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New inclusion relation of neutrosophic sets with applications and related lattice structure

Xiaohong Zhang[1,2] · Chunxin Bo[3] · Florentin Smarandache[4] · Jianhua Dai[5]

Abstract

The main purpose of this paper is to study the inclusion relations of neutrosophic sets and some applications in multiple attribute decision making. At first, the existing two definitions of inclusion relation (called type-1 and type-2 inclusion relation, respectively) of neutrosophic sets are analyzed, and the deficiencies of type-1 and type-2 inclusion relations are illustrated by examples (in fact, they are actually two extreme cases). Second, a new definition of inclusion relation of neutrosophic sets (call it type-3 inclusion relation) is introduced, and a new method of ranking of neutrosophic sets is given. The effectiveness of the ranking method is presented by some application examples in multiple attribute decision making. Finally, type-3 inclusion relation of neutrosophic sets and related lattice structure are investigated in a systematic way, the definitions of type-3 union and type-3 intersection operations are proposed, and the following important result is proved: all of neutrosophic sets based on a certainty domain constitute a generalized De Morgan algebra (non-distributive lattice) with respect to type-3 union, type-3 intersection and complement operations. From this, the essential difference between neutrosophic set and fuzzy set (and intuitionistic fuzzy set) is clarified theoretically.

Keywords Neutrosophic set · Fuzzy set · Inclusion relation · Multiple attribute decision · Lattice

1 Introduction

The notion of neutrosophic set is proposed by Smarandache in 1998 (see [20–22]). Now, neutrosophic set is gaining significant attention in solving many real life problems that involve uncertainty, impreciseness, incompleteness, inconsistent, and indeterminacy. The theory of netrosophic set has been applied in multiple attribute decision making, computational intelligence, machine learning, image processing, medical diagnosis, fault diagnosis, optimization design, algebraic systems and so on (see [2, 7, 8, 11–17, 28–33, 35–39]). Moreover, a lot of new neutrosophic theories have been proposed, for examples, neutrosophic cubic set (see [9, 10, 34]), neutrosophic soft set and neutrosophic rough set (see [3, 6, 25]).

The core idea of neutrosophic theory is to describe the uncertainty of concepts from three mutually independent aspects. In fact, this idea has many different forms of expression in the academic circle. For examples: (1) from the rough set, interval set, granular computing and other research nutrition, the famous scholar Yao creatively put forward three-way decision theory, has become a hot research topic in related fields (see [26, 27]). A very interesting research

✉ Xiaohong Zhang
zhangXiaohong@sust.edu.cn; zhangxh@shmtu.edu.cn

Chunxin Bo
201640311001@stu.shmtu.edu.cn

Florentin Smarandache
smarand@unm.edu

Jianhua Dai
david.joshua@qq.com

1   School of Arts and Sciences, Shaanxi University of Science and Technology, Xi'an 710021, People's Republic of China

2   College of Arts and Sciences, Shanghai Maritime University, Shanghai 201306, People's Republic of China

3   College of Information Engineering, Shanghai Maritime University, Shanghai 201306, People's Republic of China

4   Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA

5   Key Laboratory of High Performance Computing and Stochastic Information Processing (Ministry of Education of China), Hunan Normal University, Changsha 410081, Hunan, People's Republic of China

trend is that the connection between neutrosophic theory and three-way decision theory is concerned in [18]. (2) A picture fuzzy set (proposed by Cuong, see [1, 4–6]) can be regarded as a special neutrosophic set, so relevant research can be incorporated into the framework of neutrosophic theory. In fact, the notion of picture fuzzy set is called standard neutrosophic set in [6]. (3) Wang et al. introduced the new notions of ternary fuzzy set and ternary fuzzy number in [23]; by simple comparison, we find that the two concepts of picture fuzzy set and ternary fuzzy set are essentially equivalent. We believe that the integration of these theories will provide more effective methods and tools for solving uncertainty problems.

In the neutrosophic set, truth-membership, indeterminacy-membership, and falsity-membership are represented independently. If U is a set, a neutrosophic set defined on the universe U assigns to each element $x \in U$, a triple $(T(x), I(x), F(x))$, where $T(x)$, $I(x)$ and $F(x)$ are standard or non-standard elements of non-standard unit interval $]0^-, 1^+[= 0^- \bigcup [0. 1] \bigcup 1^+$. T is the degree of truth-membership in the set U, I is the degree of indeterminacy-membership in the set U and F is the degree of non-membership in the set U. In this paper we work with special neutrosophic sets that their degrees of memberships are standard real numbers in $[0, 1]$, they are called single valued neutrosophic set (see [24]). It can be seen from this, neutrosophic set is a powerful general formal framework that generalizes the concepts of fuzzy set and intuitionistic fuzzy set.

For a set theory, inclusion relation is a basic concept; it plays a fundamental role in defining the union and intersection operations between sets, as well as the basis for related applications. From the existing literature, there are two different definitions of the inclusion relation of neutrosophic sets. In this paper, we gave some examples to illustrate the shortcomings of the two inclusion relations, and then propose a new inclusion relation (call it type-3 inclusion relation). Based on type-3 inclusion relation, we establish a new ranking method of neutrosophic sets and apply it to multiple attribute decision making problems. At the same time, this paper also studies the union and intersection operations corresponding to type-3 inclusion relation and the related lattice structure. Using these research results, we reveal the essential difference between neutrosophic set and fuzzy set (interval-valued fuzzy set, intuitionistic fuzzy set, and so on) from an algebraic point of view.

2 Neutrosophic sets and type-1 (type-2) inclusion relations

Definition 2.1 [20, 21] Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity- membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$. That is, $T_A(x): X \to]^-0, 1^+[$, $I_A(x): X \to]^-0, 1^+[$, and $F_A(x): X \to]^-0, 1^+[$. Thus, there is no restriction on the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$, so $^-0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.

Definition 2.2 [24] Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A in X is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$. Then, a single valued neutrosophic set A can be denoted by

$$A = \left\{ \langle x, T_A(x), I_A(x), F_A(x) \rangle | x \in X \right\},$$

where $T_A(x), I_A(x), F_A(x) \in [0, 1]$ for each point x in X. Therefore, the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$ satisfies the condition $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

For convenience, "single valued neutrosophic set" is abbreviated to "neutrosophic set" later. In this paper, we use the symbol $NS(X)$ to denote the set of all single valued neutrosophic set in X.

