



# Generalized Interval Neutrosophic Rough Sets and its Application in Multi-Attribute Decision Making

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**Abstract.** Neutrosophic set (NS) was originally proposed by Smarandache to handle indeterminate and inconsistent information. It is a generalization of fuzzy sets and intuitionistic fuzzy sets. Wang and Smarandache proposed interval neutrosophic sets (INS) which is a special case of NSs and would be extensively applied to resolve practical issues. In this paper, we put forward generalized interval neutrosophic rough sets based on interval neutrosophic relations by combining interval neutrosophic sets with rough sets. We explore the hybrid model through constructive approach as well as axiomatic approach. On one hand, we define generalized interval neutrosophic lower and upper approximation operators through constructive approach. Moreover, we investigate the relevance between generalized interval neutrosophic lower (upper) approximation operators and particular interval neutrosophic relations. On the other hand, we study axiomatic characterizations of generalized interval neutrosophic approximation operators, and also show that different axiom sets of theoretical interval neutrosophic operators make sure the existence of different classes of INRs that yield the same interval neutrosophic approximation operators. Finally, we introduce generalized interval neutrosophic rough sets on two universes and a universal algorithm of multi-attribute decision making based on generalized interval neutrosophic rough sets on two universes. Besides, an example is given to demonstrate the validity of the new rough set model.

## 1. Introduction

Smarandache [24, 25] introduced neutrosophic sets (NSs) by combining non-standard analysis and a tri-component set. A NS includes three membership functions (truth-membership function, indeterminacy membership function and falsity-membership function), where every function value is a real standard or non-standard subset of the nonstandard unit interval  $]0^-, 1^+[$ . In a NS, indeterminacy is quantified explicitly, and the three membership functions are independent from each other. Riverain [19] initiated neutrosophic logics by applying the neutrosophic idea to logics. Guo et al. [9, 10] successfully applied NSs to image processing and cluster analysis. Ali and Smarandache [1] studied complex neutrosophic sets.

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For the sake of conveniently applying NSs into real world, Wang et al. [28] proposed single valued neutrosophic sets (SVNSs) which is a subclass of neutrosophic sets. Yang et al. [33] studied the single valued neutrosophic relations in detail. Biswas et al. [2] studied TOPSIS method for multi-attribute group decision-making under single-valued neutrosophic environment. Majumdar and Samanta [14] explored distance, similarity, and entropy of SVNSs. A subethood measure of SVNSs based on distance was studied by Şahin and Küçük [22]. Peng et al. [18] proposed some operations of SVNSs from a new point of view and further gave a novel approach to solve decision-making problems based on outranking relations of simplified neutrosophic numbers. Based on the combination of trapezoidal fuzzy numbers and a single valued neutrosophic set, Ye [39] introduced trapezoidal neutrosophic set and explored its application to multiple attribute decision-making. At the same time, Ye [40] also presented a simplified neutrosophic harmonic averaging projection measure and its multiple attribute decision making method with simplified neutrosophic information.

To deal with more complex problems, Wang et al. [27] introduced interval neutrosophic sets (INs) that take values on the subinterval of  $[0, 1]$ . Zhang et al. [44] studied some properties about INs and their application in multicriteria decision making problems. Subsequently, Zhang et al. [45] proposed an outranking approach for multi-criteria decision-making problems with INs. Ye [38] proposed correlation coefficients of INs, and applied it to interval neutrosophic decision-making problems. Liu and Shi [11] gave a generalized hybrid weighted average operator based on interval neutrosophic hesitant set and studied its application to multiple attribute decision making. Liu and Wang [12] proposed interval neutrosophic prioritized OWA operator on the basis of prioritized aggregated operator and prioritized ordered weighted average (POWA) operator and further studied its application to multiple attribute decision making. Ma et al. [13] proposed an interval neutrosophic linguistic multi-criteria group decision-making method and explored its application in selecting medical treatment options. Yang et al. [32] studied linear assignment method for INs. Şahin [21] introduced cross-entropy measure on INs and applied it to multicriteria decision making.

Rough set theory was established by Pawlak and it has been proved to be an efficient tool to handle imprecise information. In the development of rough set theory, there are two main methods—constructive approach and axiomatic approach. In the constructive approach, there are many primitive notions such as arbitrary binary relations on the universe, partitions or coverings of the universe, neighborhood systems and so on, then the lower and upper approximation operators can be constructed based on these existed structures [7, 8, 35, 37, 41]. On the other hand, in the axiomatic approach, one always can characterize rough approximation operators by a set of axioms [15, 26, 29, 36, 42, 43, 46].

In recent years, many scholars have focused on the research of combining neutrosophic sets with rough sets. Salama and Broumi [20] investigated the roughness of neutrosophic sets. Broumi and Smarandache [3] put forward rough neutrosophic sets as well as interval neutrosophic rough sets [4]. Yang et al. [34] proposed single valued neutrosophic rough sets which is a hybrid model of single valued neutrosophic sets and rough sets. The study of generalized interval neutrosophic rough sets based on interval neutrosophic relations is still a blank. In the present paper, we shall introduce generalized neutrosophic rough sets based on interval neutrosophic relations and explore the model from both constructive and axiomatic approaches. We also apply the new model to multi-attribute decision making problems.

The rest of the paper is organized as follows. In the next section, we briefly recall some basic notions and operations. In Section 3, we propose generalized interval neutrosophic rough sets based on interval neutrosophic relations through constructive method and some basic properties are explored. We investigate the connection between special interval neutrosophic relations and generalized interval neutrosophic lower (upper) approximation operators. Section 4 illustrates the axiomatic characterizations of generalized interval neutrosophic approximation operators. In Section 5, we introduce generalized interval neutrosophic rough sets on two universes and an algorithm of multi-attribute decision making based on the generalized model. Furthermore, we use an example to demonstrate the validity of the generalized interval neutrosophic rough set model. The last section summarizes the conclusion and gives an outlook for future research.

## 2. Preliminaries

In this section, we recall some basic notions and propositions which will be used in the paper.

### 2.1. Interval Numbers and their Operations

**Definition 2.1.** ([5, 23, 30, 31]). Let  $\tilde{a} = [a^L, a^U] = \{x | a^L \leq x \leq a^U\}$ , then  $\tilde{a}$  is said to be an interval number. If  $0 \leq a^L \leq x \leq a^U$ , then  $\tilde{a}$  is called a positive interval number.

For any two interval numbers  $\tilde{a} = [a^L, a^U]$  and  $\tilde{b} = [b^L, b^U]$ , the operations between them are given as follows:

- (1)  $\tilde{a} = \tilde{b} \iff a^L = b^L, a^U = b^U$ ;
- (2)  $\tilde{a} + \tilde{b} = [a^L + b^L, a^U + b^U]$ ;
- (3)  $\tilde{a} - \tilde{b} = [a^L - b^U, a^U - b^L]$ .

**Definition 2.2.** ([6]). Let  $L^I = \{[u, v] \in [0, 1] \times [0, 1] | u \leq v\}$  and  $\forall [u_1, v_1], [u_2, v_2] \in L^I, [u_1, v_1] \leq_{L^I} [u_2, v_2] \iff u_1 \leq u_2$  and  $v_1 \leq v_2$ . The tuple  $(L^I, \leq_{L^I})$  is referred to as a complete bounded lattice.

It is obvious that the elements in  $L^I$  are all interval numbers, so we can apply the operations of interval numbers to the elements of  $L^I$ . Thus, the smallest element of  $L^I$  is  $0_{L^I} = [0, 0]$  and the greatest element of  $L^I$  is  $1_{L^I} = [1, 1]$ . Besides, the operators  $\bar{\wedge}$  and  $\bar{\vee}$  on  $(L^I, \leq_{L^I})$  are defined as follows:

$$\begin{aligned} [u_1, v_1] \bar{\wedge} [u_2, v_2] &= [\min\{u_1, u_2\}, \min\{v_1, v_2\}], \\ [u_1, v_1] \bar{\vee} [u_2, v_2] &= [\max\{u_1, u_2\}, \max\{v_1, v_2\}], \end{aligned}$$

for any  $[u_1, v_1], [u_2, v_2] \in L^I$ .

**Definition 2.3.** ([30]). Let  $\tilde{a} = [a^L, a^U]$  and  $\tilde{b} = [b^L, b^U]$  be two interval numbers,  $l_{\tilde{a}} = a^U - a^L$  and  $l_{\tilde{b}} = b^U - b^L$ , then the degree of possibility of  $\tilde{a} \geq_{L^I} \tilde{b}$  is defined as follows:

$$p(\tilde{a} \geq_{L^I} \tilde{b}) = \max\{1 - \max(\frac{b^U - a^L}{l_{\tilde{a}} + l_{\tilde{b}}}, 0), 0\}.$$

### 2.2. Neutrosophic Sets and Interval Neutrosophic Sets

**Definition 2.4.** ([24]). Let  $U$  be a space of points (objects), with a generic element in  $U$  denoted by  $x$ . A NS  $A$  in  $U$  is characterized by a truth-membership function  $T_A$ , an indeterminacy-membership function  $I_A$  and a falsity-membership function  $F_A$ , where  $\forall x \in U, T_A(x), I_A(x)$  and  $F_A(x)$  are real standard or non-standard subsets of  $]0^-, 1^+[$ .

**Definition 2.5.** ([24]). Let  $A$  and  $B$  be two NSs in  $U$ . If  $\forall x \in U, \inf T_A(x) \leq \inf T_B(x), \sup T_A(x) \leq \sup T_B(x), \inf I_A(x) \geq \inf I_B(x), \sup I_A(x) \geq \sup I_B(x), \inf F_A(x) \geq \inf F_B(x), \sup F_A(x) \geq \sup F_B(x)$ , then we say that  $A$  is contained in  $B$ , denoted by  $A \in B$ .

In order to apply NSs conveniently, Wang et al. proposed INSs as follows.

**Definition 2.6.** ([27]). Let  $U$  be a space of points (objects), with a generic element in  $U$  denoted by  $x$ , and  $\text{Int}[0, 1]$  be the set of all closed subintervals of  $[0, 1]$ . An INS  $A$  in  $U$  is characterized by a truth-membership function  $T_A$ , an indeterminacy-membership function  $I_A$  and a falsity-membership function  $F_A$ , where  $\forall x \in U, T_A(x) = [\inf T_A(x), \sup T_A(x)]$ ,  $I_A(x) = [\inf I_A(x), \sup I_A(x)]$  and  $F_A(x) = [\inf F_A(x), \sup F_A(x)] \in \text{Int}[0, 1]$ , and  $0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3$ .

The INS  $A$  can be denoted by  $A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in U \}$  or  $A = (T_A, I_A, F_A)$ .  $\forall x \in U, A(x) = (T_A(x), I_A(x), F_A(x))$ , and  $(T_A(x), I_A(x), F_A(x))$  is called an interval neutrosophic number.

In this paper, the family of all INSs in  $U$  will be denoted by  $\text{INS}(U)$ . Let  $A$  be an INS in  $U$ . If  $\forall x \in U, \inf T_A(x) = \sup T_A(x) = 0, \inf I_A(x) = \sup I_A(x) = 1$  and  $\inf F_A(x) = \sup F_A(x) = 1$ , then we say  $A$  is an empty INS, denoted by  $\emptyset$ . If  $\forall x \in U, \inf T_A(x) = \sup T_A(x) = 1, \inf I_A(x) = \sup I_A(x) = 0$  and  $\inf F_A(x) = \sup F_A(x) = 0$ , then we say  $A$  is a full INS, denoted by  $U$ .  $\forall \alpha, \beta, \gamma \in \text{Int}[0, 1], \widehat{\alpha, \beta, \gamma}$  represents a constant INS satisfying  $T_{\widehat{\alpha, \beta, \gamma}}(x) = \alpha, I_{\widehat{\alpha, \beta, \gamma}}(x) = \beta, F_{\widehat{\alpha, \beta, \gamma}}(x) = \gamma$  for all  $x \in U$ .

