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Hesitant Neutrosophic Linguistic Sets and Their Application in Multiple Attribute Decision Making

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Abstract: In this paper, the hesitant neutrosophic linguistic set is first defined by extending a hesitant fuzzy set to accommodate linguistic terms and neutrosophic fuzzy values. Some operational laws are defined for hesitant neutrosophic linguistic fuzzy information. Several distance measures have been defined including generalized hesitant neutrosophic linguistic distance, generalized hesitant neutrosophic linguistic Hausdorff distance, and generalized hesitant neutrosophic linguistic hybrid distance. Some hesitant neutrosophic fuzzy linguistic aggregation operators based on the Choquet integral have been defined. A new multiple attribute decision making method for hesitant neutrosophic fuzzy linguistic information has been developed based on TOPSIS. In order to illustrate the feasibility and practical advantages of the new algorithm, we use it to select a company to invest. The new method is then compared with other methods.

Keywords: neutrosophic set; linguistic argument; hesitant fuzzy set; Choquet integral; multiple attribute decision making

1. Introduction

With the development of society and the economy, decision problems have become increasingly complicated. Fuzziness and uncertainty exists extensively in the decision process. Many useful tools have been developed to model fuzzy and uncertain information, including fuzzy sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, linguistic arguments, hesitant fuzzy sets, and neutrosophic sets. Many useful methods have been developed based on these tools. Smarandache [1] developed the neutrosophic set by generalizing a fuzzy set, an intuitionistic fuzzy set [2], and a hesitant fuzzy set. The neutrosophic set has received extensively attention [3–25] recently. The neutrosophic fuzzy set has been extended to accommodate interval values [7,8], linguistic variables [9,10], and trapezoidal fuzzy numbers [11]. Ye [9] proposed the concept of a single valued neutrosophic linguistic set as a generalization of the concepts of a linguistic variable and an intuitionistic linguistic set. Ye [11] combined trapezoidal fuzzy numbers with a single-valued neutrosophic set to propose a trapezoidal neutrosophic set and defined some trapezoidal neutrosophic fuzzy aggregation operators. The neutrosophic fuzzy set has been used in image segmentation [12], clustering analysis [13], supply chain management [14], etc. Guo and Sengür [12] utilized the neutrosophic set in image segmentation and proposed a novel algorithm based on neutrosophic similarity clustering. Karaaslan [13] defined the concept of a single-valued neutrosophic refined soft set by extending a single-valued neutrosophic refined set. Some aggregation operators have been developed [3,7,15–18]. Liu and Tang [3] proposed some power average aggregation operators for interval neutrosophic sets, including the interval neutrosophic power generalized weighted aggregation operator and interval neutrosophic power generalized ordered weighted aggregation operator. Zhao et al. [7] defined the generalized weighted aggregation operator for interval valued neutrosophic sets based on a traditional

generalized weighted aggregation operator. Liu and Shi [15] defined the interval neutrosophic hesitant fuzzy set and developed some new interval neutrosophic hesitant fuzzy generalized hybrid aggregation operators. Liu et al. [16] proposed some generalized neutrosophic number hamacher weighted aggregation operators. Sun et al. [17] defined a ranking index to compare two interval neutrosophic numbers and defined an interval neutrosophic Choquet integral operator. Peng et al. [18] developed the multi-valued neutrosophic sets and defined the operations of multi-valued neutrosophic numbers based on Einstein operations. Some distances, cross entropies and correlation coefficient measures have been defined [19–26]. Ye [19] defined the Hamming and Euclidean distances between interval neutrosophic sets and proposed the similarity measures based on the relationship between similarity measures and distances. Ye presented single-valued neutrosophic cross entropy in [20] and presented the correlation coefficient of single-valued neutrosophic sets based on the extension of the intuitionistic fuzzy correlation coefficient in [21]. Zhang et al. [22] proposed a new correlation coefficient measure for interval neutrosophic sets. Şahin [23] defined the interval neutrosophic cross-entropy based on extension of fuzzy cross entropy and single-valued neutrosophic cross entropy. Tian et al. [26] proposed a fuzzy cross entropy approach to calculate the discrimination measure between alternatives and the absolute ideal solutions.

A hesitant fuzzy set is another extension of the fuzzy set, which was developed by Torra and Narukawa [27,28]. In the hesitant fuzzy set, the membership degree of an element to a set is represented by several possible values. Compared with other tools to model fuzzy and uncertain information, the hesitant fuzzy set is more flexible and accurate, especially for large-scale and complex decision problems. The hesitant fuzzy set has received a great deal of attention [29–34]. Due to the fuzzy nature of human thinking, decision-makers would like to evaluate with linguistic terms. Several linguistic decision making models have been developed [35–39], including the 2-tuple linguistic model [35], uncertain linguistic variables [36], the hesitant fuzzy linguistic terms set [37], and the linguistic hesitant intuitionistic fuzzy term set [39]. With the development of the economy and society, decision problems become more and more complex, and multiple decision-makers from different fields are invited to evaluate. Decision-makers (DMs) would like to use linguistic terms in the evaluation process, but they may express some hesitation, which can be modeled properly by the neutrosophic set. Since each DM has his/her own characteristics, he/she can give proper evaluation values if he/she is familiar with the attribute and can refuse to produce any evaluation values if he/she is unfamiliar with the attribute in order to assure accuracy and reasonability of decision results. DMs can use a different number of linguistic terms and a different number of neutrosophic fuzzy values in evaluating the attribute. Then, we can get hesitant neutrosophic linguistic information. Ye [40] developed hesitant interval neutrosophic linguistic set and proposed the hesitant interval neutrosophic linguistic weighted average operator and a hesitant interval neutrosophic linguistic weighted geometric operator. It is rather difficult to use interval linguistic values in evaluation, and distances between hesitant interval neutrosophic linguistic elements have not been studied. To our best knowledge, hesitant neutrosophic linguistic sets, which are very important in decision-making, especially for complex decision-making problems requiring fast evaluation, have not been studied. Motivated by the idea of the hesitant fuzzy set, the neutrosophic set, and linguistic terms, we present hesitant neutrosophic linguistic sets. Then, we define some operational laws and some distant measures between hesitant neutrosophic linguistic sets. A new multiple attribute decision-making method based on hesitant neutrosophic linguistic information has thus been developed.

The rest of the paper is organized as follows. In Section 2, we first briefly review some concepts on the hesitant fuzzy set, the linguistic variable, and the neutrosophic set. Then, we propose a definition of the hesitant neutrosophic linguistic set and provide some operation laws of hesitant neutrosophic linguistic elements. We also define the cosine value between two hesitant neutrosophic linguistic elements to compare them. In Section 3, we present some distance measures for hesitant neutrosophic linguistic elements. In Section 4, we develop some hesitant neutrosophic linguistic aggregation operators based on the Choquet integral. In Section 5, we present a new multiple attribute group

decision-making method based on TOPSIS for hesitant neutrosophic fuzzy linguistic information. A numerical example is presented in Section 6, and some comparisons are also made. Conclusions are presented in Section 7.

2. Preliminaries

In this section, we give the definition of the hesitant neutrosophic linguistic set and present some operation laws. We also present the method to compare two hesitant neutrosophic linguistic elements by using their cosine values.

In the decision-making process, there are cases where a DM, in determining the membership of an element to a set, doubts several different values. Torra and Narukawa [27,28] generalized a fuzzy set to make each membership include several values and developed the hesitant fuzzy set (HFS). An HFS is defined in terms of a function that returns a set of membership values of each element in the domain.

Definition 1. Let X be a reference set. An HFS \tilde{A} on X is a function m that returns a subset of values in $[0, 1]$ when it is applied to X :

$$\tilde{A} = \{ \langle x, m_{\tilde{A}}(x) \rangle \mid x \in X \}$$

where $m_{\tilde{A}}(x)$ is a set of some different values in $[0, 1]$, representing the possible membership degrees of element $x \in X$ to \tilde{A} . $m_{\tilde{A}}(x)$ is called a hesitant fuzzy element (HFE).

Let $S = \{s_i \mid i = 1, \dots, g\}$ be a finite and totally ordered discrete term set, where s_i represents a possible value for a linguistic variable. For example, a set of nine terms [36] can be expressed as

$$S = \{s_1 = \text{extremely poor}, s_2 = \text{very poor}, s_3 = \text{poor}, s_4 = \text{slightly poor}, s_5 = \text{fair}, s_6 = \text{slightly good}, s_7 = \text{good}, s_8 = \text{very good}, s_9 = \text{extremely good}\}.$$

The above set satisfies the following properties:

- (1) The set is ordered: $s_i \geq s_j$, if $i \geq j$.
- (2) Max operator: $\max\{s_i, s_j\} = s_i$, if $i \geq j$.
- (3) Min operator: $\min\{s_i, s_j\} = s_i$, if $i \leq j$.
- (4) Negation operator: $\text{neg}\{s_i\} = s_{g-i}$.

In order to preserve all the information, the discrete linguistic term sets S can be extended to a continuous one: $\bar{S} = \{s_\alpha \mid s_0 \leq s_\alpha \leq s_g\}, \alpha \in [0, g]$.

Definition 2. [1] Let X be a universe of discourse, with a generic element in X denoted by x . A neutrosophic set A in X is

$$A = \{x(T_A(x), I_A(x), F_A(x)) \mid x \in X\},$$

where T_A is the truth-membership function, I_A is the indeterminacy-membership function, F_A is the real falsity-membership function. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+]$. There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, we can obtain $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

DMs would like to evaluate with linguistic terms since such terms reflect the fuzzy nature of human thinking. For complex decision-making problems, multiple experts from different fields are invited to evaluate alternatives with respect to multiple attributes. If the expert is familiar with the attribute, he/she can provide a proper evaluation value. If he/she is not familiar with the attribute, he/she can refuse to provide any evaluation value to assure the reasonability of the results. If two experts use the same truth-membership, indeterminacy-membership, or real falsity-membership, they are merged. We can then obtain the hesitant neutrosophic linguistic set as follows.

Definition 3. Let $X = \{x_1, x_2, \dots, x_n\}$ be a reference set. A hesitant neutrosophic linguistic set (HNLS) A in X is

$$A = \{ \langle x_i, h_A(x_i) \rangle \mid x_i \in X, i = 1, 2, \dots, n \}$$

where $h_A(x_i) = \{ (s_{\theta_i}, (T(s_{\theta_i}), I(s_{\theta_i}), F(s_{\theta_i}))) \}$, $T(s_{\theta_i})$ is the set of truth-membership functions, $I(s_{\theta_i})$ is the set of indeterminacy-membership functions, and $F(s_{\theta_i})$ is the set of real falsity-membership functions. $\sup(T(s_{\theta_i})) + \sup(I(s_{\theta_i})) + \sup(F(s_{\theta_i})) \leq 3$. For convenience, we call $h_A(x_i)$ the hesitant neutrosophic linguistic element (HNLE). We can use $a = (s_{\theta_i}, (T(s_{\theta_i}), I(s_{\theta_i}), and F(s_{\theta_i})))$ to represent a neutrosophic linguistic element (NLE).