Two single valued neutrosophic sets A and B are equal, written as $A = B$, if and only if $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$ and $F_A(x) = F_B(x)$ for any x in X.

For the inclusion relation of neutrosophic sets, there are two different definitions in the literature. An original definition is proposed by Smarandache (see [20–22]), we call it type-1 inclusion relation in this paper, denote it by $\subseteq_1$; another one is used in some papers (see [2, 3, 24]), we call it type-2 inclusion relation in this paper, denote it by $\subseteq_2$.

Definition 2.3 [20–22] A single valued neutrosophic set A is contained in the other single valued neutrosophic set B, denote $A \subseteq_1 B$, if and only if $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$ for any x in X.

It is easy to verify that the above inclusion relation $\subseteq_1$ is a partial ordered relation on $NS(X)$. We called it type-1 inclusion relation in this paper.

Definition 2.4 [2, 3, 24] A single valued neutrosophic set A is contained in the other single valued neutrosophic set B, denote $A \subseteq_2 B$, if and only if $T_A(x) \le T_B(x)$, $I_A(x) \le I_B(x)$, $F_A(x) \ge F_B(x)$ for any x in X.

Obviously, the above inclusion relation $\subseteq_2$ is also a partial ordered relation on $NS(X)$. We called it type-2 inclusion relation in this paper.

Proposition 2.1 *Two single valued neutrosophic sets A and B are equal, if and only if $A \subseteq_1 B$ and $B \subseteq_1 A$, if and only if $A \subseteq_2 B$ and $B \subseteq_2 A$.*

For set theory, union and intersection operations are corresponding to inclusion relation. Therefore, there are two kinds of union and intersection operations corresponding to the two inclusion relations mentioned above. Interestingly, the complement operation uses the same definition in this paper (in fact, there are indeed different ways of defining complement operation in the literature, and we will discuss this problem in another paper). The basic definitions of these concepts are listed as follows.

Definition 2.5 [20–22] The type-1 union of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cup_1 B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x) = \max(T_A(x), T_B(x)), I_C(x) = \min(I_A(x), I_B(x)),$$
$$F_C(x) = \min(F_A(x), F_B(x)), \forall x \in X.$$

Definition 2.6 [2, 3, 24] The type-2 union of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cup_2 B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x) = \max(T_A(x), T_B(x)), I_C(x) = \max(I_A(x), I_B(x)),$$
$$F_C(x) = \min(F_A(x), F_B(x)), \forall \in X.$$

Definition 2.7 [20–22] The type-1 intersection of two neutrosophic sets A and B is a neutrosophic set D, written as $D = A \cap_1 B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_D(x) = \min(T_A(x), T_B(x)), I_D(x) = \max(I_A(x), I_B(x)),$$
$$F_D(x) = \max(F_A(x), F_B(x)), \forall x \in X.$$

Definition 2.8 [2, 3, 24] The type-2 intersection of two neutrosophic sets A and B is a neutrosophic set D, written as $D = A \cap_2 B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_D(x) = \min(T_A(x), T_B(x)), I_D(x) = \min(I_A(x), I_B(x)),$$
$$F_D(x) = \max(F_A(x), F_B(x)), \forall \in X.$$

Definition 2.9 The complement of a neutrosophic set A is denote by A^c, is defined as

$$T_{A^c}(x) = F_A(x), I_{A^c}(x) = 1 - I_A(x), F_{A^c}(x) = T_A(x), \forall \in X.$$

We can prove the following important proposition (the proof is omitted).

Definition 2.10 [19] A generalized De Morgan algebra (or GM-algebra) is a universal algebra $(M, \vee, \wedge, 0, 1)$ where $(M, \wedge, \vee, 0, 1)$ is a bounded lattice and the unary operation satisfies the identities:

GM1 : $x = (x^-)^-$;
GM2 : $(x \wedge y)^- = x^- \vee y^-$;
GM3 : $1^- = 0$.

For a generalized De Morgan algebra $(M, \wedge, \vee, ^-, 0, 1)$, if $(M, \wedge, \vee, 0, 1)$ is a bounded distributive lattice, then $(M, \vee, \wedge, ^-, 0, 1)$ is a De Morgan algebra.

Proposition 2.2 *Let X be a universe. Then $(NS(X), \subseteq_1, \cup_1, \cap_1, ^c, (0, 1, 1), (1, 0, 0))$ is a De Morgan algebra, and $(NS(X), \subseteq_2, \cup_2, \cap_2, ^c, (0, 0, 1), (1, 1, 0))$ is also a De Morgan algebra.*

3 Type-3 inclusion relation of neutrosophic sets

First, we analyze the inadequacies of the type-1 and type-2 inclusion relations of neutrosophic sets.

For any $A, B \in NS(X)$,

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\};$$
$$B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle | x \in X\}.$$

By Definition 2.3 and 2.4, we know that

$$A \subseteq_1 B \Rightarrow T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x), \forall x \in X.$$
$$A \subseteq_2 B \Rightarrow T_A(x) \leq T_B(x), I_A(x) \leq I_B(x), F_A(x) \geq F_B(x), \forall x \in X.$$

Thus,

$$A \subseteq_1 B \Rightarrow T_A(x) \leq T_B(x), I_A(x) + F_A(x) \geq I_B(x) + F_B(x), \forall x \in X.$$
$$A \subseteq_2 B \Rightarrow T_A(x) + I_A(x) \leq T_B(x) + I_B(x), F_A(x) \geq F_B(x), \forall x \in X.$$

This means that type-1 and type-2 inclusion relations are actually divided three membership functions into two groups, and then determines the order relation using the method similar to intuitionistic fuzzy sets; the difference between the two types is that one calculates the uncertain part in the truth-part and the other is to calculate the uncertain part in the falsity-part. In other words, the two inclusion relations do not really take advantage of the three membership functions; they are actually two extreme ways of handling. From another point of view, since $I(x)$

represent indeterminacy, it should not be added to the $T(x)$ or $F(x)$.

Now, we give following example to illustrate the inadequacies of the two inclusion relations.