**Definition 2.7.** ([27]). Let  $A$  and  $B$  be two INSs in  $U$ . If  $\forall x \in U, T_A(x) \leq_L T_B(x), I_A(x) \geq_L I_B(x),$  and  $F_A(x) \geq_L F_B(x),$  then we say that  $A$  is contained in  $B,$  denoted by  $A \in B.$

**Definition 2.8.** ([27]). Let  $A$  be an INS in  $U.$  The complement of  $A$  is denoted by  $A^c$  and is defined as  $T_{A^c}(x) = F_A(x), I_{A^c}(x) = [1, 1] - I_A(x)$  and  $F_{A^c}(x) = T_A(x).$

For any  $y \in U,$  an INS  $1_y$  and its complement  $1_{U-\{y\}}$  are given as follows:  $\forall x \in U,$

$$T_{1_y}(x) = \begin{cases} [1, 1], & x = y \\ [0, 0], & x \neq y \end{cases}, \quad I_{1_y}(x) = F_{1_y}(x) = \begin{cases} [0, 0], & x = y \\ [1, 1], & x \neq y \end{cases};$$

$$T_{1_{U-\{y\}}}(x) = \begin{cases} [0, 0], & x = y \\ [1, 1], & x \neq y \end{cases}, \quad I_{1_{U-\{y\}}}(x) = F_{1_{U-\{y\}}}(x) = \begin{cases} [1, 1], & x = y \\ [0, 0], & x \neq y \end{cases}.$$

**Definition 2.9.** ([27]). Let  $A$  and  $B$  be two INSs in  $U.$

(1) The union of  $A$  and  $B$  is an INS  $C,$  denoted by  $C = A \cup B,$  where

$$T_C(x) = T_A(x) \vee T_B(x),$$

$$I_C(x) = I_A(x) \bar{\wedge} I_B(x),$$

$$F_C(x) = F_A(x) \bar{\wedge} F_B(x),$$

for all  $x$  in  $U.$

(2) The intersection of  $A$  and  $B$  is an INS  $D,$  denoted by  $D = A \cap B,$  where

$$T_D(x) = T_A(x) \bar{\wedge} T_B(x),$$

$$I_D(x) = I_A(x) \vee I_B(x),$$

$$F_D(x) = F_A(x) \vee F_B(x),$$

for all  $x$  in  $U.$

It is obvious that  $A \cup B$  is the smallest INS which contains both  $A$  and  $B,$  and  $A \cap B$  is the largest INS which is contained in both  $A$  and  $B.$

**Proposition 2.10.** Let  $A$  and  $B$  be two INSs in  $U,$  the following properties can be obtained:

- (1)  $A \in A \cup B$  and  $B \in A \cup B;$
- (2)  $A \cap B \in A$  and  $A \cap B \in B;$
- (3)  $(A^c)^c = A;$
- (4)  $(A \cup B)^c = A^c \cap B^c;$
- (5)  $(A \cap B)^c = A^c \cup B^c.$

*Proof.* The results are straightforward by Definitions 2.7–2.9.  $\square$

### 2.3. Operations for INNs

**Definition 2.11.** ([44]). Let  $A = \langle [\inf T_A, \sup T_A], [\inf I_A, \sup I_A], [\inf F_A, \sup F_A] \rangle$  and  $B = \langle [\inf T_B, \sup T_B], [\inf I_B, \sup I_B], [\inf F_B, \sup F_B] \rangle$  be two INNs. The operations for  $A$  and  $B$  are defined based on the Archimedean  $t$ -norm and  $t$ -conorm as follows:

$$A \oplus B = \langle [l^{-1}(l(\inf T_A) + l(\inf T_B)), l^{-1}(l(\sup T_A) + l(\sup T_B))],$$

$$[k^{-1}(k(\inf I_A) + k(\inf I_B)), k^{-1}(k(\sup I_A) + k(\sup I_B))],$$

$$[k^{-1}(k(\inf F_A) + k(\inf F_B)), k^{-1}(k(\sup F_A) + k(\sup F_B))] \rangle.$$

**Definition 2.12.** ([44]). Let  $A = \langle [\inf T_A, \sup T_A], [\inf I_A, \sup I_A], [\inf F_A, \sup F_A] \rangle$  be an INN. The score function  $s(A),$  accuracy function  $a(A),$  and certainty function  $c(A)$  of the INN  $A$  are defined as follows, respectively:

- (1)  $s(A) = [ \inf T_A + 1 - \sup I_A + 1 - \sup F_A, \sup T_A + 1 - \inf I_A + 1 - \inf F_A ],$
- (2)  $a(A) = [ \min \{ \inf T_A - \inf F_A, \sup T_A - \sup F_A \}, \max \{ \inf T_A - \inf F_A, \sup T_A - \sup F_A \} ],$
- (3)  $c(A) = [ \inf T_A, \sup T_A ].$

**Definition 2.13.** ([44]). Let  $A$  and  $B$  be two INNs. The order between them is defined as follows:

- (1) If  $p(s(A) \geq_{L'} s(B)) > 0.5$ , then  $A$  is greater than  $B$  which means  $A$  is superior to  $B$ , denoted by  $A > B$ .
- (2) If  $p(s(A) \geq_{L'} s(B)) = 0.5$  and  $p(a(A) \geq_{L'} a(B)) > 0.5$ , then  $A$  is greater than  $B$  which means  $A$  is superior to  $B$ , denoted by  $A > B$ .
- (3) If  $p(s(A) \geq_{L'} s(B)) = 0.5$ ,  $p(a(A) \geq_{L'} a(B)) = 0.5$  and  $p(c(A) \geq_{L'} c(B)) > 0.5$ , then  $A$  is greater than  $B$  which means  $A$  is superior to  $B$ , denoted by  $A > B$ .
- (4) If  $p(s(A) \geq_{L'} s(B)) = 0.5$ ,  $p(a(A) \geq_{L'} a(B)) = 0.5$  and  $p(c(A) \geq_{L'} c(B)) = 0.5$ , then  $A$  is equal to  $B$  which means  $A$  and  $B$  are indiscernible, denoted by  $A \sim B$ .

#### 2.4. Pawlak rough sets and single valued neutrosophic rough sets

**Definition 2.14.** ([16, 17]). Let  $R$  be an equivalence relation on a non-empty finite universe  $U$ . Then the pair  $(U, R)$  is referred as to a Pawlak approximation space.  $\forall X \subseteq U$ , the lower and upper approximations of  $X$  w.r.t.  $(U, R)$  are defined as follows:

$$\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\},$$

$$\overline{R}(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\},$$

where  $[x]_R = \{y \in U \mid (x, y) \in R\}$ . The pair  $(\underline{R}(X), \overline{R}(X))$  is called a Pawlak rough set.  $\underline{R}$  and  $\overline{R}$  are called lower and upper approximation operators, respectively.

**Definition 2.15.** ([28]). Let  $U$  be a space of points (objects), with a generic element in  $U$  denoted by  $x$ . A SVN  $A$  in  $U$  is described by three membership functions—a truth-membership function  $T_A$ , an indeterminacy membership function  $I_A$ , and a falsity-membership function  $F_A$ , where  $\forall x \in U, T_A(x), I_A(x), F_A(x) \in [0, 1]$ . The SVN  $A$  can be expressed as  $A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in U\}$  or  $A = (T_A, I_A, F_A)$ .  $\forall x \in U, A(x) = (T_A(x), I_A(x), F_A(x))$ , and  $(T_A(x), I_A(x), F_A(x))$  is referred as to a single valued neutrosophic number.

A SVN  $R$  in  $U \times U$  is referred to as a single valued neutrosophic relation (SVNR) in  $U$ , denoted by  $R = \{\langle (x, y), T_R(x, y), I_R(x, y), F_R(x, y) \rangle \mid (x, y) \in U \times U\}$ , where  $T_R : U \times U \rightarrow [0, 1]$ ,  $I_R : U \times U \rightarrow [0, 1]$ , and  $F_R : U \times U \rightarrow [0, 1]$  represent the truth-membership function, indeterminacy membership function, and falsity-membership function of  $R$ , respectively.

**Definition 2.16.** ([34]). Let  $R$  be a SVNR in  $U$ , the tuple  $(U, R)$  is called a single valued neutrosophic approximation space.  $\forall A \in \text{SVNS}(U)$ , the lower and upper approximations of  $A$  w.r.t.  $(U, R)$ , denoted by  $\underline{R}(A)$  and  $\overline{R}(A)$ , are two SVN  $S$ s whose membership functions are defined as:  $\forall x \in U$ ,

$$T_{\underline{R}(A)}(x) = \bigwedge_{y \in U} (F_R(x, y) \vee T_A(y)),$$

$$I_{\underline{R}(A)}(x) = \bigvee_{y \in U} ((1 - I_R(x, y)) \wedge I_A(y)),$$

$$F_{\underline{R}(A)}(x) = \bigvee_{y \in U} (T_R(x, y) \wedge F_A(y));$$

$$T_{\overline{R}(A)}(x) = \bigvee_{y \in U} (T_R(x, y) \wedge T_A(y)),$$

$$I_{\overline{R}(A)}(x) = \bigwedge_{y \in U} (I_R(x, y) \vee I_A(y)),$$

$$F_{\overline{R}(A)}(x) = \bigwedge_{y \in U} (F_R(x, y) \vee F_A(y)).$$

The pair  $(\underline{R}(A), \overline{R}(A))$  is called a single valued neutrosophic rough set of  $A$  w.r.t.  $(U, R)$ .  $\underline{R}$  and  $\overline{R}$  are referred to as the single valued neutrosophic lower and upper approximation operators, respectively.

### 3. The Constructive Approach of Generalized Interval Neutrosophic Rough Sets Based on Interval Neutrosophic Relations

#### 3.1. The notion of generalized interval neutrosophic rough sets based on interval neutrosophic relations

Broumi and Smarandache [4] put forward interval neutrosophic rough sets in which the based-relations are equivalence relations. Yang et al. [34] proposed single valued neutrosophic rough set model which is a hybrid model of single valued neutrosophic sets and rough sets. In this subsection, we will present interval neutrosophic relations and generalized interval neutrosophic rough sets based on interval neutrosophic relations.

**Definition 3.1.** ([27]). An INS in  $U \times U$  is referred to as an interval neutrosophic relation (INR) in  $U$ , denoted by  $R = \{(x, y), T_R(x, y), I_R(x, y), F_R(x, y)\} \mid (x, y) \in U \times U\}$ , where  $T_R : U \times U \rightarrow \text{Int}[0, 1]$ ,  $I_R : U \times U \rightarrow \text{Int}[0, 1]$ , and  $F_R : U \times U \rightarrow \text{Int}[0, 1]$  represent the truth-membership function, indeterminacy-membership function, and falsity-membership function of  $R$ , respectively.

Let  $R$  be an INR in  $U$ . If  $T_R(x, x) = [1, 1]$  and  $I_R(x, x) = F_R(x, x) = [0, 0]$  for all  $x \in U$ , then we say  $R$  is reflexive. If  $T_R(x, y) = T_R(y, x)$ ,  $I_R(x, y) = I_R(y, x)$  and  $F_R(x, y) = F_R(y, x)$  for all  $x, y \in U$ , then we say  $R$  is symmetric. If  $\bigvee_{y \in U} T_R(x, y) = [1, 1]$  and  $\bigwedge_{y \in U} I_R(x, y) = \bigwedge_{y \in U} F_R(x, y) = [0, 0]$  for all  $x \in U$ , then we say  $R$  is serial. If  $\bigvee_{y \in U} (T_R(x, y) \bar{\wedge} T_R(y, z)) \leq_{L'} T_R(x, z)$ ,  $I_R(x, z) \leq_{L'} \bigwedge_{y \in U} (I_R(x, y) \vee T_R(y, z))$  and  $F_R(x, z) \leq_{L'} \bigwedge_{y \in U} (F_R(x, y) \vee F_R(y, z))$  for all  $x, y, z \in U$ , then we say  $R$  is transitive.

**Definition 3.2.** Let  $R$  be an INR in  $U$ , the tuple  $(U, R)$  is referred to as an interval neutrosophic approximation space.  $\forall A \in \text{INS}(U)$ , the generalized lower and upper approximations of  $A$  w.r.t.  $(U, R)$  are two INSs, denoted by  $\underline{R}(A)$  and  $\overline{R}(A)$ , whose membership functions are defined as follows:  $\forall x \in U$ ,

$$\begin{aligned} T_{\underline{R}(A)}(x) &= \bigwedge_{y \in U} (F_R(x, y) \vee T_A(y)), \\ I_{\underline{R}(A)}(x) &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_A(y)), \\ F_{\underline{R}(A)}(x) &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_A(y)); \\ T_{\overline{R}(A)}(x) &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} T_A(y)), \\ I_{\overline{R}(A)}(x) &= \bigwedge_{y \in U} (I_R(x, y) \vee I_A(y)), \\ F_{\overline{R}(A)}(x) &= \bigwedge_{y \in U} (F_R(x, y) \vee F_A(y)). \end{aligned}$$

The pair  $(\underline{R}(A), \overline{R}(A))$  is called a generalized interval neutrosophic rough set of  $A$  w.r.t.  $(U, R)$ .  $\underline{R}$  and  $\overline{R}$  are called the generalized interval neutrosophic lower and upper approximation operators, respectively.

