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Fuzzy Equivalence on Standard and Rough Neutrosophic Sets and Applications to Clustering Analysis

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Abstract. In this paper, we propose the concept of fuzzy equivalence on standard neutrosophic sets and rough standard neutrosophic sets. We also provide some formulas for fuzzy equivalence on standard neutrosophic sets and rough standard neutrosophic sets. We also apply these formulas for cluster analysis. Numerical examples are illustrated.

Keywords: Fuzzy equivalence · Neutrosophic set · Rough set · Rough neutrosophic set · Fuzzy clustering

1 Introduction

In 1998, Smarandache introduced neutrosophic set [1]; NS is the generalization of fuzzy set [2] and intuitionistic fuzzy set [3]. Over time, the subclass of the neutrosophic set [4–6] was proposed to capture more advantages in practical applications. In 2014, Bui Cong Cuong introduced the concept of the picture fuzzy set [7]. After that, Son gave the applications of the picture fuzzy set in clustering problems in [8–19]. It is to be noted that the picture fuzzy set was regarded as a standard neutrosophic set.

Rough set theory [20] is a useful mathematical tool for data mining, especially for redundant and uncertain data [21]. On the first time, rough set is established on equivalence relation. The set of equivalence classes of the universal set, obtained by an equivalence relation, is the basis for the construction of upper and lower approximation of the subset of the universal set. Recently, rough set has been developed into the fuzzy environment and obtained several interesting results [22, 23].

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It has been realized that the combination of the neutrosophic set and rough set achieved more uncertainty in the analysis of sophisticated events in real applications [24–26]. Bui Cong Cuong et al. [27] firstly introduced some results of the standard neutrosophic soft theory. Later, Nguyen Xuan Thao et al. [28, 29] proposed the rough picture fuzzy set and the rough standard neutrosophic set which are the results of approximation of the picture fuzzy set and standard neutrosophic set, respectively, with respect to a crisp approximation space. However, the previous researches have not defined fuzzy equivalence, a basic component in the standard neutrosophic set for the approximation and inference processes.

In this paper, we introduce the concept of fuzzy equivalence for the standard neutrosophic set and the rough standard neutrosophic set. Some examples of the fuzzy equivalence for those sets and application on clustering analysis are also given. The rest of the paper is organized as follows: The rough standard neutrosophic set and fuzzy equivalence are recalled in Sect. 3. Sections 3 and 4 propose the concept of fuzzy equivalence for two standard neutrosophic sets. In Sect. 5, we give an application of clustering and Sect. 6 draws the conclusion.

2 Preliminary

Definition 1 [27]. Let U be a universal set. A standard neutrosophic set (SNS) A on the U is $A = \{(u, \mu_A(u), \eta_A(u), \gamma_A(u))|u \in U\}$, where $\mu_A(u)$ is called the "degree of positive membership of u in A," $\eta_A(u)$ is called the "degree of indeterminate/neutral membership of u in A," and $\gamma_A(u)\gamma_A(u)$ is called the "degree of negative membership of u in A," where $\mu_A(u), \eta_A(u)\ \gamma_A(u) \in [0, 1]$ satisfy the following condition:

$$0 \le \mu_A(u) + \eta_A(u) + \gamma_A(u) \le 1, \forall u \in U.$$

The family of all standard neutrosophic sets in U is denoted by SNS(U).

Definition 2 [28, 29]. For a given $A \in SNS(U)$, the mappings $\overline{RP}, \underline{RP} : SNS(U) \to SNS(U)$, in which

$$\overline{RP}(A) = \{\left(u, \mu_{\overline{RP}(A)}(u), \eta_{\overline{RP}(A)}(u), \gamma_{\overline{RP}(A)}(u)\right)|u \in U\},$$
$$\underline{RP}(A) = \{(u, \mu_{\underline{RP}(A)}(u), \eta_{\underline{RP}(A)}(u), \gamma_{\underline{RP}(A)}(u)|u \in U\},$$