Example 3.1 Let X be a universe, $A, B, C, D \in NS(X)$. If

$T_A(x) = 0.5, I_A(x) = 0.4, F_A(x) = 0.1, \forall x \in X;$
$T_B(x) = 0.5, I_B(x) = 0.3, F_B(x) = 0.1, \forall x \in X;$
$T_C(x) = 0.6, I_C(x) = 0.1, F_C(x) = 0.25, \forall x \in X;$
$T_D(x) = 0.6, I_D(x) = 0.09, F_D(x) = 0.1, \forall x \in X.$

Then
$A \subseteq_1 B$, but $B \subseteq_2 A$;
$C \subseteq_1 D$, but $C \subsetneq_2 D$ and $D \subsetneq_2 C$.

Assuming that the three membership functions in the neutrosophic sets are understood as the proportion of the votes in the voting process that are agreed, neutral (indeterminacy) or against votes, the above results show that type-1 and type-2 inclusion relations are not reasonable:

1. $T_A(x) = T_B(x) = 0.5$, $F_A(x) = F_A(x) = 0.1$, and the number of neutral, $I_A(x) = 0.4 > 0.3 = I_B(x)$, why $A \subseteq_1 B$? This is unjustifiable, it follows that the definition of type-1 inclusion relation is inadequate;
2. $T_C(x) = T_D(x) = 0.6$, $F_C(x) = 0.25 > 0.1 = F_D(x)$, and the difference between $I_C(x)$ and $I_D(x)$ is very small. Intuitively, C is less than D, but C and D can not be compared by using $\subseteq_2$, it follows that the definition of type-2 inclusion relation is inadequate.

In order to overcome the shortcomings of the above two kinds of inclusion relations, we introduce type-3 inclusion relation. The main idea is to give full play to the three membership functions and subdivide various possible situations, it can be regarded as the generalization of related methods in [1, 5].

Definition 3.1 Let A and B be two neutrosophic sets in the universe X. The type-3 inclusion relation $\subseteq_3$ is defined as follows: $A \subseteq_3 B$ if and only if

$\forall x \in X, \ \big(T_A(x) < T_B(x), F_A(x) \geq F_B(x)\big),$
$\text{or } \big(T_A(x) = T_B(x), F_A(x) F_B(x)\big),$
$\text{or } \big(T_A(x) = T_B(x), F_A(x) = F_B(x) \text{ and } I_A(x) \leq I_B(x)\big).$

For Example 3.1, we can get that $B \subseteq_3 A$ and $C \subseteq_3 D$.

The following unconventional voting example shows that the three independent membership functions in neutrosophic sets have practical application background.

Example 3.2 Take a voting of a region's deputies for example, where the voting rules are special, which allows voters to choose multiple options at the same time. Assume that there are three candidates (A, B, C) determined, and two deputies should be chosen. The provisions of the voting are given as follows:

1. For each candidate can choose in favor (draw $\sqrt{}$), neutral (draw $\bigcirc$), against (draw $\times$);
2. You can select multiple options;
3. Do not mark with any symbol if abstain from voting, that is, you may not choose any option for any candidate.

In the end, there are 100 valid ballots. The final vote count of candidate is given as follows:

A: 75 people vote for, 40 people neutral, 20 people against
B: 40 people vote for, 25 people neutral, 50 people against
C: 60 people vote for, 60 people neutral, 75 people against

We used three neutrosophic sets to represent the voting results, that is, A, B and C are regarded as neutrosophic sets in the single point set $X = \{x_0 : \text{ideal candidate}\}$:

$T_A(x_0) = 0.75, I_A(x_0) = 0.4, F_A(x_0) = 0.2;$
$T_B(x_0) = 0.6, I_B(x_0) = 0.25, F_B(x_0) = 0.5;$
$T_C(x_0) = 0.6, I_C(x_0) = 0.6, F_C(x_0) = 0.75$

By Definition 3.2, we have

$B \subseteq_3 A, C \subseteq_3 A, \text{ and } C \subseteq_3 B.$

Then the candidates A and B win two of the three qualifications.

Proposition 3.1 *Let X be a universe, $A, B, C \in NS(X)$. Then, (1) $A \subseteq_3 A$; (2) $(A \subseteq_3 B, B \subseteq_3 A) \Rightarrow A = B$; (3) $(A \subseteq_3 B, B \subseteq_3 C) \Rightarrow A \subseteq_3 C$.*

Definition 3.2 Let A and B be neutrosophic sets in X. The type-3 union, type-3 intersection and complement are defined as follows:

1. $A \cup_3 B = \begin{cases} A = \left\{ \left(x, T_A(x), I_A(x), F_A(x) \right) | x \in X \right\}, & \text{if } B \subseteq_3 A \\ B = \left\{ \left(x, T_B(x), I_B(x), F_B(x) \right) | x \in X \right\}, & \text{if } A \subseteq_3 B \\ \left\{ \left(x, \max\left(T_A(x), T_B(x) \right), 0, \min\left(F_A(x), F_B(x) \right) \right) | x \in X \right\}, & \text{otherwise} \end{cases}$

2. $A \cap_3 B = \begin{cases} A = \left\{ \left(x, T_A(x), I_A(x), F_A(x) \right) | x \in X \right\}, & \text{if } A \subseteq_3 B \\ B = \left\{ \left(x, T_B(x), I_B(x), F_B(x) \right) | x \in X \right\}, & \text{if } B \subseteq_3 A \\ \left\{ \left(x, \min\left(T_A(x), T_B(x) \right), 1, \max\left(F_A(x), F_B(x) \right) \right) | x \in X \right\}, & \text{otherwise} \end{cases}$

3. $A^c = \left\{ \left(x, F_A(x), 1 - I_A(x), T_A(x) \right) | x \in X \right\}$.

It is easy to verify the following conclusions (the proofs are omitted here, in Sect. 5, we will give the proof in a more general framework).

Proposition 3.2 *Let A, B be neutrosophic sets in X, then*

1. $A \cup_3 B$ *is the least upper bound (supremum) of A and B under the order relation $\subseteq_3$;*
2. $A \cap_3 B$ *is the greatest lower bound (infimum) of A and B under the order relation $\subseteq_3$.*

Proposition 3.3 *Let A, B, C be neutrosophic sets in X, then*

1. $A \cap_3 A = A, A \cup_3 A = A$;
2. $A \cap_3 B = B \cap_3 A, A \cup_3 B = B \cup_3 A$;
3. $(A \cap_3 B) \cap_3 C = A \cap_3 (B \cap_3 C), (A \cup_3 B) \cup_3 C = A \cup_3 (B \cup_3 C)$;
4. $A \cap_3 (B \cup_3 A) = A, A \cup_3 (B \cap_3 A) = A$;
5. $A \subseteq_3 B \Leftrightarrow A \cup_3 B = B; A \subseteq_3 B \Leftrightarrow A \cap_3 B = A$.