**Table 1:** The interval neutrosophic relation  $R$ .

$R$	$x_1$	$x_2$
$x_1$	$\langle [0.1, 0.4], [0.2, 0.3], [0.7, 0.9] \rangle$	$\langle [0.2, 0.4], [0.1, 0.2], [0.9, 1] \rangle$
$x_2$	$\langle [0.4, 0.6], [0.3, 0.4], [0.2, 0.4] \rangle$	$\langle [0.8, 0.9], [0.2, 0.3], [0, 0.1] \rangle$
$x_3$	$\langle [0.8, 0.9], [0.1, 0.2], [0.1, 0.3] \rangle$	$\langle [0.7, 0.9], [0.1, 0.3], [0.1, 0.2] \rangle$
$x_4$	$\langle [0.5, 0.8], [0.3, 0.4], [0.2, 0.3] \rangle$	$\langle [0, 0.1], [0, 0.2], [0.8, 1] \rangle$

  

$R$	$x_3$	$x_4$
$x_1$	$\langle [0.8, 1], [0.2, 0.4], [0, 0.1] \rangle$	$\langle [0.1, 0.3], [0.3, 0.4], [0.8, 1] \rangle$
$x_2$	$\langle [0.9, 1], [0.1, 0.3], [0, 0.1] \rangle$	$\langle [0, 0.1], [0.2, 0.3], [0.9, 1] \rangle$
$x_3$	$\langle [0.7, 0.8], [0.4, 0.6], [0.2, 0.3] \rangle$	$\langle [0.9, 1], [0.4, 0.6], [0.1, 0.3] \rangle$
$x_4$	$\langle [0, 0.1], [0.3, 0.4], [0.8, 0.9] \rangle$	$\langle [0, 0.2], [0.1, 0.2], [0.8, 0.9] \rangle$

**Example 3.3.** Let  $U = \{x_1, x_2, x_3, x_4\}$ .  $R \in \text{INS}(U \times U)$  is an INR given in Table 1. Assume an INS

$$A = \{ \langle x_1, [0.5, 0.8], [0.2, 0.4], [0.1, 0.3] \rangle, \langle x_2, [0.7, 0.9], [0.2, 0.4], [0.5, 0.6] \rangle, \langle x_3, [0.1, 0.2], [0.4, 0.6], [0.3, 0.7] \rangle, \langle x_4, [0.2, 0.6], [0.3, 0.5], [0.1, 0.4] \rangle \}.$$

By Definition 3.2, we can obtain the lower and upper approximations of  $A$  w.r.t.  $(U, R)$  as follows:

$$\begin{aligned} \underline{R}(A)(x_1) &= \langle [0.1, 0.2], [0.4, 0.6], [0.3, 0.7] \rangle, \quad \overline{R}(A)(x_1) = \langle [0.2, 0.4], [0.2, 0.4], [0.3, 0.7] \rangle, \\ \underline{R}(A)(x_2) &= \langle [0.1, 0.2], [0.4, 0.6], [0.5, 0.7] \rangle, \quad \overline{R}(A)(x_2) = \langle [0.7, 0.9], [0.2, 0.4], [0.2, 0.4] \rangle, \\ \underline{R}(A)(x_3) &= \langle [0.2, 0.3], [0.4, 0.6], [0.5, 0.7] \rangle, \quad \overline{R}(A)(x_3) = \langle [0.7, 0.9], [0.2, 0.4], [0.1, 0.3] \rangle, \\ \underline{R}(A)(x_4) &= \langle [0.5, 0.8], [0.4, 0.6], [0.1, 0.3] \rangle, \quad \overline{R}(A)(x_4) = \langle [0.5, 0.8], [0.2, 0.4], [0.2, 0.3] \rangle. \end{aligned}$$

**Remark 3.4.** (1) If  $R$  in Definition 3.2 is an equivalence relation, then

$$T_R(x, y) = \begin{cases} [1, 1], & y \in [x]_R \\ [0, 0], & y \notin [x]_R \end{cases}, I_R(x, y) = F_R(x, y) = \begin{cases} [0, 0], & y \in [x]_R \\ [1, 1], & y \notin [x]_R \end{cases}.$$

By Definition 3.2, we have

$$\begin{aligned} T_{\underline{R}(A)}(x) &= \bigwedge_{y \in U} (F_R(x, y) \vee T_A(y)) = \bigwedge_{y \in [x]_R} T_A(y), \\ I_{\underline{R}(A)}(x) &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_A(y)) = \bigvee_{y \in [x]_R} I_A(y), \\ F_{\underline{R}(A)}(x) &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_A(y)) = \bigvee_{y \in [x]_R} F_A(y), \\ T_{\bar{R}(A)}(x) &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} T_A(y)) = \bigvee_{y \in [x]_R} T_A(y), \\ I_{\bar{R}(A)}(x) &= \bigwedge_{y \in U} (I_R(x, y) \vee I_A(y)) = \bigwedge_{y \in [x]_R} I_A(y), \\ F_{\bar{R}(A)}(x) &= \bigwedge_{y \in U} (F_R(x, y) \vee F_A(y)) = \bigwedge_{y \in [x]_R} F_A(y), \end{aligned}$$

which means that the interval neutrosophic rough sets proposed in [4] is a special case of the generalized interval neutrosophic rough sets.

(2) If  $R$  in Definition 3.2 is degenerated to a single interval neutrosophic relation and  $A$  is degenerated to a single valued neutrosophic set, then Definition 3.2 is consistent to the notion of single valued neutrosophic rough sets proposed in [34], which means that the single valued neutrosophic rough sets proposed in [34] is a special case of the generalized interval neutrosophic rough sets.

### 3.2. The properties of generalized interval neutrosophic approximation operators

Next, we explore the properties of generalized interval neutrosophic lower and upper approximation operators.

**Theorem 3.5.** Let  $(U, R)$  be an interval neutrosophic approximation space. The interval neutrosophic lower and upper approximation operators defined in Definition 3.2 have the following properties:  $\forall A, B \in \text{INS}(U), \forall \alpha, \beta, \gamma \in \text{Int}[0, 1]$ ,

- (1)  $\underline{R}(U) = U, \bar{R}(\emptyset) = \emptyset$ ;
- (2) If  $A \subseteq B$ , then  $\underline{R}(A) \subseteq \underline{R}(B)$  and  $\bar{R}(A) \subseteq \bar{R}(B)$ ;
- (3)  $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B), \bar{R}(A \cup B) = \bar{R}(A) \cup \bar{R}(B)$ ;
- (4)  $\underline{R}(A) \cup \underline{R}(B) \subseteq \underline{R}(A \cup B), \bar{R}(A \cap B) \subseteq \bar{R}(A) \cap \bar{R}(B)$ ;
- (5)  $\underline{R}(A^c) = (\bar{R}(A))^c, \bar{R}(A^c) = (\underline{R}(A))^c$ ;
- (6)  $\underline{R}(A \cup \widehat{\alpha, \beta, \gamma}) = \underline{R}(A) \cup \widehat{\alpha, \beta, \gamma}, \bar{R}(A \cap \widehat{\alpha, \beta, \gamma}) = \bar{R}(A) \cap \widehat{\alpha, \beta, \gamma}$ ;
- (7)  $\underline{R}(\emptyset) = \emptyset \iff \underline{R}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}, \bar{R}(U) = U \iff \bar{R}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}$ .

*Proof.* (2) and (4) can be obtained straightforwardly from Definition 3.2. We just need to verify (1), (3) and (5)-(7).

(1) By Definition 3.2, we have  $\forall x \in U$ ,

$$\begin{aligned} T_{\underline{R}(U)}(x) &= \bigwedge_{y \in U} (F_R(x, y) \vee T_U(y)) = \bigwedge_{y \in U} (F_R(x, y) \vee [1, 1]) = [1, 1], \\ I_{\underline{R}(U)}(x) &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_U(y)) = \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} [0, 0]) = [0, 0], \\ F_{\underline{R}(U)}(x) &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_U(y)) = \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} [0, 0]) = [0, 0], \end{aligned}$$

Thus,  $\underline{R}(U) = U$ .

$$T_{\bar{R}(\emptyset)}(x) = \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} T_{\emptyset}(y)) = \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} [0, 0]) = [0, 0],$$

$$I_{\overline{R}(\emptyset)}(x) = \overline{\bigwedge_{y \in U} (I_R(x, y) \underline{\vee} I_{\emptyset}(y))} = \overline{\bigwedge_{y \in U} (I_R(x, y) \underline{\vee} [1, 1])} = [1, 1],$$

$$F_{\overline{R}(\emptyset)}(x) = \overline{\bigwedge_{y \in U} (F_R(x, y) \underline{\vee} F_{\emptyset}(y))} = \overline{\bigwedge_{y \in U} (F_R(x, y) \underline{\vee} [1, 1])} = [1, 1].$$

Consequently,  $\overline{R}(\emptyset) = \emptyset$ .

(3) By Definitions 2.9 and 3.2, we have  $\forall x \in U$ ,

$$\begin{aligned} T_{\underline{R}(A \cap B)}(x) &= \overline{\bigwedge_{y \in U} (F_R(x, y) \underline{\vee} T_{A \cap B}(y))} \\ &= \overline{\bigwedge_{y \in U} (F_R(x, y) \underline{\vee} (T_A(y) \overline{\wedge} T_B(y)))} \\ &= (\overline{\bigwedge_{y \in U} (F_R(x, y) \underline{\vee} (T_A(y))))} \overline{\wedge} (\overline{\bigwedge_{y \in U} (F_R(x, y) \underline{\vee} (T_B(y))))} \\ &= T_{\underline{R}(A)}(x) \overline{\wedge} T_{\underline{R}(B)}(x) \\ &= T_{\underline{R}(A) \cap \underline{R}(B)}(x), \\ I_{\underline{R}(A \cap B)}(x) &= \underline{\bigvee}_{y \in U} (([1, 1] - I_R(x, y)) \overline{\wedge} I_{A \cap B}(y)) \\ &= \underline{\bigvee}_{y \in U} (([1, 1] - I_R(x, y)) \overline{\wedge} (I_A(y) \underline{\vee} I_B(y))) \\ &= (\underline{\bigvee}_{y \in U} (([1, 1] - I_R(x, y)) \overline{\wedge} I_A(y))) \underline{\vee} (\underline{\bigvee}_{y \in U} (([1, 1] - I_R(x, y)) \overline{\wedge} I_B(y))) \\ &= I_{\underline{R}(A)}(x) \underline{\vee} I_{\underline{R}(B)}(x) \\ &= I_{\underline{R}(A) \cap \underline{R}(B)}(x), \\ F_{\underline{R}(A \cap B)}(x) &= \underline{\bigvee}_{y \in U} (T_R(x, y) \overline{\wedge} F_{A \cap B}(y)) \\ &= \underline{\bigvee}_{y \in U} (T_R(x, y) \overline{\wedge} (F_A(y) \underline{\vee} F_B(y))) \\ &= (\underline{\bigvee}_{y \in U} (T_R(x, y) \overline{\wedge} F_A(y))) \underline{\vee} (\underline{\bigvee}_{y \in U} (T_R(x, y) \overline{\wedge} F_B(y))) \\ &= F_{\underline{R}(A)}(x) \underline{\vee} F_{\underline{R}(B)}(x) \\ &= F_{\underline{R}(A) \cap \underline{R}(B)}(x). \end{aligned}$$

Therefore,  $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$ .

Similarly, we can prove that  $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$ .

(5) By Definitions 2.8 and 3.2, we have

$$\begin{aligned} T_{\underline{R}(A^c)}(x) &= \overline{\bigwedge_{y \in U} (F_R(x, y) \underline{\vee} T_{A^c}(y))} \\ &= \overline{\bigwedge_{y \in U} (F_R(x, y) \underline{\vee} F_A(y))} \\ &= F_{\overline{R}(A)}(x) \\ &= T_{(\overline{R}(A))^c}(x), \\ I_{\underline{R}(A^c)}(x) &= \underline{\bigvee}_{y \in U} (([1, 1] - I_R(x, y)) \overline{\wedge} I_{A^c}(y)) \\ &= \underline{\bigvee}_{y \in U} (([1, 1] - I_R(x, y)) \overline{\wedge} ([1, 1] - I_A(y))) \\ &= [1, 1] - \overline{\bigwedge_{y \in U} (I_R(x, y) \underline{\vee} I_A(y))} \\ &= [1, 1] - I_{\overline{R}(A)}(x) \\ &= I_{(\overline{R}(A))^c}(x), \\ F_{\underline{R}(A^c)}(x) &= \underline{\bigvee}_{y \in U} (T_R(x, y) \overline{\wedge} F_{A^c}(y)) \\ &= \underline{\bigvee}_{y \in U} (T_R(x, y) \overline{\wedge} T_A(y)) \\ &= T_{\overline{R}(A)}(x) \\ &= F_{(\overline{R}(A))^c}(x). \end{aligned}$$



$$\begin{aligned}
 (6) \quad T_{\underline{R}(A \uplus \widehat{\alpha, \beta, \gamma})}(x) &= \overline{\bigwedge_{y \in U} (F_R(x, y) \vee T_{A \uplus \widehat{\alpha, \beta, \gamma}}(y))} \\
 &= \overline{\bigwedge_{y \in U} (F_R(x, y) \vee T_A(y) \vee T_{\widehat{\alpha, \beta, \gamma}}(y))} \\
 &= \overline{\bigwedge_{y \in U} (F_R(x, y) \vee T_A(y) \vee \alpha)} \\
 &= \overline{\bigwedge_{y \in U} (F_R(x, y) \vee T_A(y)) \vee \alpha} \\
 &= T_{\underline{R}(A)}(x) \vee T_{\widehat{\alpha, \beta, \gamma}}(x) \\
 &= T_{\underline{R}(A) \uplus \widehat{\alpha, \beta, \gamma}}(x), \\
 I_{\underline{R}(A \uplus \widehat{\alpha, \beta, \gamma})}(x) &= \overline{\bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_{A \uplus \widehat{\alpha, \beta, \gamma}}(y))} \\
 &= \overline{\bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_A(y) \bar{\wedge} I_{\widehat{\alpha, \beta, \gamma}}(y))} \\
 &= \overline{\bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_A(y) \bar{\wedge} \beta)} \\
 &= \overline{\bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_A(y)) \bar{\wedge} \beta} \\
 &= I_{\underline{R}(A)}(x) \bar{\wedge} I_{\widehat{\alpha, \beta, \gamma}}(x) \\
 &= I_{\underline{R}(A) \uplus \widehat{\alpha, \beta, \gamma}}(x), \\
 F_{\underline{R}(A \uplus \widehat{\alpha, \beta, \gamma})}(x) &= \overline{\bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_{A \uplus \widehat{\alpha, \beta, \gamma}}(y))} \\
 &= \overline{\bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_A(y) \bar{\wedge} F_{\widehat{\alpha, \beta, \gamma}}(y))} \\
 &= \overline{\bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_A(y) \bar{\wedge} \gamma)} \\
 &= \overline{\bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_A(y)) \bar{\wedge} \gamma} \\
 &= F_{\underline{R}(A)}(x) \bar{\wedge} F_{\widehat{\alpha, \beta, \gamma}}(x) \\
 &= F_{\underline{R}(A) \uplus \widehat{\alpha, \beta, \gamma}}(x).
 \end{aligned}$$

Similarly, we can prove that  $\overline{R}(A \cap \widehat{\alpha, \beta, \gamma}) = \overline{R}(A) \cap \widehat{\alpha, \beta, \gamma}$ .

