000), (s_2, 0.000), (s_2, 0.000)>$
A_4	$<(s_4, 0.268), (s_4, 0.472), (s_3, -0.172)>$	$<(s_5, -0.414), (s_3, -0.172), (s_2, 0.449)>$
A_5	$<(s_3, 0.000), (s_3, 0.000), (s_1, 0.414)>$	$<(s_4, -0.449), (s_3, -0.172), (s_3, 0.162)>$
	G_3	G_4
A_1	$<(s_4, -0.449), (s_2, 0.000), (s_2, 0.449)>$	$<(s_4, 0.000), (s_1, 0.000), (s_3, 0.000)>$
A_2	$<(s_3, 0.000), (s_3, -0.172), (s_2, 0.449)>$	$<(s_4, 0.000), (s_2, 0.000), (s_3, 0.000)>$
A_3	$<(s_3, 0.172), (s_4, -0.127), (s_3, 0.172)>$	$<(s_3, -0.464), (s_3, -0.172), (s_4, 0.000)>$
A_4	$<(s_3, 0.000), (s_4, 0.000), (s_5, 0.000)>$	$<(s_5, 0.000), (s_2, -0.268), (s_2, 0.000)>$
A_5	$<(s_4, 0.000), (s_3, 0.162), (s_2, -0.268)>$	$<(s_5, 0.000), (s_3, 0.000), (s_3, -0.172)>$

Step 2. According to Table 4, we can aggregate all of the 2TLNNs r_{ij} by using the DG2TLNNWBM (DG2TLNNWGBM) operator to get the overall 2TLNNs A_i $(i = 1, 2, 3, 4, 5)$ of the green suppliers A_i. Suppose that $P = (1, 1, 1, 1)$, then the aggregating results are shown in Table 5.

Table 5. The aggregating results of the green suppliers by the DG2TLNNWBM (DG2TLNNWGBM) operator.

	DG2TLNNWBM	DG2TLNNWGBM
A_1	$<(s_4, 0.209), (s_2, 0.299), (s_2, 0.251)>$	$<(s_4, 0.192), (s_2, 0.336), (s_2, 0.262)>$
A_2	$<(s_3, 0.418), (s_3, -0.454), (s_3, -0.286)>$	$<(s_3, 0.414), (s_3, -0.446), (s_3, -0.283)>$
A_3	$<(s_3, 0.259), (s_3, -0.390), (s_3, 0.316)>$	$<(s_3, 0.250), (s_3, -0.371), (s_3, 0.326)>$
A_4	$<(s_4, 0.226), (s_3, 0.354), (s_3, 0.067)>$	$<(s_4, 0.201), (s_3, 0.396), (s_3, 0.108)>$
A_5	$<(s_4, -0.073), (s_3, 0.023), (s_2, 0.079)>$	$<(s_4, -0.098), (s_3, 0.023), (s_2, 0.095)>$

Step 3. According to the aggregating results shown in Table 5 and the score functions of the green suppliers are shown in Table 6.

Table 6. The score functions of the green suppliers.

	DG2TLNNWBM	DG2TLNNWGBM
A_1	$(s_4, -0.114)$	$(s_4, -0.135)$
A_2	$(s_3, 0.386)$	$(s_3, 0.381)$
A_3	$(s_3, 0.111)$	$(s_3, 0.098)$
A_4	$(s_3, 0.268)$	$(s_3, 0.232)$
A_5	$(s_4, -0.392)$	$(s_4, -0.405)$

Step 4. According to the score functions shown in Table 6 and the comparison formula of score functions, the ordering of the green suppliers is shown in Table 7. Note that "$>$" means "preferred to". As we can see, depending on the aggregation operators that were used, the best green supplier is A_1.

Table 7. Ordering of the green suppliers.

	Ordering
DG2TLNNWBM	$A_1 > A_5 > A_2 > A_4 > A_3$
DG2TLNNWGBM	$A_1 > A_5 > A_2 > A_4 > A_3$

6.2. Influence of the Parameter on the Final Result

In order to show the effects on the ranking results by changing parameters of P in the DG2TLNNWBM (DG2TLNNWGBM) operators, all of the results are shown in Tables 8 and 9.