Proposition 3.4 *Let A and B be neutrosophic sets in X, then*

1. $(A^c)^c = A$;
2. $(A \cap_3 B)^c = A^c \cup_3 B^c$;
3. $(A \cup_3 B)^c = A^c \cap_3 B^c$.

Theorem 3.1 *Let NS(X) be the set of all neutrosophic sets in universe X. Denote*

$$0_{\text{type}-3} = \{(x, 0, 0, 1) | x \in X\},$$
$$1_{\text{type}-3} = \{(x, 1, 1, 0) | x \in X\}.$$

Then $(NS(X), \cup_3, \cap_3, {}^c, 0_{\text{type-3}}, 1_{\text{type-3}})$ is a generalized De Morgan algebra.

4 A new ranking method with applications

The inclusion relations of neutrosophic sets are closely related to the ranking methods of the neutrosophic numbers, and the ranking methods play a basic important role in the application of neutrosophic sets in multi attribute decision making. In [22], using three functions: neutrosophic score function, neutrosophic accuracy function, and neutrosophic certainty function, Smarandache define a total order on the set of neutrosophic numbers. In this section, a new ranking method will be given with the help of the idea of type-3 inclusion relation in Sect. 3.

As a preparation, the concept of neutrosophic number and the order relation on it will be introduced.

If $t, i, f \in [0, 1]$, then (t, i, f) is called a single-valued neutrosophic number. Similar to [5], we consider the set D^* defined by

$$D^* = \left\{ x = (x_1, x_2, x_3) \mid x_1, x_2, x_3 \in [0, 1] \right\}.$$

From now on, we will assume that if $x, y \in D^*$ or $(t, i, f) \in D^*$, then x_1 (or t), x_2 (or i) and x_3 (or f) denote, respectively, the first, the second and the third component. We denote the units of D^* by $1_{D^*} = (1, 1, 0)$ and $0_{D^*} = (0, 0, 1)$, respectively.

Obviously, for every neutrosophic set

$$A = \left\{ \left(x, T_A(x), I_A(x), F_A(x) \right) | x \in X \right\},$$

It correspond with an D^*-fuzzy set, i.e., a mapping

$$A : X \to D^* : x \mapsto \left(T_A(x), I_A(x), F_A(x) \right).$$

By Definition 2.3 and 2.4, the type-1 and type-2 inclusion relation of neutrosophic sets are based on the following order relation on D^*:

$$\forall x, y \in D^*, \ x \leq_1 y \Leftrightarrow (x_1 \leq y_1) \text{ and } (x_2 \geq y_2) \text{ and } (x_3 \geq y_3).$$

$$\forall x, y \in D^*, \ x \leq_2 y \Leftrightarrow (x_1 \leq y_1) \text{ and } (x_2 \leq y_2) \text{ and } (x_3 \geq y_3).$$

By Definition 3.1, type-3 inclusion relation is based on the following order relation on D^*:

$$x \leq_3 y \Leftrightarrow \ ((x_1 < y_1) \text{ and } (x_3 \geq y_3)),$$
$$\text{or } ((x_1 = y_1) \text{ and } (x_3 > y_3)),$$
$$\text{or } ((x_1 = y_1) \text{ and } (x_3 = y_3) \text{ and } (x_2 \leq y_2)).$$

Remark 4.1 In next section, we will give a rigorous proof of the lattice properties of type-3 order relation on D^*. Note that, if for $x, y \in D^*$ that neither $x \leq_3 y$ nor $y \leq_3 x$, then x and y are incomparable, denoted as $x \|_{\leq_3} y$. Similar symbols are also applicable to type-1 and type-2 order relations.

Definition 4.1 Let (t, i, f) be a single-valued neutrosophic-number. The type-3 score function is defined by

$$s : D^* \to [0, 1], s(t, i, f) = \frac{t + (1 - f)}{2}.$$

The type-3 accuracy function is defined by

$$h : D^* \to [0, 1], h(t, i, f) = \frac{t}{t + (1 - f)}.$$

It is easy to verify the following conclusion which shows that type-3 score function and type-3 ordered relation are compatible.

Proposition 4.1 Let (t_1, i_1, f_1) and (t_2, i_2, f_2) be two single-valued neutrosophic numbers. Then

$$(t_1, i_1, f_1) \leq_3 (t_2, i_2, f_2) \Rightarrow s(t_1, i_1, f_1) \leq s(t_2, i_2, f_2).$$

It is easy to verify the following conclusion which shows that type-3 score function and type-3 accuracy function determine the 1, 3 component of the neutrosophic number (not related to the second component).

Proposition 4.2 Let (t_1, i_1, f_1) and (t_2, i_2, f_2) be two single-valued neutrosophic numbers. Then

$$s(t_1, i_1, f_1) = s(t_2, i_2, f_2) \text{ and } h(t_1, i_1, f_1) = h(t_2, i_2, f_2)$$
$$\Rightarrow (t_1 = t_2, f_1 = f_2).$$

Now, we propose a new ranking method of the neutrosophic numbers.

Definition 4.2 Let (t_1, i_1, f_1) and (t_2, i_2, f_2) be two single-valued neutrosophic numbers. Then

1. if $s(t_1, i_1, f_1) < s(t_2, i_2, f_2)$, then $(t_1, i_1, f_1) \prec (t_2, i_2, f_2)$;
2. if $s(t_1, i_1, f_1) = s(t_2, i_2, f_2)$, $h(t_1, i_1, f_1) < h(t_2, i_2, f_2)$, then $(t_1, i_1, f_1) \prec (t_2, i_2, f_2)$;
3. i f $\quad s(t_1, i_1, f_1) = s(t_2, i_2, f_2)$, $h(t_1, i_1, f_1) = h(t_2, i_2, f_2)$, $i_1 < i_2$, then $(t_1, i_1, f_1) \prec (t_2, i_2, f_2)$; $i_1 = i_2$, then $(t_1, i_1, f_1) = (t_2, i_2, f_2)$.