(7) On one hand, if  $\underline{R}(\emptyset) = \emptyset$ , then by (6), we have  $\underline{R}(\alpha, \beta, \gamma) = \underline{R}(\emptyset \uplus \widehat{\alpha, \beta, \gamma}) = \underline{R}(\emptyset) \uplus \widehat{\alpha, \beta, \gamma} = \widehat{\alpha, \beta, \gamma}$ . On the other hand, assume  $\underline{R}(\alpha, \beta, \gamma) = \widehat{\alpha, \beta, \gamma}$ , take  $\alpha = [0, 0]$  and  $\beta = \gamma = [1, 1]$ , i.e.  $\widehat{\alpha, \beta, \gamma} = \emptyset$ , then we get  $\underline{R}(\emptyset) = \emptyset$ . So  $\underline{R}(\emptyset) = \emptyset \iff \underline{R}(\alpha, \beta, \gamma) = \widehat{\alpha, \beta, \gamma}$ . Similarly, we can prove that  $\overline{R}(U) = U \iff \overline{R}(\alpha, \beta, \gamma) = \widehat{\alpha, \beta, \gamma}$ .  $\square$

**Theorem 3.6.** Let  $R_1$  and  $R_2$  be two INRs in  $U$ .  $\forall A \in \text{INS}(U)$ , we have

- (1)  $\underline{R}_1 \uplus \underline{R}_2(A) = \underline{R}_1(A) \cap \underline{R}_2(A)$ ;
- (2)  $\overline{R}_1 \uplus \overline{R}_2(A) = \overline{R}_1(A) \cup \overline{R}_2(A)$ .

*Proof.* (1) According to Definitions 2.9 and 3.2,  $\forall x \in U$ ,

$$\begin{aligned}
 T_{\underline{R}_1 \uplus \underline{R}_2(A)}(x) &= \overline{\bigwedge_{y \in U} (F_{R_1 \uplus R_2}(x, y) \vee T_A(y))} \\
 &= \overline{\bigwedge_{y \in U} ((F_{R_1}(x, y) \bar{\wedge} F_{R_2}(x, y)) \vee T_A(y))} \\
 &= \overline{\bigwedge_{y \in U} ((F_{R_1}(x, y) \vee T_A(y)) \bar{\wedge} (F_{R_2}(x, y) \vee T_A(y)))} \\
 &= (\overline{\bigwedge_{y \in U} (F_{R_1}(x, y) \vee T_A(y))}) \bar{\wedge} (\overline{\bigwedge_{y \in U} (F_{R_2}(x, y) \vee T_A(y))}) \\
 &= T_{\underline{R}_1(A)}(x) \bar{\wedge} T_{\underline{R}_2(A)}(x) \\
 &= T_{\underline{R}_1(A) \cap \underline{R}_2(A)}(x), \\
 I_{\underline{R}_1 \uplus \underline{R}_2(A)}(x) &= \overline{\bigvee_{y \in U} (([1, 1] - I_{R_1 \uplus R_2}(x, y)) \bar{\wedge} I_A(y))}
 \end{aligned}$$

$$\begin{aligned}
 &= \bigvee_{y \in U} (([1, 1] - (I_{R_1}(x, y) \bar{\wedge} I_{R_2}(x, y))) \bar{\wedge} I_A(y)) \\
 &= \bigvee_{y \in U} ((([1, 1] - I_{R_1}(x, y)) \vee ([1, 1] - I_{R_2}(x, y))) \bar{\wedge} I_A(y)) \\
 &= (\bigvee_{y \in U} (([1, 1] - I_{R_1}(x, y)) \bar{\wedge} I_A(y))) \vee (\bigvee_{y \in U} (([1, 1] - I_{R_2}(x, y)) \bar{\wedge} I_A(y))) \\
 &= I_{\underline{R_1}(A)}(x) \vee I_{\underline{R_2}(A)}(x) \\
 &= I_{\underline{R_1}(A) \bar{\cap} \underline{R_2}(A)}(x), \\
 F_{\underline{R_1} \bar{\cup} \underline{R_2}(A)}(x) &= \bigvee_{y \in U} (T_{R_1 \bar{\cup} R_2}(x, y) \bar{\wedge} F_A(y)) \\
 &= \bigvee_{y \in U} ((T_{R_1}(x, y) \vee T_{R_2}(x, y)) \bar{\wedge} F_A(y)) \\
 &= \bigvee_{y \in U} ((T_{R_1}(x, y) \bar{\wedge} F_A(y)) \vee (T_{R_2}(x, y) \bar{\wedge} F_A(y))) \\
 &= (\bigvee_{y \in U} (T_{R_1}(x, y) \bar{\wedge} F_A(y))) \vee (\bigvee_{y \in U} (T_{R_2}(x, y) \bar{\wedge} F_A(y))) \\
 &= F_{\underline{R_1}(A)}(x) \vee F_{\underline{R_2}(A)}(x) \\
 &= F_{\underline{R_1}(A) \bar{\cap} \underline{R_2}(A)}(x).
 \end{aligned}$$

Consequently,  $\underline{R_1} \bar{\cup} \underline{R_2}(A) = \underline{R_1}(A) \bar{\cap} \underline{R_2}(A)$ .

(2) According to Proposition 2.10 (5) and Theorem 3.5 (5), we have

$$\begin{aligned}
 \overline{R_1} \bar{\cup} \overline{R_2}(A) &= (R_1 \bar{\cup} R_2(A^c))^c \\
 &= (\overline{R_1}(A^c) \bar{\cap} \overline{R_2}(A^c))^c \\
 &= (\overline{R_1}(A^c))^c \bar{\cup} (\overline{R_2}(A^c))^c \\
 &= \overline{R_1}(A) \bar{\cup} \overline{R_2}(A). \quad \square
 \end{aligned}$$

**Theorem 3.7.** Let  $R_1$  and  $R_2$  be two INRs in  $U$ .  $\forall A \in \text{INS}(U)$ , we have

- (1)  $\underline{R_1}(A) \bar{\cap} \underline{R_2}(A) \subseteq \underline{R_1}(A) \bar{\cup} \underline{R_2}(A) \subseteq \underline{R_1} \bar{\cap} \underline{R_2}(A)$ ;
- (2)  $\overline{R_1} \bar{\cap} \overline{R_2}(A) \subseteq \overline{R_1}(A) \bar{\cap} \overline{R_2}(A) \subseteq \overline{R_1}(A) \bar{\cup} \overline{R_2}(A)$ .

*Proof.* (1) According to Definition 3.2,  $\forall x \in U$ ,

$$\begin{aligned}
 T_{\underline{R_1} \bar{\cap} \underline{R_2}(A)}(x) &= \bigwedge_{y \in U} (F_{R_1 \bar{\cap} R_2}(x, y) \vee T_A(y)) \\
 &= \bigwedge_{y \in U} ((F_{R_1}(x, y) \vee F_{R_2}(x, y)) \vee T_A(y)) \\
 &= \bigwedge_{y \in U} ((F_{R_1}(x, y) \vee T_A(y)) \vee (F_{R_2}(x, y) \vee T_A(y))) \\
 &\geq_{L'} (\bigwedge_{y \in U} (F_{R_1}(x, y) \vee T_A(y))) \vee (\bigwedge_{y \in U} (F_{R_2}(x, y) \vee T_A(y))) \\
 &= T_{\underline{R_1}(A)}(x) \vee T_{\underline{R_2}(A)}(x) \\
 &= T_{\underline{R_1}(A) \bar{\cup} \underline{R_2}(A)}(x), \\
 I_{\underline{R_1} \bar{\cap} \underline{R_2}(A)}(x) &= \bigvee_{y \in U} (([1, 1] - I_{R_1 \bar{\cap} R_2}(x, y)) \bar{\wedge} I_A(y)) \\
 &= \bigvee_{y \in U} (([1, 1] - (I_{R_1}(x, y) \vee I_{R_2}(x, y))) \bar{\wedge} I_A(y)) \\
 &= \bigvee_{y \in U} (([1, 1] - I_{R_1}(x, y)) \bar{\wedge} ([1, 1] - I_{R_2}(x, y)) \bar{\wedge} I_A(y)) \\
 &= \bigvee_{y \in U} ((([1, 1] - I_{R_1}(x, y)) \bar{\wedge} I_A(y)) \bar{\wedge} (([1, 1] - I_{R_2}(x, y)) \bar{\wedge} I_A(y))) \\
 &\leq_{L'} (\bigvee_{y \in U} (([1, 1] - I_{R_1}(x, y)) \bar{\wedge} I_A(y))) \bar{\wedge} (\bigvee_{y \in U} (([1, 1] - I_{R_2}(x, y)) \bar{\wedge} I_A(y))) \\
 &= I_{\underline{R_1}(A)}(x) \bar{\wedge} I_{\underline{R_2}(A)}(x) \\
 &= I_{\underline{R_1}(A) \bar{\cup} \underline{R_2}(A)}(x),
 \end{aligned}$$

$$\begin{aligned}
 F_{\underline{R}_1 \cap \underline{R}_2(A)}(x) &= \bigvee_{y \in U} (T_{\underline{R}_1 \cap \underline{R}_2}(x, y) \bar{\wedge} F_A(y)) \\
 &= \bigvee_{y \in U} ((T_{\underline{R}_1}(x, y) \bar{\wedge} T_{\underline{R}_2}(x, y)) \bar{\wedge} F_A(y)) \\
 &= \bigvee_{y \in U} ((T_{\underline{R}_1}(x, y) \bar{\wedge} F_A(y)) \bar{\wedge} (T_{\underline{R}_2}(x, y) \bar{\wedge} F_A(y))) \\
 &\leq_{L'} (\bigvee_{y \in U} (T_{\underline{R}_1}(x, y) \bar{\wedge} F_A(y))) \bar{\wedge} (\bigvee_{y \in U} (T_{\underline{R}_2}(x, y) \bar{\wedge} F_A(y))) \\
 &= F_{\underline{R}_1(A)}(x) \bar{\wedge} F_{\underline{R}_2(A)}(x) \\
 &= F_{\underline{R}_1(A) \cup \underline{R}_2(A)}(x).
 \end{aligned}$$

It is obvious that  $\underline{R}_1(A) \cap \underline{R}_2(A) \subseteq \underline{R}_1(A) \cup \underline{R}_2(A)$ . Hence, we get that  $\underline{R}_1(A) \cap \underline{R}_2(A) \subseteq \underline{R}_1(A) \cup \underline{R}_2(A) \subseteq \underline{R}_1 \cap \underline{R}_2(A)$ .

(2) According to (1) and Theorem 3.5 (5), we have

$$\begin{aligned}
 \overline{R_1 \cap R_2(A)} &= (R_1 \cap R_2(A^c))^c \\
 &\subseteq (\overline{R_1(A^c)} \cup \overline{R_2(A^c)})^c \\
 &= (\overline{R_1(A^c)})^c \cap (\overline{R_2(A^c)})^c \\
 &= \overline{R_1(A)} \cap \overline{R_2(A)}.
 \end{aligned}$$

Consequently,  $R_1 \cap R_2(A) \subseteq R_1(A) \cap R_2(A) \subseteq \overline{R_1(A)} \cup \overline{R_2(A)}$ .  $\square$

**Remark 3.8.** Let  $R_1$  and  $R_2$  be two INRs in  $U$ .  $\forall A \in \text{INS}(U)$ . If  $R_1 \in R_2$ , then  $\underline{R}_2(A) \subseteq \underline{R}_1(A)$  and  $\overline{R_1(A)} \subseteq \overline{R_2(A)}$ .

Next, we study the relationships between special INRs and generalized interval neutrosophic approximation operators.