Table 8. Ranking results for different operational parameters of the DG2TLNNWBM operator.

P	$s(A_1)$	$s(A_2)$	$s(A_3)$	$s(A_4)$	$s(A_5)$	Ordering
(1, 1, 1, 1)	$(s_4, -0.114)$	$(s_3, 0.386)$	$(s_3, 0.111)$	$(s_3, 0.268)$	$(s_4, -0.392)$	$A_1 > A_5 > A_2 > A_4 > A_3$
(2, 2, 2, 2)	$(s_5, 0.085)$	$(s_5, -0.267)$	$(s_5, -0.455)$	$(s_5, -0.263)$	$(s_5, -0.073)$	$A_1 > A_5 > A_4 > A_2 > A_3$
(3, 3, 3, 3)	$(s_5, 0.333)$	$(s_5, 0.032)$	$(s_5, -0.078)$	$(s_5, 0.163)$	$(s_5, 0.234)$	$A_1 > A_5 > A_4 > A_2 > A_3$
(4, 4, 4, 4)	$(s_5, 0.410)$	$(s_5, 0.121)$	$(s_5, 0.055)$	$(s_5, -0.326)$	$(s_5, 0.339)$	$A_1 > A_5 > A_4 > A_2 > A_3$
(5, 5, 5, 5)	$(s_5, 0.445)$	$(s_5, 0.156)$	$(s_5, 0.115)$	$(s_5, 0.404)$	$(s_5, 0.389)$	$A_1 > A_4 > A_5 > A_2 > A_3$
(6, 6, 6, 6)	$(s_5, 0.466)$	$(s_5, 0.174)$	$(s_5, 0.149)$	$(s_5, 0.448)$	$(s_5, 0.421)$	$A_1 > A_4 > A_5 > A_2 > A_3$
(7, 7, 7, 7)	$(s_5, 0.482)$	$(s_5, 0.185)$	$(s_5, 0.171)$	$(s_5, 0.476)$	$(s_5, 0.444)$	$A_1 > A_4 > A_5 > A_2 > A_3$
(8, 8, 8, 8)	$(s_5, 0.496)$	$(s_5, 0.195)$	$(s_5, 0.187)$	$(s_5, 0.495)$	$(s_5, 0.463)$	$A_1 > A_4 > A_5 > A_2 > A_3$
(9, 9, 9, 9)	$(s_6, -0.492)$	$(s_5, 0.203)$	$(s_5, 0.200)$	$(s_6, -0.490)$	$(s_5, 0.479)$	$A_4 > A_1 > A_5 > A_2 > A_3$
(10, 10, 10, 10)	$(s_5, -0.482)$	$(s_5, 0.210)$	$(s_5, 0.210)$	$(s_6, -0.479)$	$(s_5, 0.493)$	$A_4 > A_1 > A_5 > A_2 > A_3$

Table 9. Ranking results for different operational parameters of the DG2TLNNWGBM operator.