It is easy to verify the following conclusions which show that the relation $\preccurlyeq$ ($\prec$ or $=$) is a total order on D^*.

Proposition 4.3 For any (t_1, i_1, f_1), (t_2, i_2, f_2), and (t_3, i_3, f_3) in D^*, the following assertions hold:

1. $(t_1, i_1, f_1) \preccurlyeq (t_1, i_1, f_1)$;
2. if $(t_1, i_1, f_1) \preccurlyeq (t_2, i_2, f_2)$ and $(t_2, i_2, f_2) \preccurlyeq (t_1, i_1, f_1)$, then $(t_1, i_1, f_1) = (t_2, i_2, f_2)$;
3. if $(t_1, i_1, f_1) \preccurlyeq (t_2, i_2, f_2)$ and $(t_2, i_2, f_2) \preccurlyeq (t_3, i_3, f_3)$, then $(t_1, i_1, f_1) \preccurlyeq (t_3, i_3, f_3)$;
4. if $(t_1, i_1, f_1) \not\preccurlyeq (t_2, i_2, f_2)$, then $(t_2, i_2, f_2) \prec (t_1, i_1, f_1)$.

Example 4.1 Let us consider two single-valued neutrosophic numbers:

$$\alpha = (0.6,\ 0.1,\ 0.4), b = (0.5,\ 0.1,\ 0.3).$$

By Definition 4.1 and 4.2, we have

$$s(\alpha) = 0.6, s(\beta) = 0.6;$$
$$h(\alpha) = 0.5, h(\beta) = 0.417.$$

Hence, $\beta \prec \alpha$. This is in accordance with the conclusion by using neutrosophic score function, neutrosophic accuracy function, and neutrosophic certainty function in [22] (see Page 23, 24 in second extended and improved edition).

Now, we discuss some examples for multi-criteria decision-making problem of engineering alternatives by using the above ranking method of the neutrosophic numbers.

Example 4.2 Let us consider the decision-making problem adapted from [31]. There is an investment company, which wants to invest a sum of money in the best option. There is a panel with four possible alternatives to invest the money: (1) A_1 is a car company; (2) A_2 is a food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investment company must take a decision according to the following three criteria: (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is the environmental impact. Then, the weight vector of the criteria is given by $W = (0.35, 0.25, 0.4)$.

For the evaluation of an alternative A_i $(i = 1, 2, 3, 4)$ with respect to a criterion C_j $(j = 1, 2, 3)$, it is obtained from the questionnaire of a domain expert. For example, when we ask the opinion of an expert about an alternative A_1 with respect to a criterion C_1, he or she may say that the possibility in which the statement is good is 0.4 and the statement is poor is 0.3 and the degree in which he or she is not sure is 0.2. For the neutrosophic notation, it can be expressed as $\alpha_{11} = (0.4, 0.2, 0.3)$. Thus, when the four possible alternatives with respect to the above three criteria are evaluated by the expert, we can obtain the following simplified neutrosophic decision matrix D.

In [31], the author obtains the weighted arithmetic average value (aggregating simplified neutrosophic value) α_i for A_i $(i = 1, 2, 3, 4)$:

$$\alpha_1 = (0.3268, 0.2000, 0.3881),$$
$$\alpha_2 = (0.5627, 0.1414, 0.2000),$$
$$\alpha_3 = (0.4375, 0.2416, 0.2616),$$
$$\alpha_4 = (0.5746, 0.1555, 0.1663).$$

$$D = \begin{bmatrix} (0.4, 0.2, 0.3) & (0.4, 0.2, 0.3) & (0.2, 0.2, 0.5) \\ (0.6, 0.1, 0.2) & (0.6, 0.1, 0.2) & (0.5, 0.2, 0.2) \\ (0.3, 0.2, 0.3) & (0.5, 0.2, 0.3) & (0.5, 0.3, 0.2) \\ (0.7, 0.0, 0.1) & (0.6, 0.1, 0.2) & (0.4, 0.3, 0.2) \end{bmatrix}.$$

By Definition 4.1 and 4.2, we have

$$s(\alpha_1) = 0.4694, \quad s(\alpha_2) = 0.6814, \quad s(\alpha_3) = 0.5880,$$
$$s(\alpha_4) = 0.7042;$$

$$\alpha_1 \prec \alpha_3 \prec \alpha_2 \prec \alpha_4.$$

Therefore, the alternative A_4 is the best choice among all the alternatives.

Moreover, by the weighted geometric average operator, we can obtain the following computational results:

$$\alpha_1 = (0.3031, \ 0.2000, \ 0.3680),$$

$$\alpha_2 = (0.5578, \ 0.1320, \ 0.2000),$$

$$\alpha_3 = (0.4181, \ 0.2352, \ 0.2551),$$

$$\alpha_4 = (0.5385, \ 0, \ 0.1569).$$

By Definition 4.1 and 4.2, we have

$$s(\alpha_1) = 0.4676, \ s(\alpha_2) = 0.6789, \ s(\alpha_3) = 0.5815, \ s(\alpha_4) = 0.6908;$$
$$\alpha_1 \prec \alpha_3 \prec \alpha_2 \prec \alpha_4.$$

Thus, the alternative A_4 is the best choice.

Hence, we get exactly the same results as in [31].

Example 4.3 Let us consider the multi-attribute group decision making problem (about the air quality evaluation) adapted from [12]. To evaluate the air quality of Guangzhou for the 16th Asian Olympic Games, the air quality in Guangzhou for the Novembers of 2006, 2007, 2008 and 2009 were collected in order to find out the trends and to forecast the situation in 2010. There are three air-quality monitoring stations which can be seen as decision makers. There are three measured indexes, namely, SO_2 (C_1), NO_2 (C_2) and PM_{10} (C_3). The measured values from air-quality monitoring stations under these indexes are shown in Tables 1, 2 and 3 in [12], and they can be expressed by neutrosophic numbers. Let $\{S_1, S_2, S_3, S_4\} = \{$November of 2006, November of 2007, November of 2008, November of 2009$\}$ be the set of alternatives, please give the rank of air quality from 2006 to 2009.