**Theorem 3.9.** Let  $(U, R)$  be an interval neutrosophic approximation space.  $\underline{R}$  and  $\overline{R}$  are the lower and upper approximation operators defined in Definition 3.2, then we have the following results:

- (1)  $R$  is serial  $\iff \underline{R}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}, \forall \alpha, \beta, \gamma \in \text{Int}[0, 1]$ ,  
 $\iff \underline{R}(\emptyset) = \emptyset$ ,  
 $\iff \overline{R}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}, \forall \alpha, \beta, \gamma \in \text{Int}[0, 1]$ ,  
 $\iff \overline{R}(U) = U$ ;
- (2)  $R$  is reflexive  $\iff \underline{R}(A) \subseteq A, \forall A \in \text{INS}(U)$ ,  
 $\iff A \subseteq \overline{R}(A), \forall A \in \text{INS}(U)$ ;
- (3)  $R$  is symmetric  $\iff \underline{R}(1_{U-\{x\}})(y) = \underline{R}(1_{U-\{y\}})(x), \forall x, y \in U$ ,  
 $\iff \overline{R}(1_x)(y) = \overline{R}(1_y)(x), \forall x, y \in U$ ;
- (4)  $R$  is transitive  $\iff \underline{R}(A) \subseteq \underline{R}(\underline{R}(A)), \forall A \in \text{INS}(U)$ ,  
 $\iff \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A), \forall A \in \text{INS}(U)$ .

*Proof.* According to Theorem 3.5 (5), we can see that  $\underline{R}$  and  $\overline{R}$  are a pair of dual operators. Thus, we need only to consider the properties of the lower approximation operator.

(1) By Theorem 3.5 (7), it suffices to verify that

$$R \text{ is serial} \iff \underline{R}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}, \forall \alpha, \beta, \gamma \in \text{Int}[0, 1].$$

$$" \implies " \text{ If } R \text{ is serial, then for any } x \in U, \bigvee_{y \in U} T_R(x, y) = [1, 1] \text{ and } \bigwedge_{y \in U} I_R(x, y) = \bigwedge_{y \in U} F_R(x, y) = [0, 0].$$

$\forall \alpha, \beta, \gamma \in \text{Int}[0, 1], \forall x \in U$ , by Definition 3.2,

$$\begin{aligned}
 T_{\underline{R}(\widehat{\alpha, \beta, \gamma})}(x) &= \bigwedge_{y \in U} (F_R(x, y) \vee T_{\widehat{\alpha, \beta, \gamma}}(y)) \\
 &= \bigwedge_{y \in U} (F_R(x, y) \vee \alpha) \\
 &= \bigwedge_{y \in U} F_R(x, y) \vee \alpha \\
 &= [0, 0] \vee \alpha \\
 &= \alpha,
 \end{aligned}$$

$$\begin{aligned} I_{\underline{R}(\widehat{\alpha, \beta, \gamma})}(x) &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_{\widehat{\alpha, \beta, \gamma}}(y)) \\ &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} \beta) \\ &= \bigvee_{y \in U} (([1, 1] - I_R(x, y))) \bar{\wedge} \beta \\ &= ([1, 1] - \bar{\bigwedge}_{y \in U} I_R(x, y)) \bar{\wedge} \beta \\ &= [1, 1] \bar{\wedge} \beta \\ &= \beta, \end{aligned}$$

$$\begin{aligned} F_{\underline{R}(\widehat{\alpha, \beta, \gamma})}(x) &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_{\widehat{\alpha, \beta, \gamma}}(y)) \\ &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} \gamma) \\ &= \bigvee_{y \in U} T_R(x, y) \bar{\wedge} \gamma \\ &= [1, 1] \bar{\wedge} \gamma \\ &= \gamma. \end{aligned}$$

Therefore,  $\underline{R}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}$  for any  $\alpha, \beta, \gamma \in \text{Int}[0, 1]$ .

“  $\Leftarrow$  ” If  $\underline{R}(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}$  for any  $\alpha, \beta, \gamma \in \text{Int}[0, 1]$ . Take  $\alpha = [0, 0], \beta = \gamma = [1, 1]$ , then we have

$$\begin{aligned} \bigvee_{y \in U} T_R(x, y) &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} [1, 1]) \\ &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_\emptyset(y)) \\ &= F_{\underline{R}(\emptyset)}(x) \\ &= [1, 1], \end{aligned}$$

$$\begin{aligned} [1, 1] - \bar{\bigwedge}_{y \in U} I_R(x, y) &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} [1, 1]) \\ &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_\emptyset(y)) \\ &= I_{\underline{R}(\emptyset)}(x) \\ &= [1, 1], \end{aligned}$$

which implies that  $\bar{\bigwedge}_{y \in U} I_R(x, y) = [0, 0]$ .

$$\begin{aligned} \bar{\bigwedge}_{y \in U} F_R(x, y) &= \bar{\bigwedge}_{y \in U} (F_R(x, y) \vee [0, 0]) \\ &= \bar{\bigwedge}_{y \in U} (F_R(x, y) \vee T_\emptyset(y)) \\ &= T_{\underline{R}(\emptyset)}(x) \\ &= [0, 0]. \end{aligned}$$

Thus,  $R$  is serial.

(2) “  $\implies$  ” If  $R$  is reflexive, then  $T_R(x, x) = [1, 1]$  and  $I_R(x, x) = F_R(x, x) = [0, 0]$  hold for any  $x \in U$ . By Definition 3.2,  $\forall A \in \text{INS}(U), \forall x \in U$ ,

$$\begin{aligned} T_{\underline{R}(A)}(x) &= \bar{\bigwedge}_{y \in U} (F_R(x, y) \vee T_A(y)) \\ &\leq_{L'} F_R(x, x) \vee T_A(x) \\ &= [0, 0] \vee T_A(x) \\ &= T_A(x), \\ I_{\underline{R}(A)}(x) &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_A(y)) \\ &\geq_{L'} ([1, 1] - I_R(x, x)) \bar{\wedge} I_A(x) \\ &= ([1, 1] - [0, 0]) \bar{\wedge} I_A(x) \\ &= I_A(x), \end{aligned}$$

$$\begin{aligned} F_{\underline{R}(A)}(x) &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_A(y)) \\ &\geq_{L'} T_R(x, x) \bar{\wedge} F_A(x) \\ &= [1, 1] \bar{\wedge} F_A(x) \\ &= F_A(x). \end{aligned}$$

Therefore,  $\underline{R}(A) \in A$ .

“  $\Leftarrow$  ” If  $\underline{R}(A) \in A$  for any  $A \in \text{INS}(U)$ , then  $\forall x \in U$ , by taking  $A = 1_{U-\{x\}}$ , we have

$$\begin{aligned} T_R(x, x) &= (T_R(x, x) \bar{\wedge} [1, 1]) \underline{\vee} [0, 0] \\ &= (T_R(x, x) \bar{\wedge} F_{1_{U-\{x\}}}(x)) \underline{\vee} (\bigvee_{y \in U-\{x\}} (T_R(x, y) \bar{\wedge} F_{1_{U-\{x\}}}(y))) \\ &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_{1_{U-\{x\}}}(y)) \\ &= F_{\underline{R}(1_{U-\{x\}})}(x) \\ &\geq_{L'} F_{1_{U-\{x\}}}(x) \\ &= [1, 1], \end{aligned}$$

$$\begin{aligned} [1, 1] - I_R(x, x) &= (([1, 1] - I_R(x, x)) \bar{\wedge} [1, 1]) \underline{\vee} [0, 0] \\ &= (([1, 1] - I_R(x, x)) \bar{\wedge} I_{U-\{x\}}(x)) \underline{\vee} (\bigvee_{y \in U-\{x\}} (([1, 1] - I_R(x, y)) \bar{\wedge} I_{U-\{x\}}(y))) \\ &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_{U-\{x\}}(y)) \\ &= I_{\underline{R}(1_{U-\{x\}})}(x) \\ &\geq_{L'} I_{1_{U-\{x\}}}(x) \\ &= [1, 1], \end{aligned}$$

which implies that  $I_R(x, x) = [0, 0]$ ,

$$\begin{aligned} F_R(x, x) &= (F_R(x, x) \underline{\vee} [0, 0]) \bar{\wedge} [1, 1] \\ &= (F_R(x, x) \underline{\vee} T_{1_{U-\{x\}}}(x)) \bar{\wedge} (\bar{\bigwedge}_{y \in U-\{x\}} (F_R(x, y) \underline{\vee} T_{1_{U-\{x\}}}(y))) \\ &= \bar{\bigwedge}_{y \in U} (F_R(x, y) \underline{\vee} T_{1_{U-\{x\}}}(y)) \\ &= T_{\underline{R}(1_{U-\{x\}})}(x) \\ &\leq_{L'} T_{1_{U-\{x\}}}(x) \\ &= [0, 0]. \end{aligned}$$

Thus,  $R$  is reflexive.

Consequently,  $R$  is reflexive  $\iff \underline{R}(A) \in A, \forall A \in \text{INS}(U)$ .

(3) According to Definition 3.2,  $\forall x, y \in U$ ,

$$\begin{aligned} T_{\underline{R}(1_{U-\{x\}})}(y) &= \bar{\bigwedge}_{z \in U} (F_R(y, z) \underline{\vee} T_{1_{U-\{x\}}}(z)) \\ &= (F_R(y, x) \underline{\vee} T_{1_{U-\{x\}}}(x)) \bar{\wedge} (\bar{\bigwedge}_{z \in U-\{x\}} (F_R(y, z) \underline{\vee} T_{1_{U-\{x\}}}(z))) \\ &= (F_R(y, x) \underline{\vee} [0, 0]) \bar{\wedge} [1, 1] \\ &= F_R(y, x), \end{aligned}$$

$$\begin{aligned} T_{\underline{R}(1_{U-\{y\}})}(x) &= \bar{\bigwedge}_{z \in U} (F_R(x, z) \underline{\vee} T_{1_{U-\{y\}}}(z)) \\ &= (F_R(x, y) \underline{\vee} T_{1_{U-\{y\}}}(y)) \bar{\wedge} (\bar{\bigwedge}_{z \in U-\{y\}} (F_R(x, z) \underline{\vee} T_{1_{U-\{y\}}}(z))) \\ &= (F_R(x, y) \underline{\vee} [0, 0]) \bar{\wedge} [1, 1] \\ &= F_R(x, y), \end{aligned}$$

$$\begin{aligned} I_{\underline{R}(1_{U-\{x\}})}(y) &= \bigvee_{z \in U} (([1, 1] - I_R(y, z)) \bar{\wedge} I_{U-\{x\}}(z)) \\ &= (([1, 1] - I_R(y, x)) \bar{\wedge} I_{U-\{x\}}(x)) \underline{\vee} (\bigvee_{z \in U-\{x\}} (([1, 1] - I_R(y, z)) \bar{\wedge} I_{U-\{x\}}(z))) \\ &= (([1, 1] - I_R(y, x)) \bar{\wedge} [1, 1]) \underline{\vee} [0, 0] \\ &= [1, 1] - I_R(y, x), \end{aligned}$$

$$\begin{aligned}
 I_{\underline{R}(1_{U-\{y\}})}(x) &= \bigvee_{z \in U} (([1, 1] - I_R(x, z)) \bar{\wedge} I_{1_{U-\{y\}}}(z)) \\
 &= (([1, 1] - I_R(x, y)) \bar{\wedge} I_{1_{U-\{y\}}}(y)) \vee \left( \bigvee_{z \in U-\{y\}} (([1, 1] - I_R(x, z)) \bar{\wedge} I_{1_{U-\{y\}}}(z)) \right) \\
 &= (([1, 1] - I_R(x, y)) \bar{\wedge} [1, 1]) \vee [0, 0] \\
 &= [1, 1] - I_R(x, y), \\
 F_{\underline{R}(1_{U-\{x\}})}(y) &= \bigvee_{z \in U} (T_R(y, z) \bar{\wedge} F_{1_{U-\{x\}}}(z)) \\
 &= (T_R(y, x) \bar{\wedge} F_{1_{U-\{x\}}}(x)) \vee \left( \bigvee_{z \in U-\{x\}} (T_R(y, z) \bar{\wedge} F_{1_{U-\{x\}}}(z)) \right) \\
 &= (T_R(y, x) \bar{\wedge} [1, 1]) \vee [0, 0] \\
 &= T_R(y, x), \\
 F_{\underline{R}(1_{U-\{y\}})}(x) &= \bigvee_{z \in U} (T_R(x, z) \bar{\wedge} F_{1_{U-\{y\}}}(z)) \\
 &= (T_R(x, y) \bar{\wedge} F_{1_{U-\{y\}}}(y)) \bar{\wedge} \left( \bigvee_{z \in U-\{y\}} (T_R(x, z) \vee F_{1_{U-\{y\}}}(z)) \right) \\
 &= (T_R(x, y) \bar{\wedge} [1, 1]) \vee [0, 0] \\
 &= T_R(x, y),
 \end{aligned}$$

Since  $R$  is symmetric iff  $\forall x, y \in U, T_R(x, y) = T_R(y, x), I_R(x, y) = I_R(y, x)$  and  $F_R(x, y) = F_R(y, x)$ ,  $R$  is symmetric iff  $\forall x, y \in U, T_{\underline{R}(1_{U-\{x\}})}(y) = T_{\underline{R}(1_{U-\{y\}})}(x), I_{\underline{R}(1_{U-\{x\}})}(y) = I_{\underline{R}(1_{U-\{y\}})}(x)$ , and  $F_{\underline{R}(1_{U-\{x\}})}(y) = F_{\underline{R}(1_{U-\{y\}})}(x)$ , which means that  $R$  is symmetric iff  $\forall x, y \in U, \underline{R}(1_{U-\{x\}})(y) = \underline{R}(1_{U-\{y\}})(x)$ .