Definition 4. Let h, h_1 , and h_2 be HNLEs, and let $h = \{a_k\}$, $h_1 = \{a_i^{(1)}\}$, $h_2 = \{a_j^{(2)}\}$, $a_k = (s_{\theta_k}, (T(s_{\theta_k}), I(s_{\theta_k}), F(s_{\theta_k}))) \in h$, $a_i^{(1)} = (s_{\theta_i^{(1)}}, (T(s_{\theta_i^{(1)}}), I(s_{\theta_i^{(1)}}), F(s_{\theta_i^{(1)}}))) \in h_1$ and $a_j^{(2)} = (s_{\theta_j^{(2)}}, (T(s_{\theta_j^{(2)}}), I(s_{\theta_j^{(2)}}), F(s_{\theta_j^{(2)}}))) \in h_2$ be NLEs. Let $\mu_i^{(1)} \in F(s_{\theta_i^{(1)}}), \mu_j^{(2)} \in F(s_{\theta_j^{(2)}}), v_i^{(1)} \in T(s_{\theta_i^{(1)}}), v_j^{(2)} \in T(s_{\theta_j^{(2)}}), \tau_i^{(1)} \in I(s_{\theta_i^{(1)}}), \tau_j^{(2)} \in I(s_{\theta_j^{(2)}})$. $\lambda > 0$. The operational laws can be defined as follows.

- (1) $h_1 \oplus h_2 = \cup_{a_i^{(1)} \in h_1, a_j^{(2)} \in h_2} \{ (s_{\theta_i^{(1)} + \theta_j^{(2)}}, \cup \{ (\mu_i^{(1)} + \mu_j^{(2)} - \mu_i^{(1)} \mu_j^{(2)}, v_i^{(1)} v_j^{(2)}, \tau_i^{(1)} \tau_j^{(2)}) \}) \}$
- (2) $h_1 \otimes h_2 = \cup_{a_i^{(1)} \in h_1, a_j^{(2)} \in h_2} \{ (s_{\theta_i^{(1)} \theta_j^{(2)}}, \cup \{ (\mu_i^{(1)} \mu_j^{(2)}, v_i^{(1)} + v_j^{(2)} - v_i^{(1)} v_j^{(2)}, \tau_i^{(1)} + \tau_j^{(2)} - \tau_i^{(1)} \tau_j^{(2)}) \}) \}$
- (3) $\lambda h = \cup_{a_k \in h} \{ (s_{\lambda \theta_k}, \cup \{ (1 - (1 - \mu_k)^\lambda, (v_k)^\lambda, (\tau_k)^\lambda) \}) \}$
- (4) $h^\lambda = \cup_{a_k \in h} \{ (s_{\theta_k^\lambda}, \cup \{ ((\mu_k)^\lambda, 1 - (1 - v_k)^\lambda, 1 - (1 - \tau_k)^\lambda) \}) \}$.

Theorem 1. Let h, h_1, h_2 , and h_3 be HNLEs, λ, λ_1 and $\lambda_2 > 0$. Then,

- (1) $h_1 \oplus h_2 = h_2 \oplus h_1$
- (2) $h_1 \otimes h_2 = h_2 \otimes h_1$
- (3) $\lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2$
- (4) $(\lambda_1 + \lambda_2)h = \lambda_1 h \oplus \lambda_2 h$
- (5) $(h_1 \otimes h_2)^\lambda = h_1^\lambda \otimes h_2^\lambda$
- (6) $h^{\lambda_1 + \lambda_2} = h^{\lambda_1} \otimes h^{\lambda_2}$
- (7) $(h_1 \oplus h_2) \oplus h_3 = h_1 \oplus (h_2 \oplus h_3)$
- (8) $(h_1 \otimes h_2) \otimes h_3 = h_1 \otimes (h_2 \otimes h_3)$.

Different HNLEs may have different NLEs, and different NLEs may have different truth-memberships, indeterminacy-memberships, or falsity-memberships. In order to define the distances between two HNLEs more accurately, we need to extend the HNLEs until they have the same number of NLEs and until each NLE has the same number of truth-memberships, indeterminacy-memberships, and falsity-memberships. We can extend the HNLEs according to the risk attitudes of the decision-makers. If decision-makers are risk-seeking, the largest NLE can be added; if decision-makers are risk-averse, the smallest NLE can be added; if decision-makers are risk-neutral, the average value of NLEs can be added.

Definition 5. Let a_i and a_j be NLEs. $a_i = (s_{\theta_i}, (T(s_{\theta_i}), I(s_{\theta_i}), F(s_{\theta_i})))$, $a_j = (s_{\theta_j}, (T(s_{\theta_j}), I(s_{\theta_j}), F(s_{\theta_j})))$, $\mu_{ik} \in T(s_{\theta_i}), v_{il} \in I(s_{\theta_i}), \tau_{im} \in F(s_{\theta_i}), \mu_{jk} \in T(s_{\theta_j}), v_{jl} \in I(s_{\theta_j}), \tau_{jm} \in F(s_{\theta_j})$. The cosine value between a_i and a_j can be defined as follows:

$$\cos(a_i, a_j) = \frac{\theta_i \theta_j / g^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} \mu_{i\sigma(k)} \mu_{j\sigma(k)} + \frac{1}{l_i} \sum_{k=1}^{l_i} v_{i\sigma(k)} v_{j\sigma(k)} + \frac{1}{l_f} \sum_{k=1}^{l_f} \tau_{i\sigma(k)} \tau_{j\sigma(k)}}{4 \| a_i \| \cdot \| a_j \|} \tag{1}$$

where $\| a_i \| = \left(\frac{1}{4}(\theta_i^2/g^2 + \frac{1}{l_t} \sum_{k=1}^{l_t} \mu_{ik}^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} v_{ik}^2 + \frac{1}{l_f} \sum_{k=1}^{l_f} \tau_{ik}^2) \right)^{1/2}$, $\| a_j \| = \left(\frac{1}{4}(\theta_j^2/g^2 + \frac{1}{l_t} \sum_{k=1}^{l_t} \mu_{jk}^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} v_{jk}^2 + \frac{1}{l_f} \sum_{k=1}^{l_f} \tau_{jk}^2) \right)^{1/2}$, g is the number of linguistic variables in the linguistic term set, l_t is the number of truth-memberships in the NLE, l_i is the number of indeterminacy-membership in the NLE, and l_f is the number of falsity-membership in the NLE. $0 < \cos(a_i, a_j) \leq 1$. If a_j is the ideal solution $I = (s_g, \{1, \dots, 1\}, \{0, \dots, 0\}, \{0, \dots, 0\})$, then

$$\cos(a_i, I) = \frac{\theta_i/g + \frac{1}{l_t} \sum_{k=1}^{l_t} \mu_{i\sigma(k)}}{\sqrt{2(\theta_i^2/g^2 + \frac{1}{l_t} \sum_{k=1}^{l_t} \mu_{ik}^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} v_{ik}^2 + \frac{1}{l_f} \sum_{k=1}^{l_f} \tau_{ik}^2)}}.$$

Definition 6. Let a_1, a_2 be two NLEs. If $\cos(a_1, I) \leq \cos(a_2, I)$, then $a_1 \leq a_2$.

Definition 7. Let h_1, h_2 be HNLEs. $h_1 = \{a_i^{(1)}\} = \{(s_{\theta_i^{(1)}}), (T(s_{\theta_i^{(1)}}), I(s_{\theta_i^{(1)}}), F(s_{\theta_i^{(1)}})))\}$, $h_2 = \{a_i^{(2)}\} = \{(s_{\theta_i^{(2)}}), (T(s_{\theta_i^{(2)}}), I(s_{\theta_i^{(2)}}), F(s_{\theta_i^{(2)}})))\}$. The cosine value between h_1 and h_2 can be defined as follows:

$$\cos(h_1, h_2) = \frac{h_1 \cdot h_2}{\| h_1 \| \cdot \| h_2 \|} \tag{2}$$

where

$$h_1 \cdot h_2 = \frac{1}{4l} \sum_{i=1}^l ((\theta_{\sigma(i)}^{(1)} \theta_{\sigma(i)}^{(2)})/g^2 + \frac{1}{l_t} \sum_{k=1}^{l_t} (\mu_{\sigma(ik)}^{(1)} \mu_{\sigma(ik)}^{(2)}) + \frac{1}{l_i} \sum_{k=1}^{l_i} (v_{\sigma(ik)}^{(1)} v_{\sigma(ik)}^{(2)})^2 + \frac{1}{l_f} \sum_{k=1}^{l_f} (\tau_{\sigma(ik)}^{(1)} \tau_{\sigma(ik)}^{(2)})^2)$$

$$\| h_1 \| = \left(\frac{1}{4l} \sum_{i=1}^l ((\theta_i^{(1)})^2/g^2 + \frac{1}{l_t} \sum_{k=1}^{l_t} (\mu_{ik}^{(1)})^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} (v_{ik}^{(1)})^2 + \frac{1}{l_f} \sum_{k=1}^{l_f} (\tau_{ik}^{(1)})^2) \right)^{1/2}$$

$$\| h_2 \| = \left(\frac{1}{4l} \sum_{i=1}^l ((\theta_i^{(2)})^2/g^2 + \frac{1}{l_t} \sum_{k=1}^{l_t} (\mu_{ik}^{(2)})^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} (v_{ik}^{(2)})^2 + \frac{1}{l_f} \sum_{k=1}^{l_f} (\tau_{ik}^{(2)})^2) \right)^{1/2}.$$

l is the number of NLEs in the HNLEs, g is the number of linguistic variables in the linguistic term set, l_t is the number of truth-memberships in an NLE, l_i is the number of indeterminacy-membership in an NLE, l_f is the number of falsity-membership in an NLE. $0 < \cos(h_1, h_2) \leq 1$. When h_2 is the ideal solution $I' = \{(s_g, \{1, \dots, 1\}, \{0, \dots, 0\}, \{0, \dots, 0\})\}$, the bigger $\cos(h_1, I')$ between h_1 and I' is, the more consistent the direction between h_1 and I' is. Thus,

$$\cos(h_1, I') = \frac{\frac{1}{l} \sum_{i=1}^l ((\theta_{\sigma(i)}^{(1)})/g + \frac{1}{l_t} \sum_{k=1}^{l_t} \mu_{\sigma(ik)}^{(1)})}{\sqrt{\frac{2}{l} \sum_{i=1}^l ((\theta_i^{(1)})^2/g^2 + \frac{1}{l_t} \sum_{k=1}^{l_t} (\mu_{ik}^{(1)})^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} (v_{ik}^{(1)})^2 + \frac{1}{l_f} \sum_{k=1}^{l_f} (\tau_{ik}^{(1)})^2)}}. \tag{3}$$

Definition 8. Let h_1, h_2 be two HNLEs. If $\cos(h_1, I') \leq \cos(h_2, I')$, then $h_1 \leq h_2$.

3. Some Hesitant Neutrosophic Fuzzy Linguistic Distance Measures

In this section, we propose some distance measures for the hesitant neutrosophic fuzzy linguistic values and we also propose some correlated distance measures by using the Choquet integral.

The distance measures are very important since they are the basis of many well-known methods including TOPSIS, VIKOR, and ELECTRE. In this section, we develop some distance measures for hesitant neutrosophic fuzzy linguistic information.

Definition 9. Let h_1, h_2 be HNLEs. The distance measure between h_1 and h_2 can be defined as $d(h_1, h_2)$, which satisfies the following properties:

- (1) $0 \leq d(h_1, h_2) \leq 1$;
- (2) $d(h_1, h_2) = 0$ if and only if $h_1 = h_2$;
- (3) $d(h_1, h_2) = d(h_2, h_1)$.