where

$$\mu_{\overline{RP}(A)}(u) = \vee_{v\in R_S(u)}\, \mu_A(v), \eta_{\overline{RP}(A)}(u) = \wedge_{v\in R_S(u)}\, \eta_A(v),$$
$$\gamma_{\overline{RP}(A)}(u) = \wedge_{v\in R_S(u)}\, \gamma_A(v)$$

and

$$\mu_{\underline{RP}(A)}(u) = \wedge_{v\in R_S(u)}\, \mu_A(u),\ \eta_{\underline{RP}(A)}(u) = \wedge_{v\in R_S(u)}\, \eta_A(v),$$
$$\gamma_{\underline{RP}(A)}(u) = \vee_{v\in R_S(u)}\, \gamma_A(v)$$

are called to the upper and lower standard neutrosophic approximation operators, respectively, and the pair $RP(A) = \left(\underline{RP}(A), \overline{RP}(A)\right)$ is referred as the rough standard neutrosophic set of A w.r.t the approximation space (U, R), or A is called roughly defined on the approximation space (U, R). The collection of all rough standard neutrosophic sets defined on the approximation space (U, R) is denoted by $RSNS(U)$.

Definition 3 [21] A mapping $e : [0, 1]^2 \to [0, 1]$ is a fuzzy equivalence if it satisfies the following conditions:

(e1) $e(a, b) = e(b, a), \forall a, b$ in $[0, 1]$,
(e2) $e(1, 0) = 0$,
(e3) $e(b, b) = 1, \forall b$ in $[0, 1]$,
(e4) If $a \le a' \le b' \le b$, then $e(a, b) \le e(a', b')$.

Note that (e4) satisfies iff $e(a, c) \le \min\{e(a, b), e(b, c)\}$, for all $a \le b \le c$ and $a, b, c \in [0, 1]$.

3 Fuzzy Equivalence on Standard Neutrosophic Set

Definition 4 A mapping $E : SNS(U) \times SNS(U) \to [0, 1]$ is a fuzzy equivalence if it satisfies the following conditions:

(E1) $E(A, B) = E(B, A)$ for all $A, B \in SNS(U)$,
(E2) $E(A, B) = 0$ iff $A = 1_U$ and $B = 0_U$,
(E3) $E(A, A) = 1, \forall A \in SNS(U)$,
(E4) $E(A, C) \le \min\{E(A, B), E(B, C)\}$ for all $\forall A, B, C \in SNS(U)$ satisfy $A \subset B \subset C$.

Example 1 Let $U = \{x_1, x_2, \ldots, x_n\}$ be a universal set and $A, B \in SNS(U)$. A mapping $E : SNS(U) \times SNS(U) \to [0, 1]$, where

$$E(A, B) = \begin{cases} \frac{1}{n}\sum_{i=1}^{n} \frac{\min\{\mu_A(x_i), \mu_B(x_i)\} + \min\{\eta_A(x_i), \eta_B(x_i)\} + (1 - \max\{\gamma_A(x_i), \gamma_B(x_i)\})}{2} & \textit{iff } A \ne B \\ 1 & \textit{iff } A = B \end{cases}$$

is a fuzzy equivalence of A and B.

Indeed, conditions (E1), (E2), (E3), and (E4) are obvious.

Theorem 1 Let $U = \{x_1, x_2, \ldots, x_n\}$ be a universal set and $A, B \in SNS(U)$. Let a mapping $E : PFS(U) \times PFS(U) \to [0, 1]$ is defined by

$$E(A,B) = \begin{cases} \frac{1}{n}\sum_{i=1}^{n} \frac{t_1(\mu_A(x_i),\mu_B(x_i)) + t_2(\eta_A(x_i),\eta_B(x_i)) + (1 - S(\gamma_A(x_i),\gamma_B(x_i)))}{2} & \textit{iff } A \neq B \\ 1 & \textit{iff } A = B \end{cases}$$

where t_1, t_2 are t-norm on [0, 1] and s is a t-conorm on [0, 1]; then, $E(A, B)$ is a fuzzy equivalence of A and B.