(4) “ $\implies$ ” If  $R$  is transitive, then  $\bigvee_{y \in U} (T_R(x, y) \bar{\wedge} T_R(y, z)) \leq_{L'} T_R(x, z), I_R(x, z) \leq_{L'} \overline{\bigwedge_{y \in U} (I_R(x, y) \vee I_R(y, z))}$  and  $F_R(x, z) \leq_{L'} \overline{\bigwedge_{y \in U} (F_R(x, y) \vee F_R(y, z))}$  for all  $x, y, z \in U$ . According to Definition 3.2,  $\forall x \in U$ , we have

$$\begin{aligned}
 T_{\underline{R}(\underline{R}(A))}(x) &= \overline{\bigwedge_{y \in U} (F_R(x, y) \vee T_{\underline{R}(A)}(y))} \\
 &= \overline{\bigwedge_{y \in U} (F_R(x, y) \vee (\overline{\bigwedge_{z \in U} (F_R(y, z) \vee T_A(z))})} \\
 &= \overline{\bigwedge_{z \in U} \overline{\bigwedge_{y \in U} (F_R(x, y) \vee F_R(y, z) \vee T_A(z))} \\
 &= \overline{\bigwedge_{z \in U} (\overline{\bigwedge_{y \in U} (F_R(x, y) \vee F_R(y, z) \vee T_A(z))} \\
 &\geq_{L'} \overline{\bigwedge_{z \in U} (F_R(x, z) \vee T_A(z))} \\
 &= T_{\underline{R}(A)}(x), \\
 I_{\underline{R}(\underline{R}(A))}(x) &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_{\underline{R}(A)}(y)) \\
 &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} (\bigvee_{z \in U} (([1, 1] - I_R(y, z)) \bar{\wedge} I_A(z)))) \\
 &= \bigvee_{z \in U} \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} ([1, 1] - I_R(y, z)) \bar{\wedge} I_A(z)) \\
 &= \bigvee_{z \in U} (([1, 1] - \overline{\bigwedge_{y \in U} (I_R(x, y) \vee I_R(y, z))} \bar{\wedge} I_A(z)) \\
 &\leq_{L'} \bigvee_{z \in U} (([1, 1] - I_R(x, z)) \bar{\wedge} I_A(z)) \\
 &= I_{\underline{R}(A)}(x), \\
 F_{\underline{R}(\underline{R}(A))}(x) &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_{\underline{R}(A)}(y)) \\
 &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} (\bigvee_{z \in U} (T_R(y, z) \bar{\wedge} F_A(z)))) \\
 &= \bigvee_{z \in U} \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} T_R(y, z) \bar{\wedge} F_A(z)) \\
 &= \bigvee_{z \in U} (\bigvee_{y \in U} (T_R(x, y) \bar{\wedge} T_R(y, z)) \bar{\wedge} F_A(z))
 \end{aligned}$$

$$\begin{aligned} &\leq_{L'} \bigvee_{z \in U} (T_R(x, z) \bar{\wedge} F_A(z)) \\ &= F_{\underline{R}(A)}(x). \end{aligned}$$

Therefore,  $\underline{R}(A) \in \underline{R}(\underline{R}(A))$ .

“ $\Leftarrow$ ” Assume  $\underline{R}(A) \in \underline{R}(\underline{R}(A))$  for all  $A \in \text{INS}(U)$ .  $\forall x, y, z \in U$ , let  $A = 1_{U-\{z\}}$ , from the proving process of (3), we have

$$\begin{aligned} T_R(x, z) &= F_{\underline{R}(1_{U-\{z\}})}(x) \\ &\geq_{L'} \underline{F}_{\underline{R}(\underline{R}(1_{U-\{z\}}))}(x) \\ &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_{\underline{R}(1_{U-\{z\}})}(y)) \\ &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} T_R(y, z)), \\ [1, 1] - I_R(x, z) &= I_{\underline{R}(1_{U-\{z\}})}(x) \\ &\geq_{L'} \underline{I}_{\underline{R}(\underline{R}(1_{U-\{z\}}))}(x) \\ &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} I_{\underline{R}(1_{U-\{z\}})}(y)) \\ &= \bigvee_{y \in U} (([1, 1] - I_R(x, y)) \bar{\wedge} ([1, 1] - I_R(y, z))) \\ &= [1, 1] - \bar{\bigwedge}_{y \in U} (I_R(x, y) \vee I_R(y, z)), \end{aligned}$$

hence,  $I_R(x, z) \leq_{L'} \bar{\bigwedge}_{y \in U} (I_R(x, y) \vee I_R(y, z))$ ,

$$\begin{aligned} F_R(x, z) &= T_{\underline{R}(1_{U-\{z\}})}(x) \\ &\leq_{L'} \underline{T}_{\underline{R}(\underline{R}(1_{U-\{z\}}))}(x) \\ &= \bar{\bigwedge}_{y \in U} (F_R(x, y) \vee T_{\underline{R}(1_{U-\{z\}})}(y)) \\ &= \bar{\bigwedge}_{y \in U} (F_R(x, y) \vee F_R(y, z)). \end{aligned}$$

Therefore,  $R$  is transitive.  $\square$

#### 4. Axiomatic Characterizations of Generalized Interval Neutrosophic Approximation Operators

In this section, we will study the axiomatic characterizations of generalized interval neutrosophic lower and upper approximation operators by restricting a pair of abstract theoretical interval neutrosophic set operators.

**Theorem 4.1.** Let  $L: \text{INS}(U) \longrightarrow \text{INS}(U)$  be an interval neutrosophic set operator. Then, there exists an INR  $R$  in  $U$  such that  $L(A) = \underline{R}(A)$  for all  $A \in \text{INS}(U)$  iff  $L$  satisfies the following axioms (INSL1) and (INSL2) :  $\forall A, B \in \text{INS}(U)$ ,  $\alpha, \beta, \gamma \in \text{Int}[0, 1]$ ,

$$\begin{aligned} (\text{INSL1}) \quad &L(A \uplus \widehat{\alpha, \beta, \gamma}) = L(A) \uplus \widehat{\alpha, \beta, \gamma}; \\ (\text{INSL2}) \quad &L(A \bar{\cap} B) = L(A) \bar{\cap} L(B). \end{aligned}$$

*Proof.* “ $\implies$ ” It is straightforward from Theorem 3.5.

“ $\Leftarrow$ ” Suppose  $L$  satisfies axioms (INSL1) and (INSL2). By using  $L$ , we define an INR  $R = \{(x, y), T_R(x, y), I_R(x, y), F_R(x, y)\}$  as follows:

$$\begin{aligned} \forall x, y \in U, \quad &T_R(x, y) = F_{L(1_{U-\{y\}})}(x), \quad I_R(x, y) = [1, 1] - I_{L(1_{U-\{y\}})}(x), \\ &F_R(x, y) = T_{L(1_{U-\{y\}})}(x). \end{aligned}$$

Moreover, we can obtain that for all  $A \in \text{INS}(U)$ ,

$$A = \bar{\cap}_{y \in U} (1_{U-\{y\}} \uplus \widehat{A(y)}), \text{ where } A(x) = \langle T_A(x), I_A(x), F_A(x) \rangle.$$

In fact, for all  $x \in U$ , we have

$$\begin{aligned} T_{\bar{\cap}_{y \in U} (1_{U-\{y\}} \uplus \widehat{A(y)})}(x) &= \bar{\bigwedge}_{y \in U} T_{(1_{U-\{y\}} \uplus \widehat{A(y)})}(x) \\ &= \bar{\bigwedge}_{y \in U} (T_{1_{U-\{y\}}}(x) \vee T_{\widehat{A(y)}}(x)) \end{aligned}$$

$$\begin{aligned}
 &= T_{1_{U-\{x\}}}(x) \vee T_{\widehat{A(x)}}(x) \bar{\wedge} \bigwedge_{y \in U-\{x\}} (T_{1_{U-\{y\}}}(x) \vee T_{\widehat{A(y)}}(x)) \\
 &= T_A(x) \bar{\wedge} [1, 1] \\
 &= T_A(x), \\
 I_{\bigcap_{y \in U} (1_{U-\{y\}} \uplus \widehat{A(y)})}(x) &= \bigvee_{y \in U} I_{(1_{U-\{y\}} \uplus \widehat{A(y)})}(x) \\
 &= \bigvee_{y \in U} (I_{1_{U-\{y\}}}(x) \bar{\wedge} I_{\widehat{A(y)}}(x)) \\
 &= I_{1_{U-\{x\}}}(x) \bar{\wedge} I_{\widehat{A(x)}}(x) \vee \bigvee_{y \in U-\{x\}} (I_{1_{U-\{y\}}}(x) \bar{\wedge} I_{\widehat{A(y)}}(x)) \\
 &= I_A(x) \vee [0, 0] \\
 &= I_A(x),
 \end{aligned}$$

and

$$\begin{aligned}
 F_{\bigcap_{y \in U} (1_{U-\{y\}} \uplus \widehat{A(y)})}(x) &= \bigvee_{y \in U} F_{(1_{U-\{y\}} \uplus \widehat{A(y)})}(x) \\
 &= \bigvee_{y \in U} (F_{1_{U-\{y\}}}(x) \bar{\wedge} F_{\widehat{A(y)}}(x)) \\
 &= F_{1_{U-\{x\}}}(x) \bar{\wedge} F_{\widehat{A(x)}}(x) \vee \bigvee_{y \in U-\{x\}} (F_{1_{U-\{y\}}}(x) \bar{\wedge} F_{\widehat{A(y)}}(x)) \\
 &= F_A(x) \vee [0, 0] \\
 &= F_A(x),
 \end{aligned}$$

So,  $A = \bigcap_{y \in U} (1_{U-\{y\}} \uplus \widehat{A(y)})$ .

By Definition 3.2, (INSL1) and (INSL2), we have

$$\begin{aligned}
 T_{\underline{R(A)}}(x) &= \bar{\wedge}_{y \in U} (F_R(x, y) \vee T_A(y)) \\
 &= \bar{\wedge}_{y \in U} (T_{L(1_{U-\{y\}})}(x) \vee T_A(y)) \\
 &= \bar{\wedge}_{y \in U} (T_{L(1_{U-\{y\}})}(x) \vee T_{\widehat{A(y)}}(x)) \\
 &= \bar{\wedge}_{y \in U} (T_{L(1_{U-\{y\}} \uplus \widehat{A(y)})}(x)) \\
 &= \bar{\wedge}_{y \in U} (T_{L(1_{U-\{y\}} \uplus \widehat{A(y)})}(x)) \\
 &= T_{\bigcap_{y \in U} (L(1_{U-\{y\}} \uplus \widehat{A(y)}))}(x) \\
 &= T_{L(\bigcap_{y \in U} (1_{U-\{y\}} \uplus \widehat{A(y)}))}(x) \\
 &= T_{L(A)}(x), \\
 I_{\underline{R(A)}}(x) &= \bigvee_{y \in U} (([1, 1] - ([1, 1] - I_{L(1_{U-\{y\}})}(x))) \bar{\wedge} I_A(y)) \\
 &= \bigvee_{y \in U} (I_{L(1_{U-\{y\}})}(x) \bar{\wedge} I_{\widehat{A(y)}}(x)) \\
 &= \bigvee_{y \in U} (I_{L(1_{U-\{y\}} \uplus \widehat{A(y)})}(x)) \\
 &= I_{\bigcap_{y \in U} (L(1_{U-\{y\}} \uplus \widehat{A(y)}))}(x) \\
 &= I_{\bigcap_{y \in U} (L(1_{U-\{y\}} \uplus \widehat{A(y)}))}(x) \\
 &= I_{L(\bigcap_{y \in U} (1_{U-\{y\}} \uplus \widehat{A(y)}))}(x) \\
 &= I_{L(A)}(x),
 \end{aligned}$$

and

$$F_{\underline{R(A)}}(x) = \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} F_A(y))$$



$$\begin{aligned}
 &= \bigvee_{y \in U} (F_{L(1_U - \{y\})}(x) \bar{\wedge} F_{\widehat{A(y)}}(x)) \\
 &= \bigvee_{y \in U} (F_{L(1_U - \{y\}) \cup \widehat{A(y)}}(x)) \\
 &= F_{\bigcap_{y \in U} (L(1_U - \{y\}) \cup \widehat{A(y)})}(x) \\
 &= F_{\bigcap_{y \in U} (L(1_U - \{y\}) \cup \widehat{A(y)})}(x) \\
 &= F_{L(\bigcap_{y \in U} (L(1_U - \{y\}) \cup \widehat{A(y)}))}(x) \\
 &= F_{L(A)}(x)
 \end{aligned}$$

Thus, there exists an INR  $R$  such that  $L(A) = \underline{R}(A)$ .  $\square$

**Theorem 4.2.** Let  $H: INS(U) \rightarrow INS(U)$  be an interval neutrosophic set operator. Then, there exists an INR  $R$  in  $U$  such that  $H(A) = \bar{R}(A)$  for all  $A \in INS(U)$  iff  $H$  satisfies the following axioms (INSH1) and (INSH2):  $\forall A, B \in INS(U), \alpha, \beta, \gamma \in Int[0, 1]$ ,

$$\begin{aligned}
 (INSH1) & H(A \cap \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}) = H(A) \cap \widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}; \\
 (INSH2) & H(A \cup B) = H(A) \cup H(B).
 \end{aligned}$$

*Proof.* “ $\implies$ ” It is straightforward from Theorem 3.5.