Definition 10. Let h_1, h_2 be HNLEs. Then the similarity measure between h_1 and h_2 is defined as $\rho(h_1, h_2)$, which satisfies the following properties:

- (1) $0 \leq \rho(h_1, h_2) \leq 1$;
- (2) $\rho(h_1, h_2) = 1$ if and only if $h_1 = h_2$;
- (3) $\rho(h_1, h_2) = \rho(h_2, h_1)$.

Remark 1. If $d(h_1, h_2)$ is the distance between h_1 and h_2 , then $\rho(h_1, h_2) = 1 - d(h_1, h_2)$ is the similarity measure between h_1 and h_2 . Hence, we only focus on the distance measures, and the similarity measures can be determined easily.

Let h_1, h_2 be HNLEs. $h_1 = \{a_i^{(1)}\} = \{(s_{\theta_i^{(1)}}(T(s_{\theta_i^{(1)}}), I(s_{\theta_i^{(1)}}), F(s_{\theta_i^{(1)}})))\}$, $T(s_{\theta_i^{(1)}}) = \{\mu_{it_i}^{(1)}\}$, $I(s_{\theta_i^{(1)}}) = \{v_{it_i}^{(1)}\}$, $F(s_{\theta_i^{(1)}}) = \{\tau_{it_i}^{(1)}\}$ and $h_2 = \{a_i^{(2)}\} = \{(s_{\theta_i^{(2)}}(T(s_{\theta_i^{(2)}}), I(s_{\theta_i^{(2)}}), F(s_{\theta_i^{(2)}})))\}$, $T(s_{\theta_i^{(2)}}) = \{\mu_{it_i}^{(2)}\}$, $I(s_{\theta_i^{(2)}}) = \{v_{it_i}^{(2)}\}$, $F(s_{\theta_i^{(2)}}) = \{\tau_{it_i}^{(2)}\}$. $a_{\sigma(i)}^{(k)} \geq a_{\sigma(j)}^{(k)}$ for $i \geq j$. l is the number of NLEs in HNLE, g is the number of linguistic variables in the linguistic term set, l_t is the number of truth-memberships in an NLE, l_i is the number of indeterminacy-membership in an NLE, l_f is the number of falsity-membership in an NLE. Based on the operation laws of HNLEs and the extension principle, we define some distance measures between HNLEs.

The hesitant neutrosophic linguistic Euclidean distance between h_1 and h_2 can be defined as follows:

$$d_E(h_1, h_2) = \left(\frac{1}{4l} \sum_{i=1}^l \left((\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)})^2 / g^2 + \frac{1}{l_t} \sum_{k=1}^{l_t} (\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)})^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} (v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)})^2 + \frac{1}{l_f} \sum_{k=1}^{l_f} (\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)})^2 \right) \right)^{1/2}. \tag{4}$$

The hesitant neutrosophic linguistic Hamming distance between h_1 and h_2 can be defined as follows:

$$d_H(h_1, h_2) = \frac{1}{4l} \sum_{i=1}^l \left(|\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)}| / g + \frac{1}{l_t} \sum_{k=1}^{l_t} |\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)}| + \frac{1}{l_i} \sum_{k=1}^{l_i} |v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)}| + \frac{1}{l_f} \sum_{k=1}^{l_f} |\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)}| \right). \tag{5}$$

The hesitant neutrosophic linguistic Hamming–Hausdorff distance can be defined as follows:

$$d_{HH}(h_1, h_2) = \frac{1}{4} \max_{i,k} \left\{ |\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)}| / g + |\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)}| + |v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)}| + |\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)}| \right\}. \tag{6}$$

The hesitant neutrosophic linguistic Euclidean–Hausdorff distance can be defined as

$$d_{EH}(h_1, h_2) = \left(\frac{1}{4} \max_{i,k} \left\{ (\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)})^2 / g^2 + |\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)}|^2 + |v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)}|^2 + |\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)}|^2 \right\} \right)^{1/2}. \tag{7}$$

The hybrid hesitant neutrosophic linguistic Hamming distance can be defined as

$$d_{HybH}(h_1, h_2) = \frac{1}{2} \left(\frac{1}{4l} \sum_{i=1}^l \left(|\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)}| / g + \frac{1}{l_t} \sum_{k=1}^{l_t} |\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)}| + \frac{1}{l_i} \sum_{k=1}^{l_i} |v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)}| + \frac{1}{l_f} \sum_{k=1}^{l_f} |\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)}| \right) + \frac{1}{4} \max_{i,k} \left\{ |\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)}| / g + |\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)}| + |v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)}| + |\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)}| \right\} \right). \tag{8}$$

The hybrid hesitant neutrosophic linguistic Euclidean distance can be defined as

$$d_{HybE}(h_1, h_2) = \frac{1}{2} \left(\left(\frac{1}{4l} \sum_{i=1}^l \left((\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)})^2 / g^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} (\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)})^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} (v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)})^2 + \frac{1}{l_i} \sum_{k=1}^{l_i} (\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)})^2 \right) \right)^{1/2} + \left(\frac{1}{4} \max_{i,k} \left\{ (\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)})^2 / g^2 + |\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)}|^2 + |v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)}|^2 + |\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)}|^2 \right\} \right)^{1/2} \right). \tag{9}$$

The generalized hesitant neutrosophic linguistic distance can be defined as

$$d_G(h_1, h_2) = \left(\frac{1}{4l} \sum_{i=1}^l \left((\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)})^\lambda / g^\lambda + \frac{1}{l_i} \sum_{k=1}^{l_i} (\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)})^\lambda + \frac{1}{l_i} \sum_{k=1}^{l_i} (v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)})^\lambda + \frac{1}{l_i} \sum_{k=1}^{l_i} (\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)})^\lambda \right) \right)^{1/\lambda}. \tag{10}$$

If $\lambda = 1$, the generalized hesitant neutrosophic linguistic distance becomes the hesitant neutrosophic linguistic Hamming distance. If $\lambda = 2$, then the generalized hesitant neutrosophic linguistic distance becomes the hesitant neutrosophic linguistic Euclidean distance.

The generalized hesitant neutrosophic linguistic Hausdorff distance can be defined as

$$d_{GH}(h_1, h_2) = \left(\frac{1}{4} \max_{i,k} \left\{ |\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)}|^\lambda / g^\lambda + |\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)}|^\lambda + |v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)}|^\lambda + |\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)}|^\lambda \right\} \right)^{1/\lambda}. \tag{11}$$

If $\lambda = 1$, the generalized hesitant neutrosophic linguistic Hausdorff distance becomes the hesitant neutrosophic linguistic Hamming–Hausdorff distance. If $\lambda = 2$, the generalized hesitant neutrosophic linguistic Hausdorff distance becomes the hesitant neutrosophic linguistic Euclidean–Hausdorff distance.

The generalized hesitant neutrosophic linguistic hybrid distance can be defined as

$$d_{GHyb}(h_1, h_2) = \left(\frac{1}{2} \left(\frac{1}{4l} \sum_{i=1}^l \left((\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)})^\lambda / g^\lambda + \frac{1}{l_i} \sum_{k=1}^{l_i} (\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)})^\lambda + \frac{1}{l_i} \sum_{k=1}^{l_i} (v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)})^\lambda + \frac{1}{l_i} \sum_{k=1}^{l_i} (\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)})^\lambda \right) + \frac{1}{4} \max_{i,k} \left\{ |\theta_{\sigma(i)}^{(1)} - \theta_{\sigma(i)}^{(2)}|^\lambda / g^\lambda + |\mu_{\sigma(ik)}^{(1)} - \mu_{\sigma(ik)}^{(2)}|^\lambda + |v_{\sigma(ik)}^{(1)} - v_{\sigma(ik)}^{(2)}|^\lambda + |\tau_{\sigma(ik)}^{(1)} - \tau_{\sigma(ik)}^{(2)}|^\lambda \right\} \right) \right)^{1/\lambda}. \tag{12}$$

Definition 11. [41]. A fuzzy measure μ on the set X is a set function $\mu : P(X) \rightarrow [0, 1]$, satisfying the following axioms:

- (1) $\mu(\emptyset) = 0, \mu(X) = 1;$
- (2) $B \subseteq C$ implies $\mu(B) \leq \mu(C)$, for all $B, C \subseteq X;$
- (3) $\mu(B \cup C) = \mu(B) + \mu(C) + \rho\mu(B)\mu(C)$ for all $B, C \subseteq X$ and $B \cap C = \emptyset$, where $\rho \in (-1, +\infty)$.

Here, \emptyset is an empty set.

Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set. Sugeno [42] provided the following equation to determine the fuzzy measure on X , avoiding the computational complexity.

$$m(X) = \begin{cases} \frac{1}{\rho} (\prod_{i=1}^n (1 + \rho m(x_i)) - 1), & \rho \neq 0, \\ \sum_{i=1}^n m(x_i), & \rho = 0. \end{cases} \tag{13}$$

The value ρ can be uniquely determined from $m(X) = 1$ by the following equation:

$$\rho = \prod_{i=1}^n (1 + \rho m(x_i)) - 1. \tag{14}$$

Let $X = \{x_1, x_2, \dots, x_n\}$ be a reference set, let m be the fuzzy measure [41] on X , and let $H_1 = \{h_1^{(1)}, h_2^{(1)}, \dots, h_n^{(1)}\}$ and $H_2 = \{h_1^{(2)}, h_2^{(2)}, \dots, h_n^{(2)}\}$ be two collections of the HNLEs on X ,

where $h_j^{(k)} = \{a_j^{l(k)}\}$ ($j = 1, 2, \dots, n, k = 1, 2$). We then develop some weighted distances between HNLEs based on the Choquet integral.

The generalized hesitant neutrosophic linguistic correlation averaging distance between H_1 and H_2 can be defined as follows:

$$d_{GC}(H_1, H_2) = \left(\sum_{j=1}^n \frac{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))}{4l} \sum_{i=1}^l \left((\theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)})^\lambda / g^\lambda + \frac{1}{l_t} \sum_{k=1}^{l_t} (\mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)})^\lambda + \frac{1}{l_u} \sum_{k=1}^{l_u} (v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)})^\lambda + \frac{1}{l_v} \sum_{k=1}^{l_v} (\tau_{\sigma(jik)}^{(1)} - \tau_{\sigma(jik)}^{(2)})^\lambda \right) \right)^{1/\lambda} \tag{15}$$

If $\lambda = 1$, the generalized hesitant neutrosophic linguistic correlation averaging distance becomes the hesitant neutrosophic linguistic correlation averaging Hamming distance as follows:

$$d_{CH}(H_1, H_2) = \sum_{j=1}^n \frac{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))}{4l} \sum_{i=1}^l \left(|\theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)}| / g + \frac{1}{l_t} \sum_{k=1}^{l_t} |\mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)}| + \frac{1}{l_u} \sum_{k=1}^{l_u} |v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)}| + \frac{1}{l_v} \sum_{k=1}^{l_v} |\tau_{\sigma(jik)}^{(1)} - \tau_{\sigma(jik)}^{(2)}| \right) \tag{16}$$