4 Fuzzy Equivalence on Rough Standard Neutrosophic Set

Here, we propose a fuzzy equivalence of the rough standard neutrosophic sets. Let $U = \{x_1, x_2, \ldots, x_n\}$, $A, B \in SNS(U)$, and $RP(A) = \left(\underline{RP}A, \overline{RP}A\right)$, $RP(B) = \left(\underline{RP}B, \overline{RP}B\right)$. For all $x_i \in U$, denote

$$\mu_{EA}(x_i) = \left|\mu_{\overline{RP}A}(x_i) - \mu_{\underline{RP}A}(x_i)\right|;$$

$$\eta_{EA}(x_i) = \left|\eta_{\overline{RP}A}(x_i) - \eta_{\underline{RP}A}(x_i)\right|;$$

$$\gamma_{EA}(x_i) = \left|\gamma_{\overline{RP}A}(x_i) - \gamma_{\underline{RP}A}(x_i)\right|;$$

$$\mu_{AE}(x_i) = \frac{\left|\mu_{\overline{RP}A}(x_i) + \mu_{\underline{RP}A}(x_i)\right|}{2};$$

$$\eta_{AE}(x_i) = \frac{\left|\eta_{\overline{RP}A}(x_i) + \eta_{\underline{RP}A}(x_i)\right|}{2};$$

$$\gamma_{AE}(x_i) = \frac{\left|\gamma_{\overline{RP}A}(x_i) + \gamma_{\underline{RP}A}(x_i)\right|}{2};$$

$$\mu_E = \frac{1}{n}\sum_{i=1}^{n}\left\{\begin{array}{l} 1 - \frac{\left|\mu_{\overline{RP}A}(x_i) - \mu_{\overline{RP}B}(x_i)\right| + \left|\mu_{\underline{RP}A}(x_i) - \mu_{\underline{RP}B}(x_i)\right| + \left|\mu_{EA}(x_i) - \mu_{EB}(x_i)\right|}{4} \\ - \frac{\left|\mu_{AE}(x_i) - \mu_{BE}(x_i)\right|}{2} \end{array}\right\};$$

$$\eta_E = \frac{1}{n}\sum_{i=1}^{n}\left\{\begin{array}{l} 1 - \frac{\left|\eta_{\overline{RP}A}(x_i) - \eta_{\overline{RP}B}(x_i)\right| + \left|\eta_{\underline{RP}A}(x_i) - \eta_{\underline{RP}B}(x_i)\right| + \left|\eta_{EA}(x_i) - \eta_{EB}(x_i)\right|}{4} \\ - \frac{\left|\eta_{AE}(x_i) - \eta_{BE}(x_i)\right|}{2} \end{array}\right\};$$

$$\gamma_E = \frac{1}{n}\sum_{i=1}^{n}\left\{\begin{array}{l} 1 - \frac{\left|\gamma_{\overline{RP}A}(x_i) - \gamma_{\overline{RP}B}(x_i)\right| + \left|\gamma_{\underline{RP}A}(x_i) - \gamma_{\underline{RP}B}(x_i)\right| + \left|\gamma_{EA}(x_i) - \gamma_{EB}(x_i)\right|}{4} \\ - \frac{\left|\gamma_{AE}(x_i) - \gamma_{BE}(x_i)\right|}{2} \end{array}\right\}.$$

Theorem 2 The mapping $E : SNS(U) \times SNS(U) \to [0, 1]$ is defined by

$$E(A, B) = \frac{\mu_E + \eta_E + \gamma_E}{3} \text{ is a fuzzy equivalence.}$$

Proof We verify the conditions for $E(A, B)$

(E1) is obvious.
(E2) $A = 1_U; B = 0_U$. Then,

$$\begin{aligned}
&\mu_{EA}(x_i) = 0, \eta_{EA}(x_i) = 0, \gamma_{EA}(x_i) = 0;\\
&\mu_{AE}(x_i) = 1, \eta_{AE}(x_i) = 0, \eta_{AE}(x_i) = 0;\\
&\mu_{EB}(x_i) = 0, \eta_{EB}(x_i) = 0, \gamma_{EB}(x_i) = 0;\\
&\mu_{BE}(x_i) = 0,\ \eta_{BE}(x_i) = 0,\ \eta_{BE}(x_i) = 1;
\end{aligned}$$