According the method in [12], derive the collective overall values:

$$r_1 = (0.288, \ 0.396, \ 0.304),$$

$$r_2 = (0.329, \ 0.344, \ 0.310),$$

$$r_3 = (0.345, \ 0.430, \ 0.210),$$

$$r_4 = (0.417, \ 0.367, \ 0.183).$$

By Definition 4.1 and 4.2, we have

$$s(r_1) = 0.492, \ s(r_2) = 0.5095, \ s(r_3) = 0.5675, \ s(r_4) = 0.617;$$
$$r_1 \prec r_2 \prec r_3 \prec r_4.$$

So, the best alternative is S_4, i.e., the best air quality Guangzhou is November of 2009 among the Novembers of 2006, 2007, 2008, and 2009. This is in accordance with the conclusion in [12].

5 The related lattice structure

In this section, we discuss the lattice structures $(D^*, \leq_3)$ and $(NS(X), \subseteq_3)$, where $\leq_3$ is defined as follows: $\forall x, y \in D^*$,

$$x \leq_3 y \Leftrightarrow \ \ ((x_1 < y_1) \text{ and } (x_3 \geq y_3)),$$
$$\text{or } ((x_1 = y_1) \text{ and } (x_3 > y_3)),$$
$$\text{or } ((x_1 = y_1) \text{ and } (x_3 = y_3) \text{ and } (x_2 \leq y_2)).$$

Proposition 5.1 *Type-3 ordering relation $\leq_3$ on D^* is a partial ordering relation.*

Proof Assume that $x, y, z \in D^*$.

i.   Reflexivity: $x \leq_3 x$, it is obvious.
ii. Ant-symmetry: if $x \leq_3 y$ and $y \leq_3 x$, then

1. Case 1: $x_1 < y_1$ and $x_3 \geq y_3$. By the definition of $y \leq_3 x$, we can get $x_1 \geq y_1$, this is contradictory.
2. Case 2: $x_1 = y_1$ and $x_3 > y_3$. By the definition of $y \leq_3 x$, we can get $x_3 \leq y_3$, it is contradictory.
3. Case 3: $x_1 = y_1$ and $x_3 = y_3$. By $x \leq_3 y$, we have $x_2 \leq y_2$; also using $y \leq_3 x$, we have $x_2 \geq y_2$. Thus $x_2 = y_2$.

Summing up the above, if $x \leq_3 y$ and $y \leq_3 x$, then $x_1 = y_1$ and $x_2 = y_2$ and $x_3 = y_3$, that is $x = y$.

iii. Transitivity: if $x \leq_3 y$, $y \leq_3 z$, then

1. Case 1: $(x_1 < y_1, x_3 \geq y_3)$ and $(y_1 < z_1, y_3 \geq z_3)$. It follows that $x_1 < z_1$ and $x_3 \geq z_3$, thus $x \leq_3 z$.
2. Case 2: $(x_1 = y_1, x_3 > y_3)$ and $(y_1 = z_1, y_3 > z_3)$. It follows that $x_1 = z_1$ and $x_3 > z_3$, thus $x \leq_3 z$.
3. Case 3: $(x_1 = y_1, x_3 = y_3, x_2 \leq y_2)$ and $(y_1 = z_1, y_3 = z_3, y_2 \leq z_2)$. It follows that $x_1 = z_1$, $x_3 = z_3$ and $x_2 \leq z_2$, thus $x \leq_3 z$.
4. Case 4: $(x_1 < y_1, x_3 \geq y_3)$ and $(y_1 = z_1, y_3 > z_3)$. It follows that $x_1 < z_1$ and $x_3 > z_3$, thus $x \leq_3 z$.
5. Case 5: $(x_1 < y_1, x_3 \geq y_3)$ and $(y_1 = z_1, y_3 = z_3, y_2 \leq z_2)$. It follows that $x_1 < z_1$, $x_3 \geq z_3$ and $y_2 \leq z_2$, thus $x \leq_3 z$.
6. Case 6: $(x_1 = y_1, x_3 > y_3)$ and $(y_1 < z_1, y_3 \geq z_3)$. It follows that $x_1 < z_1$ and $x_3 > z_3$, thus $x \leq_3 z$.
7. Case 7: $(x_1 = y_1, x_3 > y_3)$ and $(y_1 = z_1, y_3 = z_3, y_2 \leq z_2)$. It follows that $x_1 = z_1$, $x_3 > z_3$ and $y_2 \leq z_2$, thus $x \leq_3 z$.
8. Case 8: $(x_1 = y_1, x_3 = y_3, x_2 \leq y_2)$ and $(y_1 < z_1, y_3 \geq z_3)$. It follows that $x_1 < z_1$, $x_3 \geq z_3$ and $x_2 \leq y_2$, thus $x \leq_3 z$.
9. Case 9: $(x_1 = y_1, x_3 = y_3, x_2 \leq y_2)$ and $(y_1 = z_1, y_3 > z_3)$, then $x_1 = z_1$, $x_3 > z_3$ and $x_2 \leq y_2$, thus $x \leq_3 z$.

Summing up the above, if $x \leq_3 y$ and $y \leq_3 z$, then $x \leq_3 z$. The proof is completed.

Proposition 5.2 *For any x, y* $\in D^*$, *we define*

$$x \wedge_3 y = \begin{cases} x, & \text{if } x \leq_3 y \\ y, & \text{if } y \leq_3 x \\ (\min(x_1, y_1), 1, \max(x_3, y_3)), & \text{otherwise} \end{cases}$$

$$x \vee_3 y = \begin{cases} y, & \text{if } x \leq_3 y \\ x, & \text{if } y \leq_3 x \\ (\max(x_1, y_1), 0, \min(x_3, y_3)), & \text{otherwise} \end{cases}$$

Then $x \wedge_3 y$ is the greatest lower bound (infimum) of x, y, denote by $\inf(x, y)$; $x \vee_3 y$ is the least upper bound (supremum) of x, y, denote by $\sup(x, y)$. That is, $(D^*, \leq_3)$ is a lattice.

Proof If $x \leq_3 y$ or $y \leq_3 x$, by the definition of "$\wedge_3$", $x \wedge_3 y$ is the greatest lower bound of x, y, that is, $x \wedge_3 y = \inf(x, y)$. Moreover, if $x \leq_3 y$ or $y \leq_3 x$, then $x \vee_3 y$ is the least upper bound of x, y, that is, $x \vee_3 y = \sup(x, y)$.