If $\lambda = 2$, the generalized hesitant neutrosophic linguistic correlation averaging distance becomes the hesitant neutrosophic linguistic correlation averaging Euclidean distance as follows:

$$d_{CE}(H_1, H_2) = \left(\sum_{j=1}^n \frac{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))}{4l} \sum_{i=1}^l \left((\theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)})^2 / g^2 + \frac{1}{l_t} \sum_{k=1}^{l_t} (\mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)})^2 + \frac{1}{l_u} \sum_{k=1}^{l_u} (v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)})^2 + \frac{1}{l_v} \sum_{k=1}^{l_v} (\tau_{\sigma(jik)}^{(1)} - \tau_{\sigma(jik)}^{(2)})^2 \right) \right)^{1/2} \tag{17}$$

The generalized hesitant neutrosophic linguistic correlation averaging Housdorff distance is defined as follows:

$$d_{GCH}(H_1, H_2) = \left(\sum_{j=1}^n \frac{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))}{4} \max_{i,k} \left\{ |\theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)}| / g^\lambda + |\mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)}|^\lambda + |v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)}|^\lambda + |\tau_{\sigma(jik)}^{(1)} - \tau_{\sigma(jik)}^{(2)}|^\lambda \right\} \right)^{1/\lambda} \tag{18}$$

If $\lambda = 1$, the generalized hesitant neutrosophic linguistic correlation averaging Housdorff distance becomes the hesitant neutrosophic linguistic correlation averaging Hamming–Housdorff distance

$$d_{CHH}(H_1, H_2) = \left(\sum_{j=1}^n \frac{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))}{4} \max_{i,k} \left\{ |\theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)}| / g + |\mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)}| + |v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)}| + |\tau_{\sigma(jik)}^{(1)} - \tau_{\sigma(jik)}^{(2)}| \right\} \right) \tag{19}$$

If $\lambda = 2$, the generalized hesitant neutrosophic linguistic correlation averaging Housdorff distance becomes the hesitant neutrosophic linguistic correlation averaging Euclidean–Housdorff distance:

$$d_{CEH}(H_1, H_2) = \left(\sum_{j=1}^n \frac{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))}{4} \max_{i,k} \left\{ |\theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)}|^2 / g^2 + |\mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)}|^2 + |v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)}|^2 + |\tau_{\sigma(jik)}^{(1)} - \tau_{\sigma(jik)}^{(2)}|^2 \right\} \right)^{1/2} \tag{20}$$

The generalized hesitant neutrosophic linguistic hybrid correlation averaging distance between H_1 and H_2 can be defined as follows:

$$d_{GHC}(H_1, H_2) = \left(\frac{1}{2} \left(\sum_{j=1}^n \frac{1}{4} (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})) \left(\frac{1}{l} \sum_{i=1}^l \left((\theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)})^\lambda / g^\lambda + \frac{1}{l_t} \sum_{k=1}^{l_t} (\mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)})^\lambda + \frac{1}{l_u} \sum_{k=1}^{l_u} (v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)})^\lambda + \frac{1}{l_v} \sum_{k=1}^{l_v} (\tau_{\sigma(jik)}^{(1)} - \tau_{\sigma(jik)}^{(2)})^\lambda \right) + \max_{i,k} \left\{ |\theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)}| / g^\lambda + |\mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)}|^\lambda + |v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)}|^\lambda + |\tau_{\sigma(jik)}^{(1)} - \tau_{\sigma(jik)}^{(2)}|^\lambda \right\} \right) \right)^{1/\lambda} \tag{21}$$

If $\lambda = 1$, the generalized hesitant neutrosophic linguistic hybrid correlation averaging distance becomes the hesitant neutrosophic linguistic hybrid correlation Hamming distance:

$$d_{CHyb}(H_1, H_2) = \frac{1}{2} \left(\sum_{j=1}^n \frac{1}{4} (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})) \left(\frac{1}{l} \sum_{i=1}^l \left((\theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)}) / g + \right. \right. \right. \\ \left. \left. \frac{1}{l_t} \sum_{k=1}^{l_t} (\mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)}) + \frac{1}{l_u} \sum_{k=1}^{l_u} (v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)}) + \frac{1}{l_v} \sum_{k=1}^{l_v} (\tau_{\sigma(jik)}^{(1)} - \right. \right. \\ \left. \left. \tau_{\sigma(jik)}^{(2)}) \right) + \max_{i,k} \left\{ \left| \theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)} \right| / g + \left| \mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)} \right| \right. \right. \\ \left. \left. + \left| v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)} \right| + \left| \tau_{\sigma(jik)}^{(1)} - \tau_{\sigma(jik)}^{(2)} \right| \right\} \right). \tag{22}$$

If $\lambda = 2$, the generalized hesitant neutrosophic linguistic hybrid correlation averaging distance becomes the hesitant neutrosophic linguistic hybrid correlation Euclidean distance:

$$d_{CHE}(H_1, H_2) = \left(\frac{1}{2} \left(\sum_{j=1}^n \frac{1}{4} (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})) \left(\frac{1}{l} \sum_{i=1}^l \left((\theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)})^2 / g^2 + \right. \right. \right. \right. \\ \left. \left. \frac{1}{l_t} \sum_{k=1}^{l_t} (\mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)})^2 + \frac{1}{l_u} \sum_{k=1}^{l_u} (v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)})^2 + \frac{1}{l_v} \sum_{k=1}^{l_v} (\tau_{\sigma(jik)}^{(1)} - \right. \right. \\ \left. \left. \tau_{\sigma(jik)}^{(2)})^2 \right) + \max_{i,k} \left\{ \left| \theta_{\sigma(ji)}^{(1)} - \theta_{\sigma(ji)}^{(2)} \right|^2 / g^2 + \left| \mu_{\sigma(jik)}^{(1)} - \mu_{\sigma(jik)}^{(2)} \right|^2 \right. \right. \\ \left. \left. + \left| v_{\sigma(jik)}^{(1)} - v_{\sigma(jik)}^{(2)} \right|^2 + \left| \tau_{\sigma(jik)}^{(1)} - \tau_{\sigma(jik)}^{(2)} \right|^2 \right\} \right) \right)^{1/2}. \tag{23}$$

4. Some Hesitant Neutrosophic Fuzzy Linguistic Aggregation Operators Based on the Choquet Integral

In this section, we present some hesitant neutrosophic fuzzy linguistic aggregation operators considering correlation by using the Choquet integral.

Definition 12. Let $X = \{x_1, x_2, \dots, x_n\}$ be a reference set, and let m be the fuzzy measure [41] on X . Let $h_j = \{a_j^k\}$ ($j = 1, 2, \dots, n$) be a collection of the HNLEs, then the hesitant neutrosophic fuzzy linguistic correlation averaging (HNLCA) operator can be defined as follows:

$$HNLCA(h_1, h_2, \dots, h_n) = \sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})) h_{\sigma(j)} \tag{24}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $h_{\sigma(1)} \geq h_{\sigma(2)} \geq \dots \geq h_{\sigma(n)}$. $A_{\sigma(i)} = \{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(i)}\}$, $A_{\sigma(0)} = \emptyset$, and \emptyset is an empty set.

The aggregated value of the HNLCA operator is also an HNLE.

$$HNLCA(h_1, h_2, \dots, h_n) = \sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})) h_{\sigma(j)} \\ = \bigcup_{a_j^k \in h_j} \left\{ (s_{\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})) \theta_{\sigma(j)}^k}, \bigcup \{ (1 - \prod_{j=1}^n (1 - \mu_{\sigma(j)}^k)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})}, \right. \\ \left. \prod_{j=1}^n (v_{\sigma(j)}^k)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})}, \prod_{j=1}^n (\tau_{\sigma(j)}^k)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})} \} \right\}.$$

Definition 13. Let $X = \{x_1, x_2, \dots, x_n\}$ be a reference set, and let m be the fuzzy measure on X . Let $h_j = \{a_j^k\}$ ($j = 1, 2, \dots, n$) be a collection of the HNLEs. Thus, the hesitant neutrosophic fuzzy linguistic correlation geometric averaging (HNLCGA) operator can be defined as follows:

$$HNLCGA(h_1, h_2, \dots, h_n) = \prod_{j=1}^n (h_{\sigma(j)})^{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))}, \tag{25}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $h_{\sigma(1)} \geq h_{\sigma(2)} \geq \dots \geq h_{\sigma(n)}$. $A_{\sigma(i)} = \{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(i)}\}$, $A_{\sigma(0)} = \emptyset$, and \emptyset is an empty set.

The aggregated value of the HNLCGA operator is also an HNLE.

$$\begin{aligned} \text{HNLCGA}(h_1, h_2, \dots, h_n) &= \prod_{j=1}^n (h_{\sigma(j)})^{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))} \\ &= \bigcup_{a_j^k \in h_j} \left\{ \left(\left(\prod_{j=1}^n (\theta_{\sigma(j)}^k)^{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))} \right) \cup \left\{ \left(\prod_{j=1}^n (\mu_{\sigma(j)}^k)^{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))} \right) \right. \right. \\ &\quad \left. \left. 1 - \prod_{j=1}^n (1 - \nu_{\sigma(j)}^k)^{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))}, 1 - \prod_{j=1}^n (1 - \tau_{\sigma(j)}^k)^{(m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))} \right\} \right\}. \end{aligned}$$

We further develop the generalized hesitant neutrosophic fuzzy linguistic correlation averaging (GHNLCGA) operator as follows.

Definition 14. Let $X = \{x_1, x_2, \dots, x_n\}$ be a reference set, and let m be the fuzzy measure on X . If $h_j = \{a_j^k\}$ ($j = 1, 2, \dots, n$) is a collection of the HNLEs, then the GHNLCGA operator can be defined as follows:

$$\text{GHNLCGA}(h_1, h_2, \dots, h_n) = \left(\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})) (h_{\sigma(j)})^\lambda \right)^{1/\lambda}, \tag{26}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $h_{\sigma(1)} \geq h_{\sigma(2)} \geq \dots \geq h_{\sigma(n)}$. $A_{\sigma(i)} = \{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(i)}\}$, $A_{\sigma(0)} = \emptyset$, \emptyset is an empty set, and $\lambda > 0$.