So that $\mu_E = \eta_E = \gamma_E = 0$ and $E(A, B) = \frac{\mu_E + \eta_E + \gamma_E}{3} = 0$
(E3) is obvious.
(E4) Note that, if $0 \le a \le b \le c \le 1$, then $|a - c| \ge |a - b|$ and $1 \ge |a - c| \ge |c - b| \ge 0$ so that $0 \le 1 - |a - c| \le 1 - |a - b| \le 1$ and $0 \le 1 - |a - c| \le 1 - |c - b| \le 1$. Because $A \subseteq B \subseteq C$, then $\underline{RP}A \subseteq \underline{RP}B$, $\overline{RP}A \subseteq \overline{RP}B$, and $\underline{RP}B \subseteq \underline{RP}C$, $\overline{RP}B \subseteq \overline{RP}C$. Hence, $E(A, C) \le \min\{E(A, C), E(B, C)\}$. □
Now, let $U = \{x_1, x_2, \ldots, x_n\}$, $A, B \in SNS(U)$, and $RP(A) = \left(\underline{RP}A, \overline{RP}A\right)$, $RP(B) = \left(\underline{RP}B, \overline{RP}B\right)$. For all $x_i \in U$, denote

$$\bar{E}(A, B)(x_i) = \left|\mu_{\overline{RP}A}(x_i) - \mu_{\overline{RP}A}(x_i)\right| + \left|\eta_{\overline{RP}A}(x_i) - \left|\eta_{\overline{RP}B}(x_i)\right| + \left|\gamma_{\overline{RP}A}(x_i) - \gamma_{\overline{RP}B}(x_i)\right|,$$
$$\underline{E}(A, B)(x_i) = \left|\mu_{\underline{RP}A}(x_i) - \mu_{\underline{RP}B}(x_i)\right| + \left|\eta_{\underline{RP}A}(x_i) - \eta_{\underline{RP}B}(x_i)\right| + \left|\gamma_{\underline{RP}A}(x_i) - \gamma_{\underline{RP}B}(x_i)\right|.$$

Theorem 3 The mapping $E : SNS(U) \times SNS(U) \to [0, 1]$ is defined by

$$E(A, B) = \frac{1}{n} \sum_{i=1}^{n} \left[1 - \frac{\bar{E}(A, B)(x_i) + \underline{E}(A, B)(x_i)}{2}\right]$$

is a fuzzy equivalence.

Proof Similar to proof of Theorem 2.

5 An Application to Clustering Analysis

Example 2 Suppose there are three types of products $D = \{D_1, D_2 D_3\}$ and ten regular customers $U = \{u_1, u_2, \ldots, u_{10}\}$. R is an equivalence and $U/R = \{X_1 = \{u_1, u_3, u_9\};$

$X_2 = \{u_2, u_7, u_{10}\}; X_3 = \{u_4\}; X_4 = \{u_5, u_8\}, X_5 = \{u_{10}\}\}$. This division can be based on age or income. We consider that each customer evaluates the product by the linguistic labels {Good, Not-Rated, Not-Good}. Thus, each customer is a neutrosophic set on the products set (Table 1). Now, we can look at similar levels of customer groups in order to strategically sell products. Therefore, one can consider the sets of equivalence classes of customers for the three product categories above. In Table 2, we obtain a rough neutrosophic information system, in which for each X_i, the upper line is $\overline{RP}X_i$ and the lower line is $\underline{RP}\,X_i (i = 1, .., 5)$.