Now, we suppose that $x ||_{\leq_3} y$. Thus, by the definition of "$\wedge_3$" and "$\vee_3$",

$$x \wedge_3 y = (\min(x_1, y_1), 1, \max(x_3, y_3)),$$
$$x \vee_3 y = (\max(x_1, y_1), 0, \min(x_3, y_3)).$$

i. To prove $x \wedge_3 y = \inf(x, y)$: Denote
$z = (z_1, z_2, z_3) = (\min(x_1, y_1), 1, \max(x_3, y_3))$.
Then $x_1 \geq \min(x_1, y_1) = z_1$, $x_3 \leq \max(x_3, y_3) = z_3$.
If $x_1 > z_1$ and $x_3 \leq z_3$, then $z \leq_3 x$.
If $x_1 = z_1$ and $x_3 < z_3$, then $z \leq_3 x$.
If $x_1 = z_1$ and $x_3 = z_3$, then $y_1 \geq x_1$, $y_3 \leq x_3$, it follows that $x \leq_3 y$ or $y \leq_3 x$. It is a contradiction with the condition $x ||_{\leq_3} y$. Hence we can get $z \leq_3 x$. Similarly, we can get $z \leq_3 y$. This means that z is the lower bound of x and y.

Next, we need prove that z is the greatest lower bound of x and y.

Assume $a = (a_1, a_2, a_3) \in D^*$ such that $a \leq_3 x$ and $a \leq_3 y$.

1. Case 1: $(a_1 < x_1, a_3 \geq x_3)$ and $(a_1 < y_1, a_3 \geq y_3)$. It follows that $a_1 < \min(x_1, y_1) = z_1$ and $a_3 \geq \max(x_3, y_3) = z_3$, thus $a \leq_3 z$.
2. Case 2: $(a_1 = x_1, a_3 > x_3)$ and $(a_1 = y_1, a_3 > y_3)$. It follows that $a_1 = \min(x_1, y_1) = z_1$ and $a_3 > \max(x_3, y_3) = z_3$, thus $a \leq_3 z$.
3. Case 3: $(a_1 = x_1, a_3 = x_3, a_2 \leq x_2)$ and $(a_1 = y_1, a_3 = y_3, a_2 \leq y_2)$. It follows that $a_1 = \min(x_1, y_1) = z_1$, $a_3 = \max(x_3, y_3) = z_3$ and $a_2 \leq \max(x_2, y_2) \leq 1 = z_2$. Thus $a \leq_3 z$.
4. Case 4: $(a_1 = x_1, a_3 > x_3)$ and $(a_1 < y_1, a_3 \geq y_3)$. It follows that $a_1 = x_1 = \min(x_1, y_1) = z_1$ and $a_3 \geq \max(x_3, y_3) = z_3$.
If $a_1 = x_1 = \min(x_1, y_1) = z_1$ and $a_3 > \max(x_3, y_3) = z_3$,

then $a \leq_3 z$; If $a_1 = x_1 = \min(x_1, y_1) = z_1$ and $a_3 = \max(x_3, y_3) = z_3$, from this and the hidden condition $a_2 \leq 1 = z_2$, we get $a \leq_3 z$.
5. Case 5: $(a_1 < x_1, a_3 \geq x_3)$ and $(a_1 = y_1, a_3 > y_3)$. It follows that $a_1 = y_1 = \min(x_1, y_1) = z_1$ and $a_3 \geq \max(x_3, y_3) = z_3$. Similar to Case 4, we can get $a \leq_3 z$.
6. Case 6: $(a_1 < x_1, a_3 \geq x_3)$ and $(a_1 = y_1, a_3 = y_3, a_2 \leq y_2)$. It follows that $y_1 < x_1$ and $y_3 \geq x_3$, so $y \leq_3 x$, it is a contradiction with hypothesis $x ||_{\leq_3} y$.
7. Case 7: $(a_1 = x_1, a_3 > x_3)$ and $(a_1 = y_1, a_3 = y_3, a_2 \leq y_2)$. It follows that $y_1 = x_1$ and $y_3 > x_3$, so $y \leq_3 x$, it is a contradiction with hypothesis $x ||_{\leq_3} y$.
8. Case 8: $(a_1 = x_1, a_3 = x_3, a_2 \leq x_2)$ and $(a_1 < y_1, a_3 \geq y_3)$. It follows that $x_1 < y_1$ and $x_3 \geq y_3$, so $x \leq_3 y$, it is a contradiction with hypothesis $x ||_{\leq_3} y$.
9. Case 9: $(a_1 = x_1, a_3 = x_3, a_2 \leq x_2)$ and $(a_1 = y_1, a_3 > y_3)$. It follows that $x_1 = y_1$ and $x_3 > y_3$, so $x \leq_3 y$, it is a contradiction with hypothesis $x ||_{\leq_3} y$.

Therefore, we always have $a \leq_3 z$. That is, $z = (\min(x_1, y_1), 1, \max(x_3, y_3))$ is the greatest lower bound of x, y.

ii. To prove $x \vee_3 y = \sup(x, y)$: Denote

$w = (w_1, w_2, w_3) = (\max(x_1, y_1), 0, \min(x_3, y_3))$.
Then $x_1 \leq \max(x_1, y_1) = w_1$, $x_3 \geq \min(x_3, y_3) = w_3$.
If $x_1 < w_1$ and $x_3 \geq w_3$, then $x \leq_3 w$.
If $x_1 = w_1$ and $x_3 > w_3$, then $x \leq_3 w$.
If $x_1 = w_1$ and $x_3 = w_3$, then $y_1 \leq x_1$, $y_3 \geq x_3$, so $y \leq_3 x$ or $x \leq_3 y$, it is a contradiction with hypothesis $x ||_{\leq_3} y$. Hence we always have $x \leq_3 w$.

Similarly, we can prove $y \leq_3 w$. Therefore, w is an upper bound of x and y.

Next, we need prove w is the least upper bound of x and y. Suppose $b = (b_1, b_2, b_3) \in D^*$ such that $x \leq_3 b$, $y \leq_3 b$.