The aggregated value of the GHNLCGA operator is also an HNLE.

$$\begin{aligned} \text{GHNLCGA}(h_1, h_2, \dots, h_n) &= \left(\sum_{j=1}^n m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}) (h_{\sigma(j)})^\lambda \right)^{1/\lambda} \\ &= \bigcup_{a_j^k \in h_j} \left\{ \left(\left(\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})) (\theta_{\sigma(j)}^k)^\lambda \right)^{1/\lambda} \cup \left\{ \left(\left(1 - \prod_{j=1}^n (1 - (\mu_{\sigma(j)}^k)^\lambda)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})} \right) \right)^{1/\lambda} \right. \right. \\ &\quad \left. \left. 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \nu_{\sigma(j)}^k)^\lambda)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \tau_{\sigma(j)}^k)^\lambda)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})} \right)^{1/\lambda} \right\} \right\}. \end{aligned}$$

Theorem 2. (Commutativity). If $(h'_1, h'_2, \dots, h'_n)$ is any permutation of (h_1, h_2, \dots, h_n) , then

$$\text{GHNLCGA}(h'_1, h'_2, \dots, h'_n) = \text{GHNLCGA}(h_1, h_2, \dots, h_n).$$

Proof of Theorem 2.

$$\begin{aligned} \text{GHNLCGA}(h_1, h_2, \dots, h_n) &= \left(\sum_{j=1}^n m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}) (h_{\sigma(j)})^\lambda \right)^{1/\lambda} \\ &= \bigcup_{a_j^k \in h_j} \left\{ \left(\left(\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})) (\theta_{\sigma(j)}^k)^\lambda \right)^{1/\lambda} \cup \left\{ \left(\left(1 - \prod_{j=1}^n (1 - (\mu_{\sigma(j)}^k)^\lambda)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})} \right) \right)^{1/\lambda} \right. \right. \\ &\quad \left. \left. 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \nu_{\sigma(j)}^k)^\lambda)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \tau_{\sigma(j)}^k)^\lambda)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})} \right)^{1/\lambda} \right\} \right\} \end{aligned}$$

and

$$\begin{aligned} \text{GHNLCGA}(h'_1, h'_2, \dots, h'_n) &= \left(\sum_{j=1}^n m(A'_{\sigma(j)}) - m(A'_{\sigma(j-1)}) (h'_{\sigma(j)})^\lambda \right)^{1/\lambda} \\ &= \bigcup_{a_j^k \in h_j} \left\{ \left(\left(\sum_{j=1}^n (m(A'_{\sigma(j)}) - m(A'_{\sigma(j-1)})) (\theta_{\sigma(j)}^k)^\lambda \right)^{1/\lambda} \cup \left\{ \left(\left(1 - \prod_{j=1}^n (1 - (\mu_{\sigma(j)}^k)^\lambda)^{m(A'_{\sigma(j)}) - m(A'_{\sigma(j-1)})} \right) \right)^{1/\lambda} \right. \right. \\ &\quad \left. \left. 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \nu_{\sigma(j)}^k)^\lambda)^{m(A'_{\sigma(j)}) - m(A'_{\sigma(j-1)})} \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \tau_{\sigma(j)}^k)^\lambda)^{m(A'_{\sigma(j)}) - m(A'_{\sigma(j-1)})} \right)^{1/\lambda} \right\} \right\}. \end{aligned}$$

If $(h'_1, h'_2, \dots, h'_n)$ is any permutation of (h_1, h_2, \dots, h_n) , then we have $h_{\sigma(j)} = h'_{\sigma(j)}$ and $m(A_{\sigma(j)}) = m(A'_{\sigma(j)})$.

$$\text{GHNLCGA}(h'_1, h'_2, \dots, h'_n) = \text{GHNLCGA}(h_1, h_2, \dots, h_n).$$

□

Theorem 3. (Idempotency). If $h_j = h, j = 1, 2, \dots, n, h_j = \{a_j^k\}, a_j^k = (s_{\theta_j^k}, (T(s_{\theta_j^k}), I(s_{\theta_j^k}), F(s_{\theta_j^k}))), T(s_{\theta_j^k}) = \{\mu_j^{(k)}\}, I(s_{\theta_j^k}) = \{\nu_j^{(k)}\}, F(s_{\theta_j^k}) = \{\tau_j^{(k)}\},$ then

$$GHNLC A(h_1, h_2, \dots, h_n) = h.$$

Proof of Theorem 3.

$$\begin{aligned} GHNLC A(h_1, h_2, \dots, h_n) &= (\sum_{j=1}^n m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))(h_{\sigma(j)})^{\lambda})^{1/\lambda} \\ &= \bigcup_{a_j^k \in h_j} \{ (s_{(\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))(\theta_{\sigma(j)}^k)^{\lambda})^{1/\lambda}}, \cup \{ ((1 - \prod_{j=1}^n (1 - (\mu_{\sigma(j)}^k)^{\lambda})^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})})^{1/\lambda}, \\ &1 - (1 - \prod_{j=1}^n (1 - (1 - \nu_{\sigma(j)}^k)^{\lambda})^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})})^{1/\lambda}, 1 - (1 - \prod_{j=1}^n (1 - (1 - \tau_{\sigma(j)}^k)^{\lambda})^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})})^{1/\lambda}) \} \}. \end{aligned}$$

If $h_j = h, j = 1, 2, \dots, n,$ then $a_j^k = a^k = (s_{\theta^k}, (T(s_{\theta^k}), I(s_{\theta^k}), F(s_{\theta^k}))), T(s_{\theta^k}) = \{\mu^{(k)}\}, I(s_{\theta^k}) = \{\nu^{(k)}\}, F(s_{\theta^k}) = \{\tau^{(k)}\}.$ Thus,

$$\begin{aligned} S_{(\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))(\theta^k)^{\lambda})^{1/\lambda}} &= S_{((\theta^k)^{\lambda} \sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})))^{1/\lambda}} = S_{((\theta^k)^{\lambda})^{1/\lambda}} = S_{\theta^k} \\ (1 - \prod_{j=1}^n (1 - (\mu_{\sigma(j)}^k)^{\lambda})^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})})^{1/\lambda} &= (1 - \prod_{j=1}^n (1 - (\mu^{(k)})^{\lambda})^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})})^{1/\lambda} \\ = (1 - (1 - (\mu^{(k)})^{\lambda})^{\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))})^{1/\lambda} &= (1 - (1 - (\mu^{(k)})^{\lambda})^{\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))})^{1/\lambda} \\ = (1 - (1 - (\mu^{(k)})^{\lambda}))^{1/\lambda} &= \mu^{(k)}. \end{aligned}$$

$$\begin{aligned} &1 - (1 - \prod_{j=1}^n (1 - (1 - \nu_{\sigma(j)}^{(k)})^{\lambda})^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})})^{1/\lambda} \\ &= 1 - (1 - \prod_{j=1}^n (1 - (1 - \nu^{(k)})^{\lambda})^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})})^{1/\lambda} \\ &= 1 - (1 - (1 - (1 - \nu^{(k)})^{\lambda})^{\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))})^{1/\lambda} \\ &= 1 - (1 - (1 - (1 - \nu^{(k)})^{\lambda}))^{1/\lambda} = 1 - ((1 - \nu^{(k)})^{\lambda})^{1/\lambda} = \nu^{(k)}. \end{aligned}$$

$$\begin{aligned} &1 - (1 - \prod_{j=1}^n (1 - (1 - \tau_{\sigma(j)}^{(k)})^{\lambda})^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})})^{1/\lambda} \\ &= 1 - (1 - \prod_{j=1}^n (1 - (1 - \tau^{(k)})^{\lambda})^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})})^{1/\lambda} \\ &= 1 - (1 - (1 - (1 - \tau^{(k)})^{\lambda})^{\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))})^{1/\lambda} \\ &= 1 - (1 - (1 - (1 - \tau^{(k)})^{\lambda}))^{1/\lambda} = 1 - ((1 - \tau^{(k)})^{\lambda})^{1/\lambda} = \tau^{(k)}. \end{aligned}$$

Hence, $GHNLC A(h_1, h_2, \dots, h_n) = h.$ □

Definition 15. Let $X = \{x_1, x_2, \dots, x_n\}$ be a reference set, and let m be the fuzzy measure on $X.$ If $h_j = \{a_j^k\}$ ($j = 1, 2, \dots, n$) is a collection of the HNLEs, then the QHNLC A operator can be defined as follows:

$$QHNLC A(h_1, h_2, \dots, h_n) = g^{-1}(\sum_{j=1}^n (m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}))g(h_{\sigma(j)})) \tag{27}$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1, 2, \dots, n)$ such that $h_{\sigma(1)} \geq h_{\sigma(2)} \geq \dots \geq h_{\sigma(n)}.$ $A_{\sigma(i)} = \{x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(i)}\}, A_{\sigma(0)} = \emptyset,$ \emptyset is an empty set, and $g(x)$ is a strictly monotonic continuous function.

5. The Hesitant Neutrosophic Fuzzy Linguistic Decision-Making Method Based on TOPSIS

For a multiple attribute decision-making problem with p different periods t_k ($k = 1, 2, \dots, p$). Let $\{A_1, A_2, \dots, A_m\}$ be the set of alternatives, and let $\{C_1, C_2, \dots, C_n\}$ be the set of attributes. $h_{ij}^k = \{(s_{\theta_{ij}^k}, (T(s_{\theta_{ij}^k}), I(s_{\theta_{ij}^k}), F(s_{\theta_{ij}^k})))\}$ is the hesitant neutrosophic linguistic evaluation value given

by DMs in evaluating alternative A_i with respect to the attribute C_j in period k , where $s_{\theta_{ij}^k} \in S$, $\mu_{ij}^{kt} \in T(s_{\theta_{ij}^k})$, $\nu_{ij}^{kl} \in I(s_{\theta_{ij}^k})$, $\tau_{ij}^{km} \in F(s_{\theta_{ij}^k})$, and $0 \leq \sup \mu_{ij}^{kt} + \sup \nu_{ij}^{kl} + \sup \tau_{ij}^{km} \leq 3$. The decision matrix is formed as $D^k = (h_{ij}^k)_{m \times n}$ ($k = 1, 2, \dots, p$).

Algorithm

Step 1. The DMs evaluate alternatives A_i ($i = 1, 2, \dots, m$) with respect to attributes C_j ($j = 1, 2, \dots, n$) in period k ($k = 1, 2, \dots, p$) with hesitant neutrosophic fuzzy linguistic elements and decision matrices are formed $H^k = (h_{ij}^k)_{m \times n}$.

$$H^k = \begin{pmatrix} h_{11}^k & h_{12}^k & \dots & h_{1n}^k \\ h_{21}^k & h_{22}^k & \dots & h_{2n}^k \\ \vdots & \vdots & \ddots & \vdots \\ h_{m1}^k & h_{m2}^k & \dots & h_{mn}^k \end{pmatrix}.$$

Step 2. Confirm the fuzzy density of each period and determine ρ by using the equation $\rho = \prod_{i=1}^n (1 + \rho m(t_i)) - 1$.

Step 3. Calculate the evaluation values $h_{ij}^1, h_{ij}^2, \dots, h_{ij}^p$ in different periods into collective h_{ij} by using the GHNLCa operator as follows:

$$\begin{aligned} h_{ij} &= \text{GHNLCa}(h_{ij}^1, h_{ij}^2, \dots, h_{ij}^p) \\ &= \bigcup_{a_{ij}^k \in h_{ij}^k} \left\{ (s_{\sum_{k=1}^p (m(A_{\sigma(k)}) - m(A_{\sigma(k-1)})) (\theta_{ij}^{(k)})^\lambda})^{1/\lambda}, \bigcup \left\{ \left((1 - \prod_{k=1}^p (1 - (\mu_{ij}^{kt})^\lambda)^{m(A_{\sigma(k)}) - m(A_{\sigma(k-1)})} \right)^{1/\lambda}, \right. \right. \\ &\quad \left. \left. 1 - (1 - \prod_{k=1}^p (1 - (1 - \nu_{ij}^{kl})^\lambda)^{m(A_{\sigma(k)}) - m(A_{\sigma(k-1)})} \right)^{1/\lambda}, 1 - (1 - \prod_{k=1}^p (1 - (1 - \tau_{ij}^{km})^\lambda)^{m(A_{\sigma(k)}) - m(A_{\sigma(k-1)})} \right)^{1/\lambda} \right\} \right\} \end{aligned}$$

where $A_{\sigma(k)} = \{t_1, t_2, \dots, t_k\}$, $A_0 = \phi$. Thus, the decision matrix is formed as $H = (h_{ij})_{m \times n}$, $h_{ij} = \{(s_{\theta_{ij}}, (T(s_{\theta_{ij}}), I(s_{\theta_{ij}}), F(s_{\theta_{ij}})))\}$.