Table 1. A neutrosophic information system

U	D_1	D_2	D_3
u_1	(0.2, 0.3, 0.5)	(0.15, 0.6, 0.2)	(0.4, 0.05, 0.5)
u_2	(0.3, 0.1, 0.5)	(0.3, 0.3, 0.3)	(0.35, 0.1, 0.4)
u_3	(0.6, 0, 0.4)	(0.3, 0.05, 0.6)	(0.1, 0.45, 0.4)
u_4	(0.15, 0.1, 0.7)	(0.1, 0.05, 0.8)	(0.2, 0.4, 0.3)
u_5	(0.05, 0,2, 0.7)	(0.2, 0.4, 0.3)	(0.05, 0.4, 0.5)
u_6	(0.1, 0.3, 0.5)	(0.2, 0.3, 0.4)	(1, 0, 0)
u_7	(0.25, 0.3, 0.4)	(1, 0, 0)	(0.3, 0.3, 0.4)
u_8	(0.1, 0.6, 0.2)	(0.25, 0.3, 0.4)	(0.4, 0, 0.6)
u_9	(0.45, 0,1, 0.45)	(0.25, 0.4, 0.3)	(0.2, 0.5, 0.3)
u_{10}	(0.05, 0.05, 0.9)	(0.4, 0.2, 0.3)	(0.05, 0.7, 0.2)

Table 2. A rough neutrosophic information system

	D_1	D_2	D_3
X_1	(0.6, 0, 0.4)	(0.3, 0.05, 0.2)	(0.4,0.05,0.3)
	(0.2, 0, 0.5)	(0.15, 0.05, 0.6)	(0.1,0.05,0.5)
X_2	(0.3, 0.05, 0.4)	(1, 0, 0)	(0.35, 0.1, 0.2)
	(0.05, 0.05, 0.9)	(0.3, 0.1, 0.3)	(0.05, 0.1, 0.4)
X_3	(0.15, 0.1, 0.7)	(0.1, 0.05, 0.8)	(0.2, 0.4, 0.3)
	(0.15, 0.1, 0.7)	(0.1, 0.05, 0.8)	(0.2, 0.4, 0.3)
X_4	(0.1, 0.2, 0.2)	(0.25, 0.3, 0.3)	(0.4, 0, 0.5)
	(0.05, 0.2, 0.7)	(0.2, 0.3, 0.4)	(0.05, 0, 0.6)
X_5	(0.1, 0.3, 0.5)	(0.2, 0.3, 0.4)	(1, 0, 0)
	(0.1, 0.3, 0.5)	(0.2, 0.3, 0.4)	(1, 0, 0)

We calculate the similarity relations on $\{X_1, X_2, X_3, X_4, X_5\}$ (based on Theorem 2) as follows:

$$R_1 = \begin{bmatrix} 1.0000 & & & & \\ 0.8111 & 1.0000 & & & \\ 0.7708 & 0.7208 & 1.0000 & & \\ 0.8278 & 0.7750 & 0.7458 & 1.0000 & \\ 0.7069 & 0.6347 & 0.7000 & 0.7819 & 1.0000 \end{bmatrix}$$

Use the maximum tree method for fuzzy clustering analysis. Firstly, the Kruskal method is used to draw the largest tree:

$$3 \underset{0.7458}{\rightarrow} 4 \underset{0.775}{\rightarrow} 2 \underset{0.7819}{\rightarrow} 1 \underset{0.8278}{\rightarrow} 5$$

The tree implies that for $\alpha \in [0, 1]$, we can classify the $U/R = \{X_1, X_2, X_3, X_4, X_5\}$ by fuzzy equivalence $E(X_i, X_j) \geq \alpha$, where $X_i, X_j \in U/R$ as follows:

+ If $0 \leq \alpha \leq 0.7458$ then it has a cluster $\{X_1, X_2, X_3, X_4, X_5\}$.
+ If $0.7458 < \alpha \leq 0.775$ then we have two clusters $\{X_3\}, \{X_1, X_2, X_4, X_5\}$.
+ If $0.775 < \alpha \leq 0.7819$ then we have three clusters $\{X_3\}, \{X_4\}, \{X_1, X_2, X_5\}$.
+ If $0.7819 < \alpha \leq 0.8278$ then we have four clusters $\{X_3\}, \{X_4\}, \{X_2\}, \{X_1, X_5\}$.
+ If $0.8278 < \alpha \leq 1$ then we have five clusters $\{X_3\}, \{X_4\}, \{X_2\}, \{X_1\}, \{X_5\}$.