“ $\impliedby$ ” Suppose  $H$  satisfies axioms (INSH1) and (INSH2). By using  $H$ , we define an INR  $R = \{(x, y), T_R(x, y), I_R(x, y), F_R(x, y)\}$  as follows:

$$T_R(x, y) = T_{H(1_y)}(x), \quad I_R(x, y) = I_{H(1_y)}(x), \quad F_R(x, y) = F_{H(1_y)}(x).$$

Moreover, we can obtain that for all  $A \in INS(U)$ ,

$$A = \bigcup_{y \in U} (1_y \cap \widehat{A(y)}).$$

In fact, for all  $x \in U$ , we have

$$\begin{aligned}
 T_{\bigcup_{y \in U} (1_y \cap \widehat{A(y)})}(x) &= \bigvee_{y \in U} T_{1_y \cap \widehat{A(y)}}(x) \\
 &= \bigvee_{y \in U} (T_{1_y}(x) \bar{\wedge} T_{\widehat{A(y)}}(x)) \\
 &= T_{1_x}(x) \bar{\wedge} T_{\widehat{A(x)}}(x) \underline{\vee} \bigvee_{y \in U - \{x\}} (T_{1_y}(x) \bar{\wedge} T_{\widehat{A(y)}}(x)) \\
 &= T_A(x) \underline{\vee} [0, 0] \\
 &= T_A(x),
 \end{aligned}$$

$$\begin{aligned}
 I_{\bigcup_{y \in U} (1_y \cap \widehat{A(y)})}(x) &= \bigwedge_{y \in U} (I_{1_y \cap \widehat{A(y)}}(x)) \\
 &= \bigwedge_{y \in U} (I_{1_y}(x) \underline{\vee} I_{\widehat{A(y)}}(x)) \\
 &= I_{1_x}(x) \underline{\vee} I_{\widehat{A(x)}}(x) \bar{\wedge} \bigwedge_{y \in U - \{x\}} (I_{1_y}(x) \underline{\vee} I_{\widehat{A(y)}}(x)) \\
 &= I_A(x) \bar{\wedge} [1, 1] \\
 &= I_A(x),
 \end{aligned}$$

and

$$\begin{aligned}
 F_{\bigcup_{y \in U} (1_y \cap \widehat{A(y)})}(x) &= \bigwedge_{y \in U} (F_{1_y \cap \widehat{A(y)}}(x)) \\
 &= \bigwedge_{y \in U} (F_{1_y}(x) \underline{\vee} F_{\widehat{A(y)}}(x)) \\
 &= F_{1_x}(x) \underline{\vee} F_{\widehat{A(x)}}(x) \bar{\wedge} \bigwedge_{y \in U - \{x\}} (F_{1_y}(x) \underline{\vee} F_{\widehat{A(y)}}(x)) \\
 &= F_A(x) \bar{\wedge} [1, 1] \\
 &= F_A(x),
 \end{aligned}$$

$$\text{So, } A = \bigcup_{y \in U} (1_y \cap \widehat{A(y)}).$$

By Definition 3.2, (INSH1) and (INSH2), we have

$$\begin{aligned}
 T_{\overline{R}(A)}(x) &= \bigvee_{y \in U} (T_R(x, y) \bar{\wedge} T_A(y)) \\
 &= \bigvee_{y \in U} (T_{H(1_y)}(x) \bar{\wedge} T_A(y)) \\
 &= \bigvee_{y \in U} (T_{H(1_y)}(x) \bar{\wedge} T_{\widehat{A(y)}}(x)) \\
 &= \bigvee_{y \in U} (T_{H(1_y) \cap \widehat{A(y)}}(x)) \\
 &= \bigvee_{y \in U} (T_{H(1_y \cap \widehat{A(y)})}(x)) \\
 &= T_{\bigcup_{y \in U} (H(1_y \cap \widehat{A(y))})}(x) \\
 &= T_{H(\bigcup_{y \in U} (1_y \cap \widehat{A(y)}))}(x) \\
 &= T_{H(A)}(x),
 \end{aligned}$$

$$\begin{aligned}
 I_{\overline{R}(A)}(x) &= \overline{\bigwedge_{y \in U} (I_R(x, y) \vee I_A(y))} \\
 &= \overline{\bigwedge_{y \in U} (I_{H(1_y)}(x) \vee I_A(y))} \\
 &= \overline{\bigwedge_{y \in U} (I_{H(1_y)}(x) \vee I_{\widehat{A(y)}}(x))} \\
 &= \overline{\bigwedge_{y \in U} (I_{H(1_y) \cap \widehat{A(y)}}(x))} \\
 &= \overline{\bigwedge_{y \in U} (I_{H(1_y \cap \widehat{A(y)})}(x))} \\
 &= I_{\bigcup_{y \in U} (H(1_y \cap \widehat{A(y)}))}(x) \\
 &= I_{H(\bigcup_{y \in U} (1_y \cap \widehat{A(y)}))}(x) \\
 &= I_{H(A)}(x),
 \end{aligned}$$

and

$$\begin{aligned}
 F_{\overline{R}(A)}(x) &= \overline{\bigwedge_{y \in U} (F_R(x, y) \vee F_A(y))} \\
 &= \overline{\bigwedge_{y \in U} (F_{H(1_y)}(x) \vee F_A(y))} \\
 &= \overline{\bigwedge_{y \in U} (F_{H(1_y)}(x) \vee F_{\widehat{A(y)}}(x))} \\
 &= \overline{\bigwedge_{y \in U} (F_{H(1_y) \cap \widehat{A(y)}}(x))} \\
 &= \overline{\bigwedge_{y \in U} (F_{H(1_y \cap \widehat{A(y)})}(x))} \\
 &= F_{\bigcup_{y \in U} (H(1_y \cap \widehat{A(y)}))}(x) \\
 &= F_{H(\bigcup_{y \in U} (1_y \cap \widehat{A(y)}))}(x) \\
 &= F_{H(A)}(x).
 \end{aligned}$$

Therefore, there exists an INR  $R$  such that  $H(A) = \overline{R}(A)$ .  $\square$

**Remark 4.3.** If  $L, H : INS(U) \rightarrow INS(U)$  satisfy (INSL1), (INSL2) and (INSU1), (INSU2), respectively. Then,  $L(A) = (H(A^c))^c$  and  $H(A) = (L(A^c))^c$ . In this case,  $L$  and  $H$  are called a pair of dual operators. Furthermore, if  $L$  and  $H$  are dual operators, then (INSL1), (INSL2) are equivalent to (INSU1), (INSU2).

*Proof.* It follows immediately from Theorem 3.5.  $\square$

Next, we investigate axiomatic characterizations of other special generalized interval neutrosophic approximation operators.

**Theorem 4.4.** Let  $L, H : INS(U) \longrightarrow INS(U)$  be a pair of dual operators, then there exists a serial INR  $R$  in  $U$  such that  $L(A) = \underline{R}(A)$ ,  $H(A) = \overline{R}(A)$  for all  $A \in INS(U)$  iff  $L$  satisfies axioms (INSL1), (INSL2) and one of the following equivalent axioms about  $L$ , or equivalently  $H$  satisfies (INSU1), (INSU2) and one of the following equivalent axioms about  $H$ :

- (INSL3)  $L(\emptyset) = \emptyset$ ;
- (INSU3)  $H(U) = U$ ;
- (INSL4)  $L(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}$ , for all  $\alpha, \beta, \gamma \in Int[0, 1]$ ;
- (INSU4)  $H(\widehat{\alpha, \beta, \gamma}) = \widehat{\alpha, \beta, \gamma}$ , for all  $\alpha, \beta, \gamma \in Int[0, 1]$ .

*Proof.* It follows immediately from Theorems 3.9 (1), 4.1 and 4.2.  $\square$

**Theorem 4.5.** Let  $L, H : INS(U) \longrightarrow INS(U)$  be a pair of dual operators, then there exists a reflexive INR  $R$  in  $U$  such that  $L(A) = \underline{R}(A)$ ,  $H(A) = \overline{R}(A)$  for all  $A \in INS(U)$  iff  $L$  satisfies axioms (INSL1), (INSL2) and (INSL5), or equivalently  $H$  satisfies (INSU1), (INSU2) and (INSU5):

- (INSL5)  $L(A) \in A$ ;
- (INSU5)  $A \in H(A)$ .

*Proof.* It follows immediately from Theorems 3.9 (2), 4.1 and 4.2.  $\square$

**Theorem 4.6.** Let  $L, H : INS(U) \longrightarrow INS(U)$  be a pair of dual operators, then there exists a symmetric INR  $R$  in  $U$  such that  $L(A) = \underline{R}(A)$ ,  $H(A) = \overline{R}(A)$  for all  $A \in INS(U)$  iff  $L$  satisfies axioms (INSL1), (INSL2) and (INSL6), or equivalently  $H$  satisfies (INSU1), (INSU2) and (INSU6):

- (INSL6)  $L(1_{U-\{y\}})(x) = L(1_{U-\{x\}})(y)$ ,  $\forall x, y \in U$ ;
- (INSU6)  $H(1_y)(x) = H(1_x)(y)$ ,  $\forall x, y \in U$ .

*Proof.* It follows immediately from Theorems 3.9 (3), 4.1 and 4.2.  $\square$

**Theorem 4.7.** Let  $L, H : INS(U) \longrightarrow INS(U)$  be a pair of dual operators, then there exists a transitive INR  $R$  in  $U$  such that  $L(A) = \underline{R}(A)$ ,  $H(A) = \overline{R}(A)$  for all  $A \in INS(U)$  iff  $L$  satisfies axioms (INSL1), (INSL2) and (INSL7), or equivalently  $H$  satisfies (INSU1), (INSU2) and (INSU7):

- (INSL7)  $L(A) \in L(L(A))$ ,  $\forall A \in INS(U)$ ;
- (INSU7)  $H(H(A)) \in H(A)$ ,  $\forall A \in INS(U)$ .

*Proof.* It follows immediately from Theorems 3.9 (4), 4.1 and 4.2.  $\square$

## 5. An Application of Generalized Interval Neutrosophic Rough Sets in Multi-Attribute Decision Making

### 5.1. An algorithm to medical diagnosis based on generalized interval neutrosophic rough sets

In order to conveniently apply generalized interval neutrosophic rough sets to real world, it is necessary to extend the generalized interval neutrosophic rough sets on one universe in Section 4 to two universes case.

**Definition 5.1.** Let  $U, V$  be two spaces of points (objects). An INS  $R$  in  $U \times V$  is referred to as an INR from  $U$  to  $V$ , denoted by  $R = \{\langle (x, y), T_R(x, y), I_R(x, y), F_R(x, y) \rangle \mid (x, y) \in U \times V\}$ , where  $T_R, I_R, F_R : U \times V \longrightarrow Int [0, 1]$  represent the truth-membership function, indeterminacy-membership function and falsity-membership function of  $R$ , respectively.

**Definition 5.2.** Let  $R$  be an INR from  $U$  to  $V$ , the tuple  $(U, V, R)$  is referred to as an interval neutrosophic approximation space on two universes.  $\forall A \in \text{INS}(V)$ , the lower and upper approximations of  $A$  w.r.t.  $(U, V, R)$  are two INSs in  $U$ , denoted by  $\underline{R}(A)$  and  $\overline{R}(A)$ , where  $\forall x \in U$ :

$$\begin{aligned} T_{\underline{R}(A)}(x) &= \bigwedge_{y \in V} (F_R(x, y) \vee T_A(y)), \\ I_{\underline{R}(A)}(x) &= \bigvee_{y \in V} (([1, 1] - I_R(x, y)) \wedge I_A(y)), \\ F_{\underline{R}(A)}(x) &= \bigvee_{y \in V} (T_R(x, y) \wedge F_A(y)); \\ T_{\overline{R}(A)}(x) &= \bigvee_{y \in V} (T_R(x, y) \wedge T_A(y)), \\ I_{\overline{R}(A)}(x) &= \bigwedge_{y \in V} (I_R(x, y) \vee I_A(y)), \\ F_{\overline{R}(A)}(x) &= \bigwedge_{y \in V} (F_R(x, y) \vee F_A(y)). \end{aligned}$$

The pair  $(\underline{R}(A), \overline{R}(A))$  is referred to as a generalized interval neutrosophic rough set on two universes.

Based on Definition 2.11, we can give the sum of two INSs as follows.

**Definition 5.3.** Let  $A$  and  $B$  be two INSs in  $U$ , the sum of  $A$  and  $B$  is defined as:

$$A \boxplus B = \{ \langle x, A(x) \oplus B(x) \rangle \mid x \in U \}.$$

Note that we can compare two interval numbers by Definitions 2.12 and 2.13. Moreover, by Definitions 5.2 and 5.3, we can apply generalized interval neutrosophic rough sets on two universes to multi-attribute decision making problems.