Step 4. Extend the collective decision matrix according to risk attitudes of DMs until all HNLEs have the same number of NLEs and until each NLE has the same number of truth-memberships, indeterminacy-memberships, and falsity-memberships. The extended decision matrix is formed as $H' = (h'_{ij})_{m \times n}$.

Step 5. Determine the hesitant neutrosophic linguistic positive ideal solution (HNLPIS), denoted as h^+ , and the hesitant neutrosophic linguistic negative ideal solution (HNLNIS), denoted as h^- , as follows:

$$h^+ = (h_1^+, h_2^+, \dots, h_n^+) = (\max_i h'_{i1}, \max_i h'_{i2}, \dots, \max_i h'_{in}), \tag{28}$$

$$h^- = (h_1^-, h_2^-, \dots, h_n^-) = (\min_i h'_{i1}, \min_i h'_{i2}, \dots, \min_i h'_{in}), \tag{29}$$

where $h_j^+ = \{(s_{\theta_j^{t+}}, (T(s_{\theta_j^{t+}}), I(s_{\theta_j^{t+}}), F(s_{\theta_j^{t+}})))\}$, $T(s_{\theta_j^{t+}}) = \{\mu_{ijk}^{t+}\}$, $I(s_{\theta_j^{t+}}) = \{\nu_{ijk}^{t+}\}$, $F(s_{\theta_j^{t+}}) = \{\tau_{ijk}^{t+}\}$, $h_j^- = \{(s_{\theta_j^{t-}}, (T(s_{\theta_j^{t-}}), I(s_{\theta_j^{t-}}), F(s_{\theta_j^{t-}})))\}$, $T(s_{\theta_j^{t-}}) = \{\mu_{ijk}^{t-}\}$, $I(s_{\theta_j^{t-}}) = \{\nu_{ijk}^{t-}\}$, $F(s_{\theta_j^{t-}}) = \{\tau_{ijk}^{t-}\}$.

Step 6. Confirm the fuzzy density $m(C_i)$ of each attribute. Determine ρ by using Equation (14) and calculate fuzzy measures of attribute sets.

Step 7. Based on the generalized hesitant neutrosophic linguistic correlation weighted averaging distance, calculate the distances of each alternative's collective evaluation values to the HNLPIS h^+ and HNLNIS h^- , respectively.

$$d_{GC}^{i+}(h_i, h^+) = \left(\sum_{j=1}^n (m(B_{\sigma(j)}) - m(B_{\sigma(j-1)})) \frac{1}{4l} \sum_{i=1}^l \left(\frac{1}{l_\theta} \sum_{t=1}^{l_\theta} (|\theta_{\sigma(ji)}^t - \theta_{\sigma(ji)}^{t+}|^\lambda / g^\lambda + \frac{1}{l_t} \sum_{k=1}^{l_t} |\mu_{\sigma(jik)}^t - \mu_{\sigma(jik)}^{t+}|^\lambda + \frac{1}{l_u} \sum_{k=1}^{l_u} |v_{\sigma(jik)}^t - v_{\sigma(jik)}^{t+}|^\lambda + \frac{1}{l_v} \sum_{k=1}^{l_v} |\tau_{\sigma(jik)}^t - \tau_{\sigma(jik)}^{t+}|^\lambda) \right) \right)^{1/\lambda} \quad (30)$$

$$d_{GC}^{i-}(h_i, h^-) = \left(\sum_{j=1}^n (m(B_{\sigma(j)}) - m(B_{\sigma(j-1)})) \left(\frac{1}{4l} \sum_{i=1}^l \frac{1}{l_\theta} \sum_{t=1}^{l_\theta} (|\theta_{\sigma(ji)}^t - \theta_{\sigma(ji)}^{t-}|^\lambda / g^\lambda + \frac{1}{l_t} \sum_{k=1}^{l_t} |\mu_{\sigma(jik)}^t - \mu_{\sigma(jik)}^{t-}|^\lambda + \frac{1}{l_u} \sum_{k=1}^{l_u} |v_{\sigma(jik)}^t - v_{\sigma(jik)}^{t-}|^\lambda + \frac{1}{l_v} \sum_{k=1}^{l_v} |\tau_{\sigma(jik)}^t - \tau_{\sigma(jik)}^{t-}|^\lambda) \right) \right)^{1/\lambda} \quad (31)$$

where $d_{GC}^{1+}(h_{\sigma(i1)}, h_{\sigma(1)}^+) \geq d_{GC}^{2+}(h_{\sigma(i2)}, h_{\sigma(2)}^+) \geq \dots \geq d_{GC}^{n+}(h_{\sigma(in)}, h_{\sigma(n)}^+)$, $d_{GC}^{1-}(h_{\sigma(i1)}, h_{\sigma(1)}^-) \geq d_{GC}^{2-}(h_{\sigma(i2)}, h_{\sigma(2)}^-) \geq \dots \geq d_{GC}^{n-}(h_{\sigma(in)}, h_{\sigma(n)}^-)$, $B_{\sigma(j)} = \{C_{\sigma(1)}, C_{\sigma(2)}, \dots, C_{\sigma(j)}\}$, and $B_{\sigma(0)} = \phi$, $C_{\sigma(j)}$ is the attribute corresponding to the j th largest $d_{GC}^{i+}(h_{ij}, h_j^+)$ in d_{GC}^{i+} or the j th largest $d_{GC}^{i-}(h_{ij}, h_j^-)$ in d_{GC}^{i-} .

Step 8. Calculate closeness coefficients CC_i ($i = 1, 2, \dots, m$) for alternatives A_i ($i = 1, 2, \dots, m$) as follows:

$$CC_i = \frac{d_{GC}^{i-}(h_i, h^-)}{d_{GC}^{i+}(h_i, h^-) + d_{GC}^{i-}(h_i, h^-)}, i = 1, 2, \dots, m. \quad (32)$$

Step 9. Rank alternatives according to the ranking of closeness coefficients and select the optimal alternative.

6. Numerical Example

6.1. Practical Example

A company wants to invest a sum of money in an industry (adapted from [43]). The experts mainly consider the following attributes: C_1 —the price of the product; C_2 —the quality of the product; C_3 —the risk; C_4 —the delivery time. After pre-evaluation, there are five alternatives left for further evaluation. Alternatives are evaluated with respect to attributes using hesitant neutrosophic linguistic information. They consider information in three periods ($t_i, i = 1, 2, \dots, 3$). The new method is used to rank alternatives.

Step 1. The hesitant neutrosophic linguistic matrices D^k ($k = 1, 2, 3$) are given by experts in evaluating alternatives with respect to attributes in different periods as in Tables 1–3.

Step 2. Determine the fuzzy density of different t_i and λ . Assume $m(t_1) = 0.3, m(t_2) = 0.4$, and $m(t_3) = 0.5$. By using Equation (14), we can get $\rho = -0.3793$. Then we can get $m(t_1, t_2) = 0.6545, m(t_1, t_3) = 0.7431, m(t_2, t_3) = 0.8241$, and $m(t_1, t_2, t_3) = 1.0$.

Step 3. Aggregate the collective evaluation values by using the GHNLCA operator. We can take different λ values. Here, we consider the HNLCA operator by taking $\lambda = 1$ in the GHNLCA operator. The results are shown in Table 4.

Step 4. Assume decision-makers are risk-averse. Thus, the smallest linguistic evaluation value, the smallest truth-membership, the largest indeterminacy-membership, and the largest falsity-membership can be added until all the HNLEs have the same number of NLEs and until each NLE has the same number of truth-memberships, indeterminacy-memberships, and falsity-memberships. The results are shown in Table 5.

Step 5. We take the HNLPIs and the HNLNIs as follows:

HNLPIs: $h^+ = \{ \{(s_9, (\{1, 1\}, \{0, 0\}, \{0, 0\})), (s_9, (\{1, 1\}, \{0, 0\}, \{0, 0\}))\}, \{(s_9, (\{1, 1\}, \{0, 0\}, \{0, 0\})), (s_9, (\{1, 1\}, \{0, 0\}, \{0, 0\}))\}, \{(s_9, (\{1, 1\}, \{0, 0\}, \{0, 0\})), (s_9, (\{1, 1\}, \{0, 0\}, \{0, 0\}))\}, \{(s_9, (\{1, 1\}, \{0, 0\}, \{0, 0\})), (s_9, (\{1, 1\}, \{0, 0\}, \{0, 0\}))\}, \{(s_9, (\{1, 1\}, \{0, 0\}, \{0, 0\})), (s_9, (\{1, 1\}, \{0, 0\}, \{0, 0\}))\} \}$.

HNLNIs: $h^- = \{ \{(s_1, (\{0, 0\}, \{1, 1\}, \{1, 1\})), (s_1, (\{0, 0\}, \{1, 1\}, \{1, 1\}))\}, \{(s_1, (\{0, 0\}, \{1, 1\}, \{1, 1\})), (s_1, (\{0, 0\}, \{1, 1\}, \{1, 1\}))\}, \{(s_1, (\{0, 0\}, \{1, 1\}, \{1, 1\})), (s_1, (\{0, 0\}, \{1, 1\}, \{1, 1\}))\}, \{(s_1, (\{0, 0\}, \{1, 1\}, \{1, 1\})), (s_1, (\{0, 0\}, \{1, 1\}, \{1, 1\}))\} \}$.

Step 6. Assume $m(C_1) = 0.25$, $m(C_2) = 0.35$, $m(C_3) = 0.30$, and $m(C_4) = 0.20$. We can get $\rho = -0.2330$ by using Equation (14). Then, we can determine the fuzzy measures of the attribute sets as follows:

$$m(\{C_1\}) = 0.25, m(\{C_2\}) = 0.35, m(\{C_3\}) = 0.30, m(\{C_4\}) = 0.20, m(\{C_1, C_2\}) = 0.5796, m(\{C_1, C_3\}) = 0.5325, m(\{C_1, C_4\}) = 0.4383, m(\{C_2, C_3\}) = 0.6255, m(\{C_2, C_4\}) = 0.5337, m(\{C_3, C_4\}) = 0.4860, m(\{C_1, C_2, C_3\}) = 0.8391, m(\{C_1, C_2, C_4\}) = 0.7526, m(\{C_1, C_3, C_4\}) = 0.7077, m(\{C_2, C_3, C_4\}) = 0.7964, m(\{C_1, C_2, C_3, C_4\}) = 1.0.$$

Step 7. Calculate the correlated distances of each alternative’s evaluation values to the HNLPIs and the HNLNIs by generalized hesitant neutrosophic linguistic correlation weighted averaging distance. Here, we can take different λ in Equations (30) and (31). If $\lambda = 2$, we can get

$$d_{CE}^{1+} = 0.5249, d_{CE}^{2+} = 0.5344, d_{CE}^{3+} = 0.5082, d_{CE}^{4+} = 0.5190, d_{CE}^{5+} = 0.5249, d_{CE}^{1-} = 0.5107, d_{CE}^{2-} = 0.5231, d_{CE}^{3-} = 0.5432, d_{CE}^{4-} = 0.5624, d_{CE}^{5-} = 0.5439.$$

Step 8. Calculate closeness coefficients of alternatives as

$$CC_1 = 0.4932, CC_2 = 0.4947, CC_3 = 0.5166, CC_4 = 0.5200, CC_5 = 0.5262.$$

Step 9. Rank alternatives according to the ranking of CC_i to get $A_5 > A_4 > A_3 > A_2 > A_1$ and the optimal alternative is A_5 .