Now, we calculate the similarity relations on $\{X_1, X_2, X_3, X_4, X_5\}$ (based on Theorem 3) as follows:

$$R_2 = \begin{bmatrix} 1.0000 & & & & \\ 0.8792 & 1.0000 & & & \\ 0.8542 & 0.8167 & 1.0000 & & \\ 0.8813 & 0.8563 & 0.85 & 1.0000 & \\ 0.85 & 0.7667 & 0.775 & 0.8521 & 1.0000 \end{bmatrix}$$

Use the maximum tree method for fuzzy clustering analysis. Firstly, the Kruskal method is used to draw the largest tree:

$$3 \underset{0.85}{\rightarrow} 4 \underset{0.8521}{\rightarrow} 2 \underset{0.8563}{\rightarrow} 1 \underset{0.8813}{\rightarrow} 5$$

The tree implies that for $\alpha \in [0, 1]$, we can classify the $U/R = \{X_1, X_2, X_3, X_4, X_5\}$ by fuzzy equivalence $E(X_i, X_j) \geq \alpha$,, where $X_i, X_j \in U/R$ as follows:

+ If $0 \leq \alpha \leq 0.85$ then it has a cluster $\{X_1, X_2, X_3, X_4, X_5\}$.
+ If $0.85 < \alpha \leq 0.8521$ then we have two clusters $\{X_3\}, \{X_1, X_2, X_4, X_5\}$.
+ If $0.8521 < \alpha \leq 0.8563$ then we have three clusters $\{X_3\}, \{X_4\}, \{X_1, X_2, X_5\}$.
+ If $0.8563 < \alpha \leq 0.8813$ then we have four clusters $\{X_3\}, \{X_4\}, \{X_2\}, \{X_1, X_5\}$.
+ If $0.8813 < \alpha \leq 1 < \alpha \leq 1$ then we have five clusters $\{X_3\}, \{X_4\}, \{X_2\}, \{X_1\}, \{X_5\}$.

The clustering analysis on the rough standard neutrosophic set is analogously done. We find that the clustering result by Theorems 2 and 3 is giving the same clustering results. But theoretically, computation using Theorem 3 is simpler than Theorem 2.

6 Conclusions

We have introduced the preliminary results of the fuzzy equivalences on the standard neutrosophic set and the rough standard neutrosophic set. Using these definitions, we can perform clustering analysis on the datasets of neutrosophic sets.

Further studies regarding this research can be expanded of fuzzy equivalence of topological spaces and metrics. With that, we can also build fuzzy equivalent matrix models for clustering problems for real applications.

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References

1. Smarandache, F.: Neutrosophy. Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998; http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf(last edition online).
2. Zadeh, L. A.: Fuzzy Sets. Information and Control 8(3) (1965) 338–353.
3. Atanassov, K.: Intuitionistic Fuzzy Sets. Fuzzy set and systems 20 (1986) 87–96.
4. Wang, H., Smarandache, F., Zhang, Y.Q. et al: Interval NeutrosophicSets and Logic: Theory and Applications in Computing. Hexis, Phoenix, AZ (2005).
5. Wang, H.,Smarandache, F., Zhang, Y.Q.,et al., Single Valued NeutrosophicSets. Multispace and Multistructure 4 (2010) 410–413.
6. Ye, J.: A Multi criteria Decision-Making Method Using Aggregation Operators for Simplified Neutrosophic Sets. Journal of Intelligent & Fuzzy Systems 26 (2014) 2459–2466.
7. Cuong, B.C.: Picture Fuzzy Sets. Journal of Computer Science and Cybernetics 30(4) (2014) 409–420.
8. Cuong, B.C., Son, L.H., Chau, H.T.M.: Some Context Fuzzy Clustering Methods for Classification Problems. Proceedings of the 1st International Symposium on Information and Communication Technology (2010) 34–40.
9. Son, L.H., Thong, P.H.: Some Novel Hybrid Forecast Methods Based On Picture Fuzzy Clustering for Weather Nowcasting from Satellite Image Sequences. Applied Intelligence 46 (1) (2017) 1–15.
10. Son, L.H., Tuan, T.M.: A cooperative semi-supervised fuzzy clustering framework for dental X-ray image segmentation. Expert Systems With Applications 46 (2016) 380–393.
11. Son, L.H., Viet, P.V., Hai, P.V.: Picture Inference System: A New Fuzzy Inference System on Picture Fuzzy Set. Applied Intelligence (2017) https://doi.org/10.1007/s10489-016-0856-1.
12. Son, L.H.: A Novel Kernel Fuzzy Clustering Algorithm for Geo-Demographic Analysis. Information Sciences 317 (2015) 202–223.
13. Son, L.H.: Generalized Picture Distance Measure and Applications to Picture Fuzzy Clustering. Applied Soft Computing 46 (2016) 284–295.