In what follows, we will consider medical diagnosis problems based on generalized interval neutrosophic rough sets on two universes. Suppose that the universe  $U = \{x_1, x_2, \dots, x_m\}$  represents a set of diseases, and the universe  $V = \{y_1, y_2, \dots, y_n\}$  represents a set of symptoms. Let  $R \in \text{INR}(U \times V)$  be an INR from  $U$  to  $V$ , where  $\forall (x_i, y_j) \in U \times V$ ,  $R(x_i, y_j)$  represents the degree that the disease  $x_i$  ( $x_i \in U$ ) has the symptom  $y_j$  ( $y_j \in V$ ). Given a patient  $A$ , the symptoms of the patient (also denoted by  $A$ ) are illustrated by an INS  $A$  in the universe  $V$ . In the following, we give a six-steps algorithm to diagnose what kind of disease the patient  $A$  is suffering from.

**Algorithm**

Step 1. According to Definition 5.2, we calculate the lower and upper approximations of  $A$ , namely  $\underline{R}(A)$  and  $\overline{R}(A)$ .

Step 2. According to Definition 5.3, we calculate  $\underline{R}(A) \boxplus \overline{R}(A)$ .

Step 3. According to Definition 2.12, for all  $i \in \{1, 2, \dots, m\}$ , we calculate  $s((\underline{R}(A) \boxplus \overline{R}(A))(x_i))$ ,  $a((\underline{R}(A) \boxplus \overline{R}(A))(x_i))$  and  $c((\underline{R}(A) \boxplus \overline{R}(A))(x_i))$ , respectively.

Step 4. According to Definition 2.13, for all  $i \in \{1, 2, \dots, m\}$ , we compare all the  $s((\underline{R}(A) \boxplus \overline{R}(A))(x_i))$ ,  $a((\underline{R}(A) \boxplus \overline{R}(A))(x_i))$  and  $c((\underline{R}(A) \boxplus \overline{R}(A))(x_i))$ .

Step 5. The optimal choice is  $x_k$  if there doesn't exist  $i \in \{1, 2, \dots, k-1, k+1, \dots, m\}$  such that  $(\underline{R}(A) \boxplus \overline{R}(A))(x_i) > (\underline{R}(A) \boxplus \overline{R}(A))(x_k)$ .

Step 6. If  $k$  has several values, then we take every  $x_k$  as the optimal choice which means that the patient is suffering from all the diseases  $\{x_k\}$  at the same time.

**5.2. An illustrative example**

In this subsection, an example for medical diagnosis is illustrated as the demonstration of the established algorithm proposed in Subsection 5.1.

We take into account the medical diagnosis problem partly adopted from [43] and adjust the hesitant fuzzy environment to neutrosophic environment. Let  $U = \{x_1, x_2, x_3, x_4\}$  be four diseases (where  $x_i$  ( $i = 1, 2, 3, 4$ ) represent "common cold", "malaria" "typhoid", and "stomach disease", respectively), and the

universe  $V = \{y_1, y_2, y_3, y_4, y_5\}$  be five symptoms (where  $y_j$  ( $j = 1, 2, 3, 4, 5$ ) represent “fever”, “headache”, “stomachache”, “cough”, and “chest-pain”, respectively). Let  $R$  be an INR from  $U$  to  $V$  which is actually a medical knowledge statistic data of the relationship of the disease  $x_i$  ( $x_i \in U$ ) and the symptom  $y_j$  ( $y_j \in V$ ). The statistic data is provided in Table 2.

Assume that the symptoms of a patient  $A$  are illustrated by an INS in the universe  $V$  as follows:

$$A = \{\langle y_1, [0.8, 0.9], [0.2, 0.3], [0.1, 0.3] \rangle, \langle y_2, [0.7, 0.9], [0.1, 0.2], [0.1, 0.2] \rangle, \langle y_3, [0.7, 0.8], [0.2, 0.4], [0.1, 0.3] \rangle, \langle y_4, [0.1, 0.2], [0.3, 0.4], [0.8, 0.9] \rangle, \langle y_5, [0, 0.1], [0.1, 0.3], [0.8, 1] \rangle\}.$$

**Table 2:** The interval neutrosophic relation  $R$  between the symptoms and diseases.

$R$	$x_1$	$x_2$
$y_1$	$\langle [0.4, 0.5], [0.2, 0.3], [0.3, 0.4] \rangle$	$\langle [0.8, 0.9], [0.1, 0.2], [0, 0.1] \rangle$
$y_2$	$\langle [0.5, 0.6], [0.3, 0.4], [0.2, 0.3] \rangle$	$\langle [0.8, 0.9], [0.2, 0.3], [0, 0.1] \rangle$
$y_3$	$\langle [0, 0.1], [0.1, 0.2], [0.8, 0.9] \rangle$	$\langle [0, 0.2], [0.1, 0.3], [0.7, 0.9] \rangle$
$y_4$	$\langle [0.7, 0.8], [0.3, 0.4], [0.2, 0.3] \rangle$	$\langle [0, 0.1], [0, 0.2], [0.8, 1] \rangle$
$y_5$	$\langle [0.4, 0.5], [0.5, 0.6], [0.6, 0.7] \rangle$	$\langle [0, 0.1], [0.1, 0.2], [0.9, 1] \rangle$

  

$R$	$x_3$	$x_4$
$y_1$	$\langle [0.8, 1], [0.2, 0.4], [0, 0.1] \rangle$	$\langle [0.1, 0.3], [0.3, 0.4], [0.8, 1] \rangle$
$y_2$	$\langle [0.9, 1], [0.1, 0.3], [0, 0.1] \rangle$	$\langle [0, 0.1], [0.2, 0.3], [0.9, 1] \rangle$
$y_3$	$\langle [0.7, 0.8], [0.4, 0.6], [0.2, 0.3] \rangle$	$\langle [0.9, 1], [0.4, 0.6], [0.1, 0.3] \rangle$
$y_4$	$\langle [0, 0.1], [0.3, 0.4], [0.8, 0.9] \rangle$	$\langle [0, 0.2], [0.1, 0.2], [0.8, 0.9] \rangle$
$y_5$	$\langle [0, 0.2], [0.2, 0.4], [0.7, 1] \rangle$	$\langle [0.1, 0.4], [0.2, 0.5], [0.7, 0.8] \rangle$

According to Definition 5.2, we can obtain that

$$\begin{aligned} \underline{R}(A) &= \{\langle x_1, [0.2, 0.3], [0.3, 0.4], [0.7, 0.8] \rangle, \langle x_2, [0.7, 0.9], [0.3, 0.4], [0.1, 0.3] \rangle, \\ &\quad \langle x_3, [0.7, 0.8], [0.3, 0.4], [0.1, 0.3] \rangle, \langle x_4, [0.7, 0.8], [0.3, 0.4], [0.1, 0.4] \rangle\}, \\ \overline{R}(A) &= \{\langle x_1, [0.5, 0.6], [0.2, 0.3], [0.2, 0.3] \rangle, \langle x_2, [0.8, 0.9], [0.1, 0.3], [0.1, 0.2] \rangle, \\ &\quad \langle x_3, [0.8, 0.9], [0.1, 0.3], [0.1, 0.2] \rangle, \langle x_4, [0.7, 0.8], [0.2, 0.3], [0.1, 0.3] \rangle\}. \end{aligned}$$

Let  $k(x) = -\log(x)$ , then  $k^{-1}(x) = e^{-x}$ ,  $l(x) = -\log(1 - x)$ , and  $l^{-1}(x) = 1 - e^{-1(x)}$ . By Definitions 2.11 and 5.3, we have

$$\begin{aligned} \underline{R}(A) \boxplus \overline{R}(A) &= \{\langle x_1, [0.60, 0.72], [0.06, 0.12], [0.14, 0.24] \rangle, \langle x_2, [0.94, 0.99], [0.03, 0.12], [0.01, 0.06] \rangle, \\ &\quad \langle x_3, [0.94, 0.98], [0.03, 0.12], [0.01, 0.06] \rangle, \langle x_4, [0.91, 0.96], [0.06, 0.12], [0.01, 0.12] \rangle\}. \end{aligned}$$

According to Definition 2.13, we can calculate the score functions, accuracy functions, and certainty functions of the INN  $(\underline{R}(A) \boxplus \overline{R}(A))(x_i)$  ( $i = 1, 2, 3, 4$ ) as follows:

$$\begin{aligned} s((\underline{R}(A) \boxplus \overline{R}(A))(x_1)) &= [2.24, 2.52], a((\underline{R}(A) \boxplus \overline{R}(A))(x_1)) = [0.46, 0.48], \\ c((\underline{R}(A) \boxplus \overline{R}(A))(x_1)) &= [0.60, 0.72]; \\ s((\underline{R}(A) \boxplus \overline{R}(A))(x_2)) &= [2.76, 2.95], a((\underline{R}(A) \boxplus \overline{R}(A))(x_2)) = [0.93, 0.93], \\ c((\underline{R}(A) \boxplus \overline{R}(A))(x_2)) &= [0.94, 0.99]; \\ s((\underline{R}(A) \boxplus \overline{R}(A))(x_3)) &= [2.76, 2.95], a((\underline{R}(A) \boxplus \overline{R}(A))(x_3)) = [0.92, 0.93], \\ c((\underline{R}(A) \boxplus \overline{R}(A))(x_3)) &= [0.94, 0.98]; \\ s((\underline{R}(A) \boxplus \overline{R}(A))(x_4)) &= [2.67, 2.89], a((\underline{R}(A) \boxplus \overline{R}(A))(x_4)) = [0.84, 0.90], \\ c((\underline{R}(A) \boxplus \overline{R}(A))(x_4)) &= [0.91, 0.96]. \end{aligned}$$

By Definition 2.13, we can compute that

$$\begin{aligned} p(s((\underline{R}(A) \boxplus \overline{R}(A))(x_2)) \geq_{L'} s((\underline{R}(A) \boxplus \overline{R}(A))(x_1))) &= 1, \\ p(s((\underline{R}(A) \boxplus \overline{R}(A))(x_4)) \geq_{L'} s((\underline{R}(A) \boxplus \overline{R}(A))(x_1))) &= 1, \end{aligned}$$

$$p(s(\underline{R}(A) \boxplus \overline{R}(A))(x_2) \geq_{L'} s(\underline{R}(A) \boxplus \overline{R}(A))(x_4)) = 1,$$

$$p(s(\underline{R}(A) \boxplus \overline{R}(A))(x_2) \geq_{L'} s(\underline{R}(A) \boxplus \overline{R}(A))(x_3)) = 0.5.$$

It follows that

$$s(\underline{R}(A) \boxplus \overline{R}(A))(x_2) > s(\underline{R}(A) \boxplus \overline{R}(A))(x_4) > s(\underline{R}(A) \boxplus \overline{R}(A))(x_1),$$

$$s(\underline{R}(A) \boxplus \overline{R}(A))(x_3) > s(\underline{R}(A) \boxplus \overline{R}(A))(x_4) > s(\underline{R}(A) \boxplus \overline{R}(A))(x_1).$$

In order to compare  $(\underline{R}(A) \boxplus \overline{R}(A))(x_2)$  and  $(\underline{R}(A) \boxplus \overline{R}(A))(x_3)$ , we calculate

$$p(a(\underline{R}(A) \boxplus \overline{R}(A))(x_2) \geq_{L'} a(\underline{R}(A) \boxplus \overline{R}(A))(x_3)) = 1,$$

which means that  $(\underline{R}(A) \boxplus \overline{R}(A))(x_2) > (\underline{R}(A) \boxplus \overline{R}(A))(x_3)$ .

Thus, we can conclude

$$(\underline{R}(A) \boxplus \overline{R}(A))(x_2) > (\underline{R}(A) \boxplus \overline{R}(A))(x_3) > (\underline{R}(A) \boxplus \overline{R}(A))(x_4) > (\underline{R}(A) \boxplus \overline{R}(A))(x_1).$$

That is to say, the patient  $A$  is suffering from malaria  $x_2$ .

Compared with the model and algorithm given in [34], the proposed model and algorithm in the present paper is more flexible which means that it can deal with more complex data and information since single valued neutrosophic sets is a special case of interval neutrosophic sets. From the analysis above, we can see that the algorithm based on generalized neutrosophic rough sets on two universes suits more general decision-making environment.

## 6. Conclusion

In this paper, we propose the hybrid model—generalized interval neutrosophic rough sets based on interval neutrosophic relations by combining two powerful tools of handling information—interval neutrosophic sets and rough sets. Furthermore, we investigate the generalized interval neutrosophic rough sets from both constructive and axiomatic approaches in detail. Then, generalized interval neutrosophic rough sets on two universes are introduced for wider application of generalized interval neutrosophic rough sets. After that, we provide an algorithm to handle decision making problem in medical diagnosis based on generalized interval neutrosophic rough sets on two universes. Finally, we present a numerical example to demonstrate the validness of the proposed generalized interval neutrosophic rough sets. For the future prospects, we will devote to explore the application of the proposed model to data mining and attribute reduction.

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