If another λ value is used in Step 7, we can get distances, coefficients, and alternative rankings as in Table 6.

Table 1. Decision matrix D^1 .

Alternative	C_1	C_2
A_1	$\{(s_5, \{0.3, 0.4\}, \{0.2\}, \{0.3\})\}$	$\{(s_4, \{0.4\}, \{0.3\}, \{0.2\})\}$
A_2	$\{(s_6, \{0.6\}, \{0.1\}, \{0.2\}), (s_7, \{0.3\}, \{0.4\}, \{0.3\})\}$	$\{(s_5, \{0.6\}, \{0.1\}, \{0.2\})\}$
A_3	$\{(s_4, \{0.3\}, \{0.2\}, \{0.3\})\}$	$\{(s_3, \{0.5\}, \{0.2\}, \{0.3\}), (s_4, \{0.3\}, \{0.2\}, \{0.5\})\}$
A_4	$\{(s_4, \{0.7\}, \{0.0\}, \{0.1\}), (s_5, \{0.6\}, \{0.2\}, \{0.2\})\}$	$\{(s_7, \{0.6\}, \{0.1\}, \{0.2\})\}$
A_5	$\{(s_3, \{0.5\}, \{0.3, 0.4\}, \{0.2\})\}$	$\{(s_2, \{0.3\}, \{0.3, 0.4\}, \{0.5\})\}$
	C_3	C_4
A_1	$\{(s_2, \{0.2, 0.3\}, \{0.3\}, \{0.5, 0.6\})\}$	$\{(s_5, \{0.5\}, \{0.2\}, \{0.4\}), (s_7, \{0.4\}, \{0.2\}, \{0.3\})\}$
A_2	$\{(s_3, \{0.5\}, \{0.3\}, \{0.2\})\}$	$\{(s_6, \{0.7, 0.8\}, \{0.1\}, \{0.1\})\}$
A_3	$\{(s_5, \{0.6\}, \{0.1\}, \{0.3\})\}$	$\{(s_2, \{0.4\}, \{0.2\}, \{0.3\})\}$
A_4	$\{(s_2, \{0.7\}, \{0.2\}, \{0.1, 0.2\})\}$	$\{(s_3, \{0.5\}, \{0.3\}, \{0.2\})\}$
A_5	$\{(s_8, \{0.4\}, \{0.3\}, \{0.4\})\}$	$\{(s_4, \{0.5\}, \{0.4\}, \{0.3\}), (s_5, \{0.3\}, \{0.3\}, \{0.5\})\}$

Table 2. Decision matrix D^2 .

Alternative	C_1	C_2
A_1	$\{(s_6, \{0.2\}, \{0.4\}, \{0.3\}), (s_7, \{0.5\}, \{0.1\}, \{0.2\})\}$	$\{(s_5, \{0.2\}, \{0.2, 0.4\}, \{0.5\})\}$
A_2	$\{(s_4, \{0.4\}, \{0.1\}, \{0.3\})\}$	$\{(s_6, \{0.4\}, \{0.3\}, \{0.5\})\}$
A_3	$\{(s_7, \{0.5\}, \{0.3, 0.4\}, \{0.1\})\}$	$\{(s_4, \{0.2\}, \{0.4\}, \{0.3\})\}$
A_4	$\{(s_3, \{0.5\}, \{0.2\}, \{0.2\})\}$	$\{(s_6, \{0.5\}, \{0.2, 0.3\}, \{0.2\})\}$
A_5	$\{(s_2, \{0.6\}, \{0.4\}, \{0.3\})\}$	$\{(s_4, \{0.6\}, \{0.3\}, \{0.4\})\}$
	C_3	C_4
A_1	$\{(s_3, \{0.4\}, \{0.4\}, \{0.3\})\}$	$\{(s_3, \{0.5\}, \{0.2\}, \{0.4\})\}$
A_2	$\{(s_4, \{0.3\}, \{0.2\}, \{0.4\})\}$	$\{(s_5, \{0.6\}, \{0.1, 0.2\}, \{0.3\})\}$
A_3	$\{(s_3, \{0.5\}, \{0.3\}, \{0.2\})\}$	$\{(s_2, \{0.3\}, \{0.2\}, \{0.3\})\}$
A_4	$\{(s_4, \{0.4\}, \{0.2\}, \{0.1\})\}$	$\{(s_3, \{0.4\}, \{0.1\}, \{0.3\}), (s_4, \{0.3\}, \{0.2\}, \{0.2\})\}$
A_5	$\{(s_5, \{0.7, 0.8\}, \{0.3\}, \{0.3\})\}$	$\{(s_6, \{0.5\}, \{0.3\}, \{0.4\})\}$

Table 3. Decision matrix D^3 .

Alternative	C_1	C_2
A_1	$\{(s_3, \{0.3\}, \{0.4\}, \{0.5\})\}$	$\{(s_3, \{0.3\}, \{0.4\}, \{0.2\})\}$
A_2	$\{(s_5, \{0.5\}, \{0.2\}, \{0.3\})\}$	$\{(s_4, \{0.7\}, \{0.2\}, \{0.2\})\}$
A_3	$\{(s_4, \{0.4\}, \{0.1\}, \{0.3\}), (s_5, \{0.6\}, \{0.2\}, \{0.1\})\}$	$\{(s_5, \{0.6\}, \{0.1\}, \{0.3\})\}$
A_4	$\{(s_6, \{0.6\}, \{0.1\}, \{0.2\})\}$	$\{(s_4, \{0.5\}, \{0.2\}, \{0.2\})\}$
A_5	$\{(s_3, \{0.4\}, \{0.2\}, \{0.3\})\}$	$\{(s_6, \{0.7, 0.8\}, \{0.2\}, \{0.4\})\}$
	C_3	C_4
A_1	$\{(s_6, \{0.2\}, \{0.1, 0.3\}, \{0.6\})\}$	$\{(s_5, \{0.4\}, \{0.3\}, \{0.3\})\}$
A_2	$\{(s_2, \{0.7\}, \{0.2\}, \{0.1\})\}$	$\{(s_4, \{0.5\}, \{0.2\}, \{0.2\}), (s_5, \{0.6\}, \{0.1\}, \{0.2\})\}$
A_3	$\{(s_7, \{0.6\}, \{0.2\}, \{0.2\})\}$	$\{(s_2, \{0.7\}, \{0.1\}, \{0.2\})\}$
A_4	$\{(s_4, \{0.4, 0.5\}, \{0.1\}, \{0.2\})\}$	$\{(s_3, \{0.6\}, \{0.2, 0.4\}, \{0.3\})\}$
A_5	$\{(s_2, \{0.6\}, \{0.2\}, \{0.4\}), (s_3, \{0.5\}, \{0.2\}, \{0.3\})\}$	$\{(s_7, \{0.8\}, \{0.2\}, \{0.1\})\}$

Table 4. Collective decision matrix D .

Alternative	C_1	C_2
A_1	$\{(s_{4.6635}, \{0.2661, 0.2992\}, \{0.3249\}, \{0.3579\}), (s_{5.0180}, \{0.3787, 0.4068\}, \{0.1988\}, \{0.3100\})\}$	$\{(s_{4.0090}, \{0.2992\}, \{0.2870, 0.3669\}, \{0.2768\})\}$
A_2	$\{(s_{4.9455}, \{0.5012\}, \{0.1271\}, \{0.2656\})\}$	$\{(s_{5.009}, \{0.5819\}, \{0.1876\}, \{0.2768\})\}$
A_3	$\{(s_{5.0635}, \{0.4109\}, \{0.1817, 0.2013\}, \{0.2032\}), (s_{5.4090}, \{0.4879\}, \{0.2309, 0.2557\}, \{0.1390\})\}$	$\{(s_{4.0455}, \{0.4532\}, \{0.2013\}, \{0.3000\}), (s_{4.3455}, \{0.3951\}, \{0.2013\}, \{0.3497\})\}$
A_4	$\{(s_{4.3365}, \{0.6029\}, \{0.0000\}, \{0.1625\}), (s_{4.6365}, \{0.5671\}, \{0.1574\}, \{0.2000\})\}$	$\{(s_{5.3090}, \{0.5324\}, \{0.1625, 0.1876\}, \{0.2000\})\}$
A_5	$\{(s_{2.6455}, \{0.4340\}, \{0.2888, 0.3148\}, \{0.2656\})\}$	$\{(s_{4.0910}, \{0.5716, 0.6276\}, \{0.2608, 0.2843\}, \{0.4277\})\}$
	C_3	C_4
A_1	$\{(s_{3.7365}, \{0.2776, 0.3059\}, \{0.2273, 0.3322\}, \{0.4443\})\}$	$\{(s_{4.291}, \{0.4657\}, \{0.2301\}, \{0.3622\}), (s_{4.891}, \{0.4376\}, \{0.2301\}, \{0.3322\})\}$
A_2	$\{(s_{3.0090}, \{0.5278\}, \{0.2259\}, \{0.2013\})\}$	$\{(s_{4.9545}, \{0.6037, 0.6491\}, \{0.1271, 0.1625\}, \{0.1876\})\}$
A_3	$\{(s_{4.9820}, \{0.5671\}, \{0.1876\}, \{0.2259\})\}$	$\{(s_{2.0000}, \{0.5013\}, \{0.1574\}, \{0.2608\})\}$
A_4	$\{(s_{3.4000}, \{0.5126, 0.5424\}, \{0.1574\}, \{0.1271, 0.1564\})\}$	$\{(s_{3.0000}, \{0.5062\}, \{0.1767, 0.2245\}, \{0.2656\}), (s_{3.3545}, \{0.4785\}, \{0.2259, 0.2870\}, \{0.2301\})\}$
A_5	$\{(s_{4.8635}, \{0.5921, 0.6467\}, \{0.2608\}, \{0.3612\}), (s_{5.2090}, \{0.5594, 0.6184\}, \{0.2608\}, \{0.3270\})\}$	$\{(s_{5.7455}, \{0.6357\}, \{0.2843\}, \{0.2273\}), (s_{6.0455}, \{0.5970\}, \{0.2608\}, \{0.2649\})\}$

Table 5. Extended hesitant neutrosophic fuzzy linguistic decision matrix \tilde{D} .