14. Son, L.H.: Measuring Analogousness in Picture Fuzzy Sets: From Picture Distance Measures to Picture Association Measures. Fuzzy Optimization and Decision Making (2017) https://doi.org/10.1007/s10700-016-9249-5.
15. Son, L.H.: DPFCM: A novel distributed picture fuzzy clustering method on picture fuzzy sets. Expert systems with applications 42 (2015) 51–66.
16. Thong, P.H., Son, L.H., Fujita, H.: Interpolative Picture Fuzzy Rules: A Novel Forecast Method for Weather Nowcasting. Proceeding of the 2016 IEEE International Conference on Fuzzy Systems (2016) 86–93.
17. Thong, P.H., Son, L.H.: A Novel Automatic Picture Fuzzy Clustering Method Based On Particle Swarm Optimization and Picture Composite Cardinality. Knowledge-Based Systems 109 (2016) 48–60.
18. Thong, P.H., Son, L.H.: Picture Fuzzy Clustering for Complex Data. Engineering Applications of Artificial Intelligence 56 (2016) 121–130.
19. Thong, P.H., Son, L.H.: Picture Fuzzy Clustering: A New Computational Intelligence Method. Soft Computing 20(9) (2016) 3544–3562.
20. Pawlak, Z.: Rough Sets. International Journal of Computer and Information Sciences 11 (5) (1982) 341–356.
21. Fodor, J., Yager, R. R.: Fuzzy Set Theoretic Operations and Quantifers. Fundermentals of Fuzzy Sets. Klwuer (2000).
22. Dubois, D., Prade, H.: Rough Fuzzy Sets and Fuzzy Rough Sets. International Journal of General Systems 17 (1990) 191–209.
23. Yao, Y.Y: Combination of Rough and Fuzzy Sets Based on $\alpha-$ level sets. Rough sets and Data mining: analysis for imprecise data. Kluwer Academic Publisher, Boston (1997) 301–321.
24. Broumi, S. and Smarandache, F.: Rough neutrosophic sets. Italian Journal of Pure and Applied Mathematics, N.32, (2014) 493–502.
25. Broumi, S. and Smarandache, F.: Lower and upper soft interval valued neutrosophic rough approximations of an IVNSS-relation, Sisom& Acoustics, (2014) 8 pages.
26. Broumi, S. and Smarandache, F.: Interval–Valued Neutrosophic Soft Rough Set, International Journal of Computational Mathematics. Volume 2015 (2015), Article ID 232919, 13 pages http://dx.doi.org/10.1155/2015/232919.
27. Cuong, B. C., Phong, P. H. and Smarandache, F.: Standard Neutrosophic Soft Theory: Some First Results. Neutrosophic Sets and Systems 12 (2016) 80–91.
28. Thao, N. X., Dinh, N. V.: Rough Picture Fuzzy Set and Picture Fuzzy Topologies. Journal of Science computer and Cybernetics 31 (3) (2015) 245–254.
29. Thao, N.X., Cuong. B. C., Smarandache, F.: Rough Standard Neutrosophic Sets: An Application on Standard Neutrosophic Information Systems. International Conference on Communication, Management and Information Technology, in press.