P	$s(A_1)$	$s(A_2)$	$s(A_3)$	$s(A_4)$	$s(A_5)$	Ordering
(1, 1, 1, 1)	$(s_4, -0.135)$	$(s_3, 0.381)$	$(s_3, 0.098)$	$(s_3, 0.232)$	$(s_4, -0.405)$	$A_1 > A_5 > A_2 > A_4 > A_3$
(2, 2, 2, 2)	$(s_3, -0.046)$	$(s_3, -0.422)$	$(s_2, 0.283)$	$(s_2, 0.300)$	$(s_3, -0.242)$	$A_1 > A_5 > A_2 > A_4 > A_3$
(3, 3, 3, 3)	$(s_3, -0.430)$	$(s_2, 0.326)$	$(s_2, 0.018)$	$(s_2, -0.122)$	$(s_2, 0.450)$	$A_1 > A_5 > A_2 > A_3 > A_4$
(4, 4, 4, 4)	$(s_2, 0.380)$	$(s_2, 0.218)$	$(s_2, -0.102)$	$(s_2, -0.352)$	$(s_2, 0.310)$	$A_1 > A_5 > A_2 > A_3 > A_4$
(5, 5, 5, 5)	$(s_2, 0.271)$	$(s_2, 0.156)$	$(s_2, -0.174)$	$(s_2, -0.498)$	$(s_2, 0.233)$	$A_1 > A_5 > A_2 > A_3 > A_4$
(6, 6, 6, 6)	$(s_2, 0.200)$	$(s_2, 0.111)$	$(s_2, -0.225)$	$(s_1, 0.400)$	$(s_2, 0.186)$	$A_1 > A_5 > A_2 > A_3 > A_4$
(7, 7, 7, 7)	$(s_2, 0.150)$	$(s_2, 0.073)$	$(s_2, -0.263)$	$(s_1, 0.326)$	$(s_2, 0.153)$	$A_5 > A_1 > A_2 > A_3 > A_4$
(8, 8, 8, 8)	$(s_2, 0.113)$	$(s_2, 0.040)$	$(s_2, -0.294)$	$(s_1, 0.269)$	$(s_2, 0.130)$	$A_5 > A_1 > A_2 > A_3 > A_4$
(9, 9, 9, 9)	$(s_2, 0.085)$	$(s_2, 0.011)$	$(s_2, -0.320)$	$(s_1, 0.225)$	$(s_2, 0.113)$	$A_5 > A_1 > A_2 > A_3 > A_4$
(10, 10, 10, 10)	$(s_2, 0.063)$	$(s_2, -0.016)$	$(s_2, -0.343)$	$(s_1, 0.189)$	$(s_2, 0.099)$	$A_5 > A_1 > A_2 > A_3 > A_4$

6.3. Comparative Analysis

Then, we compare our proposed method with other existing methods, including the LNNWAA operator and the LNNWGA operator proposed by Fang & Ye [53] and cosine measures of linguistic neutrosophic numbers [54]. The comparative results are shown in Table 10.

Table 10. Ordering of the green suppliers.

	Ordering
LNNWAA [53]	$A_1 > A_5 > A_4 > A_2 > A_3$
LNNWGA [53]	$A_1 > A_5 > A_4 > A_2 > A_3$
$C^{w_1}{}_{LNNs}$ [54]	$A_1 > A_5 > A_2 > A_3 > A_4$
$C^{w_2}{}_{LNNs}$ [54]	$A_1 > A_5 > A_2 > A_4 > A_3$

From above, we can that we get the same results to show the practicality and effectiveness of the proposed approaches. However, the existing aggregation operators, such as the LNNWAA operator and the LNNWGA operator, do not consider the information about the relationship between the arguments being aggregated, and thus cannot eliminate the influence of unfair arguments on the decision result. Our proposed DG2TLNNWBM and DG2TLNNWGBM operators consider the information about the relationship among arguments being aggregated.

7. Conclusions

In this paper, we investigate the MADM problems with 2TLNNs. Then, we utilize the Bonferroni mean (BM) operator, generalized Bonferroni mean (GBM) operator, and dual generalized Bonferroni mean (DGBM) operator to develop some Bonferroni mean aggregation operators with 2TLNNs: 2-tuple linguistic neutrosophic number weighted Bonferroni mean (2TLNNWBM) operator, 2-tuple linguistic neutrosophic number weighted geometric Bonferroni mean (2TLNNWGBM) operator, generalized 2-tuple linguistic neutrosophic number weighted Bonferroni mean (G2TLNNWBM) operator, generalized 2-tuple linguistic neutrosophic number weighted geometric Bonferroni mean (G2TLNNWGBM) operator, dual generalized 2-tuple linguistic neutrosophic number weighted Bonferroni mean (DG2TLNNWBM) operator, and dual generalized 2-tuple linguistic neutrosophic number weighted geometric Bonferroni mean (DG2TLNNWGBM) operator. The prominent characteristic of these proposed operators are studied. Then, we have utilized these operators to develop some approaches to solve the MADM problems with 2TLNNs. Finally, a practical example for green supplier selection in green supply chain management is given to verify the developed approach and to demonstrate its practicality and effectiveness. In the future, the application of the proposed aggregating operators of 2TLNNs needs to be explored in the decision making, risk analysis, and many other fields under uncertain environments [55–77].

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