Alternative	C_1	C_2
A_1	$\{(s_{4.6635}, \{0.2661, 0.2992\}, \{0.3249, 0.3669\}, \{0.3579, 0.4443\}), (s_{5.0180}, \{0.3787, 0.4068\}, \{0.1988, 0.3669\}, \{0.3100, 0.4443\})\}$	$\{(s_{4.0090}, \{0.2661, 0.2992\}, \{0.2870, 0.3669\}, \{0.2768, 0.4443\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$
A_2	$\{(s_{4.9455}, \{0.2661, 0.5012\}, \{0.1271, 0.3669\}, \{0.2656, 0.4443\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$	$\{(s_{5.009}, \{0.2661, 0.5819\}, \{0.1876, 0.3669\}, \{0.2768, 0.4443\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$
A_3	$\{(s_{5.0635}, \{0.2661, 0.4109\}, \{0.1817, 0.2013\}, \{0.2032, 0.4443\}), (s_{5.4090}, \{0.2661, 0.4879\}, \{0.2309, 0.2557\}, \{0.1390, 0.4443\})\}$	$\{(s_{4.0455}, \{0.2661, 0.4532\}, \{0.2013, 0.3669\}, \{0.3000, 0.4443\}), (s_{4.3455}, \{0.2661, 0.3951\}, \{0.2013, 0.3669\}, \{0.3497, 0.4443\})\}$
A_4	$\{(s_{4.3365}, \{0.2661, 0.6029\}, \{0.0000, 0.3669\}, \{0.1625, 0.4443\}), (s_{4.6365}, \{0.2661, 0.5671\}, \{0.1574, 0.3669\}, \{0.2000, 0.4443\})\}$	$\{(s_{5.3090}, \{0.2661, 0.5324\}, \{0.1625, 0.1876\}, \{0.2000, 0.4443\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$
A_5	$\{(s_{2.6455}, \{0.2661, 0.4340\}, \{0.2888, 0.3148\}, \{0.2656, 0.4443\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$	$\{(s_{4.0910}, \{0.5716, 0.6276\}, \{0.2608, 0.2843\}, \{0.4277, 0.4443\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$
	C_3	C_4
A_1	$\{(s_{3.7365}, \{0.2776, 0.3059\}, \{0.2273, 0.3322\}, \{0.4443, 0.4443\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$	$\{(s_{4.291}, \{0.2661, 0.4657\}, \{0.2301, 0.3669\}, \{0.3622, 0.4443\}), (s_{4.891}, \{0.2661, 0.4376\}, \{0.2301, 0.3669\}, \{0.3322, 0.4443\})\}$
A_2	$\{(s_{3.0090}, \{0.2661, 0.5278\}, \{0.2259, 0.3669\}, \{0.2013, 0.4443\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$	$\{(s_{4.9545}, \{0.6037, 0.6491\}, \{0.1271, 0.1625\}, \{0.1876, 0.4443\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$
A_3	$\{(s_{4.9820}, \{0.2661, 0.5671\}, \{0.1876, 0.3669\}, \{0.2259, 0.4443\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$	$\{(s_{2.0000}, \{0.2661, 0.5013\}, \{0.1574, 0.2608\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$
A_4	$\{(s_{3.4000}, \{0.5126, 0.5424\}, \{0.1574, 0.3669\}, \{0.1271, 0.1564\}), (s_{2.0000}, \{0.2661, 0.2661\}, \{0.3669, 0.3669\}, \{0.4443, 0.4443\})\}$	$\{(s_{3.0000}, \{0.2661, 0.5062\}, \{0.1767, 0.2245\}, \{0.2656, 0.4443\}), (s_{3.3545}, \{0.2661, 0.4785\}, \{0.2259, 0.2870\}, \{0.2301, 0.4443\})\}$
A_5	$\{(s_{4.8635}, \{0.5921, 0.6467\}, \{0.2608, 0.3669\}, \{0.3612, 0.4443\}), (s_{5.2090}, \{0.5594, 0.6184\}, \{0.2608, 0.3669\}, \{0.3270, 0.4443\})\}$	$\{(s_{5.7455}, \{0.2661, 0.6357\}, \{0.2843, 0.3669\}, \{0.2273, 0.4443\}), (s_{6.0455}, \{0.2661, 0.5970\}, \{0.2608, 0.3669\}, \{0.2649, 0.4443\})\}$

$$d_i(h'_i, h'^-) = \left(\sum_{j=1}^n (m(B_{\sigma(j)}) - m(B_{\sigma(j-1)})) \frac{1}{3l} \sum_{i=1}^l \left(\frac{1}{l_\theta} \sum_{t=1}^{l_\theta} (|\theta_{\sigma(ji)}^t - 1|^\lambda + \frac{1}{l_t} \sum_{k=1}^{l_t} |\mu_{\sigma(jik)}^t - 0|^\lambda + \frac{1}{l_t} \sum_{k=1}^{l_t} |\tau_{\sigma(jik)}^t - 1|^\lambda) \right) \right)^{1/\lambda} \tag{35}$$

Calculate the closeness coefficients of alternatives by using Equation (32). The results are also shown in Table 7. From the results, we can get different ranking results if indeterminacy memberships are not considered. In most cases, A_4 is the optimal alternative and A_5 becomes the suboptimal alternative. If $\lambda = 10$, A_3 is the optimal alternative, and A_5 is ranked as the second to last.

Table 7. Decision results using different λ with hesitant intuitionistic fuzzy linguistic elements (HIFLEs).

λ	CC_i	A_1	A_2	A_3	A_4	A_5	Ranking
$\lambda = 0.1$	CC_i	0.4110	0.4295	0.4556	0.4787	0.4678	$A_4 > A_5 > A_3 > A_2 > A_1$
$\lambda = 0.5$	CC_i	0.4158	0.4369	0.4589	0.4809	0.4718	$A_4 > A_5 > A_3 > A_2 > A_1$
$\lambda = 1$	CC_i	0.4212	0.4443	0.4626	0.4838	0.4756	$A_4 > A_5 > A_3 > A_2 > A_1$
$\lambda = 2$	CC_i	0.4300	0.4541	0.4686	0.4891	0.4800	$A_4 > A_5 > A_3 > A_2 > A_1$
$\lambda = 5$	CC_i	0.4444	0.4659	0.4791	0.4997	0.4796	$A_4 > A_5 > A_3 > A_2 > A_1$
$\lambda = 10$	CC_i	0.4775	0.4812	0.5026	0.5021	0.4705	$A_3 > A_4 > A_2 > A_5 > A_1$

If the linguistic arguments are not considered, the hesitant neutrosophic linguistic elements reduce to the hesitant neutrosophic elements [45] $h''_{ij} = \{ (T^k_{ij}, I^k_{ij}, F^k_{ij}) \}$. The hesitant neutrosophic elements are first aggregated into collective ones by using the generalized hesitant neutrosophic fuzzy correlated averaging (GHNFCA) operator as follows:

$$\begin{aligned} \text{GHNFCA}(h''_1, h''_2, \dots, h''_n) &= \left(\sum_{j=1}^n m(A_{\sigma(j)}) - m(A_{\sigma(j-1)}) (h''_{\sigma(j)})^\lambda \right)^{1/\lambda} \\ &= \bigcup_{a^k_j \in h_j} \left\{ \bigcup \left\{ \left((1 - \prod_{j=1}^n (1 - (\mu_{\sigma(j)}^k)^\lambda)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})}) \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \nu_{\sigma(j)}^k)^\lambda)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})}) \right)^{1/\lambda}, 1 - \left(1 - \prod_{j=1}^n (1 - (1 - \tau_{\sigma(j)}^k)^\lambda)^{m(A_{\sigma(j)}) - m(A_{\sigma(j-1)})}) \right)^{1/\lambda} \right\} \right\}. \end{aligned}$$

The DMs are also assumed to be risk-averse and the smallest truth-membership, the largest indeterminacy-membership, and the largest falsity-membership are added. Determine the hesitant neutrosophic fuzzy positive ideal solution h''^+ and hesitant neutrosophic fuzzy negative ideal solution h''^- . Calculate the distances of collective neutrosophic fuzzy values to h''^+ and h''^- by the following equations:

$$d_i(h''_i, h''^+) = \left(\sum_{j=1}^n (m(B_{\sigma(j)}) - m(B_{\sigma(j-1)})) \frac{1}{3l} \sum_{i=1}^l \left(\frac{1}{l_\theta} \sum_{t=1}^{l_\theta} (|\mu_{\sigma(ji)}^t - 1|^\lambda + \frac{1}{l_t} \sum_{k=1}^{l_t} |\nu_{\sigma(jik)}^t - 0|^\lambda + \frac{1}{l_t} \sum_{k=1}^{l_t} |\tau_{\sigma(jik)}^t - 0|^\lambda) \right) \right)^{1/\lambda} \tag{36}$$

$$d_i(h''_i, h''^-) = \left(\sum_{j=1}^n (m(B_{\sigma(j)}) - m(B_{\sigma(j-1)})) \frac{1}{3l} \sum_{i=1}^l \left(\frac{1}{l_\theta} \sum_{t=1}^{l_\theta} (|\mu_{\sigma(ji)}^t - 0|^\lambda + \frac{1}{l_t} \sum_{k=1}^{l_t} |\nu_{\sigma(jik)}^t - 1|^\lambda + \frac{1}{l_t} \sum_{k=1}^{l_t} |\tau_{\sigma(jik)}^t - 1|^\lambda) \right) \right)^{1/\lambda} \tag{37}$$

Other steps are the same as the proposed methods. If $\lambda = 2$, we can get the results as $CC_1 = 0.7661$, $CC_1 = 0.7229$, $CC_1 = 0.7686$, $CC_1 = 0.6880$, and $CC_1 = 0.7256$. The alternatives can be ranked as $A_3 > A_1 > A_5 > A_2 > A_4$.

Results of different methods are listed in Table 8 if $\lambda = 2$. From the results, we can see that different ranking results can be obtained using different methods. In the method of [45], only hesitant neutrosophic fuzzy values have been considered. In the method of [44], the memberships are in the form of intuitionistic fuzzy values. In fact, DMs would like to evaluate with linguistic arguments in the evaluation process. By using hesitant neutrosophic fuzzy elements to model fuzziness and uncertainty, more accuracy results can be obtained. Decision results are more reasonable in the proposed method.

Based on comparison with existing methods, therefore, this paper presents a new way of solving multiple attribute decision-making in a hesitant neutrosophic linguistic environment.

Table 8. Results of different methods.

Method	Ranking
Method from [44]	$A_4 > A_5 > A_3 > A_2 > A_1$
Method from [45]	$A_3 > A_1 > A_5 > A_2 > A_4$
Proposed method	$A_5 > A_4 > A_3 > A_2 > A_1$

7. Conclusions

In this paper, we first introduce a definition of the hesitant neutrosophic fuzzy linguistic set. We then define some distance measures including the hesitant neutrosophic linguistic Euclidean distance, the hesitant neutrosophic linguistic Hamming distance, and the generalized hesitant neutrosophic linguistic distance. We also define some correlated distance measures including the generalized hesitant neutrosophic linguistic correlation averaging distance and the generalized hesitant neutrosophic linguistic hybrid correlation averaging distance. We present some correlated aggregation operators for hesitant neutrosophic fuzzy linguistic set including the HNLCA operator, the HNLCA operator, the GHNLCA operator, and the QHNLCA operator. A new hesitant neutrosophic fuzzy linguistic multiple attribute decision-making method has been developed based on the new distance measures and the new aggregation operators. We apply the new method to solve investment problems. Two special cases of the hesitant neutrosophic fuzzy linguistic set from existing works have been used in the numerical example to illustrate the difference between the proposed method and several other methods.

The proposed method has the following advantages. First, the hesitant neutrosophic fuzzy linguistic set is the extension of the hesitant fuzzy set, the neutrosophic set, and the linguistic term set, which can be applied in decision problems with imprecise, uncertain, incomplete, and inconsistent information. Compared with other tools, the hesitant neutrosophic fuzzy linguistic set is more flexible and accurate, and many existing tools are special cases of the hesitant neutrosophic fuzzy linguistic set. Second, correlation exists extensively in the decision-making process, which can be modeled by using the Choquet integral. We combine the Choquet integral with the hesitant neutrosophic fuzzy linguistic theory to develop some distance measures and aggregation operators. Finally, the new multiple attribute group decision-making method is based on the TOPSIS and the Choquet integral for hesitant neutrosophic fuzzy linguistic information, which can be used to solve complicated decision problems.

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