1. Case 1: $(x_1 < b_1, x_3 \geq b_3)$ and $(y_1 < b_1, y_3 \geq b_3)$. It follows that $b_1 > \max(x_1, y_1) = w_1$ and $b_3 \leq \min(x_3, y_3) = w_3$. Thus $w \leq_3 b$.
2. Case 2: $(b_1 = x_1, x_3 > b_3)$ and $(b_1 = y_1, y_3 > b_3)$. It follows that $b_1 = \max(x_1, y_1) = w_1$, $b_3 < \min(x_3, y_3) = w_3$, thus $w \leq_3 b$.
3. Case 3: $(b_1 = x_1, b_3 = x_3, x_2 \leq b_2)$ and $(b_1 = y_1, b_3 = y_3, y_2 \leq b_2)$. It follows that $b_1 = \max(x_1, y_1) = w_1$, $b_3 = \min(x_3, y_3) = w_3$ and $b_2 \geq \max(x_2, y_2) \geq 0 = w_2$, thus $w \leq_3 b$.
4. Case 4: $(b_1 = x_1, x_3 > b_3)$ and $(y_1 < b_1, y_3 \geq b_3)$. It follows that $b_1 \geq \max(x_1, y_1) = w_1$ and $b_3 \leq \min(x_3, y_3) = w_3$. If $(b_1 > \max(x_1, y_1) = w_1, b_3 \leq \min(x_3, y_3) = w_3)$ or $(b_1 = \max(x_1, y_1) = w_1, b_3 < \min(x_3, y_3) = w_3)$, then $w \leq_3 b$; If $b_1 = \max(x_1, y_1) = w_1$ and $b_3 = \min(x_3, y_3) = w_3$, according the hidden condition $b_2 \geq 0 = w_2$, we can get $w \leq_3 b$.
5. Case 5: $(x_1 < b_1, x_3 \geq b_3)$ and $(b_1 = y_1, y_3 > b_3)$. It follows that $b_1 \geq \max(x_1, y_1) = w_1$ and $b_3 \leq \min(x_3, y_3) = w_3$, similar to Case 4, we can get $w \leq_3 b$.

6. Case 6: $(x_1 < b_1, x_3 \geq b_3)$ and $(b_1 = y_1, b_3 = y_3, y_2 \leq b_2)$. It follows that $x_1 < y_1$ and $x_3 \geq y_3$, so $x \leq_3 y$, it is a contradiction with hypothesis $x||_{\leq_3} y$.

7. Case 7: $(b_1 = x_1, x_3 > b_3)$ and $(b_1 = y_1, b_3 = y_3, y_2 \leq b_2)$. It follows that $y_1 = x_1$ and $y_3 < x_3$, so $x \leq_3 y$, it is a contradiction with hypothesis $x||_{\leq_3} y$.

8. Case 8: $(b_1 = x_1, b_3 = x_3, x_2 \leq b_2)$ and $(y_1 < b_1, y_3 \geq b_3)$. It follows that $y_1 < x_1$ and $y_3 \geq x_3$, so $y \leq_3 x$, it is a contradiction with hypothesis $x||_{\leq_3} y$.

9. Case 9: $(b_1 = x_1, b_3 = x_3, x_2 \leq b_2)$ and $(b_1 = y_1, y_3 > b_3)$. It follows that $y_1 = x_1$ and $y_3 > x_3$, so $y \leq_3 x$, it is a contradiction with hypothesis $x||_{\leq_3} y$.

Therefore, we always have $w \leq_3 b$. This means that $w = (\max(x_1, y_1), 0, \min(x_3, y_3))$ is the least upper bound of x, y.

Combing (i) and (ii), we get that $x \wedge_3 y = \inf(x, y)$, $x \vee_3 y = \sup(x, y)$, and $(D^*, \leq_3)$ is a lattice. The proof is completed.

Definition 5.1 For any $x, y \in D^*$, the type-3 union $x \vee_3 y$ and type-3 intersection $x \wedge_3 y$ are defined as Proposition 5.2; the complement is defined as following:

$$x^c = (x_1, x_2, x_3)^c = (x_3, 1 - x_2, x_1).$$

It is easy to verify the following conclusions.

Proposition 5.3 For any $x, y, z \in D^*$, we have

1. $x \wedge_3 x = x, \quad x \vee_3 x = x$;
2. $x \wedge_3 y = y \wedge_3 x, \quad x \vee_3 y = y \vee_3 x$;
3. $(x \wedge_3 y) \wedge_3 z = x \wedge_3 (y \wedge_3 z), (x \vee_3 y) \vee_3 z = x \vee_3 (y \vee_3 z)$;
   $x \wedge_3 (y \vee_3 x) = x, \quad x \vee_3 (y \wedge_3 x) = x$;
4. $x \leq_3 y \Leftrightarrow x \vee_3 y = y; x \leq_3 y \Leftrightarrow x \wedge_3 y = x$;

5. $(x^c)^c = x$.

Proposition 5.4 Let $x, y \in D^*$, then

1. $(x \wedge_3 y)^c = x^c \vee_3 y^c$;
2. $(x \vee_3 y)^c = x^c \wedge_3 y^c$.

Proof Assume that $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in D^*$. By Definition 5.1, we have

$x^c = (x_3, 1 - x_2, x_1)$, $y^c = (y_3, 1 - y_2, y_1)$.
If $y \leq_3 x$, then
Case 1: $y_1 < x_1$ and $y_3 \geq x_3$. It follows that $x^c \leq_3 y^c$. Thus $(x \vee_3 y)^c = y^c = x^c \vee_3 y^c$.
Case 2: $y_1 = x_1$ and $y_3 > x_3$. It follows that $x^c \leq_3 y^c$. Thus $(x \vee_3 y)^c = y^c = x^c \vee_3 y^c$.
Case 3: $y_1 = x_1$, $y_3 = x_3$ and $y_2 \leq x_2$. It follows that $1 - x_2 \leq 1 - y_2$. Thus $x^c \leq_3 y^c$, and $(x \vee_3 y)^c = y^c = x^c \vee_3 y^c$.

Similarly, if $x \leq_3 y$, then $(x \vee_3 y)^c = x^c = x^c \vee_3 y^c$.
If neither $y \leq_3 x$ nor $x \leq_3 y$, then

$$x \wedge_3 y = (\min(x_1, y_1), 1, \max(x_3, y_3))$$

Thus

$$(x \wedge_3 y)^c = (\max(x_3, y_3), 0, \min(x_1, y_1)),$$

$$x^c \vee_3 y^c = (x_3, 1 - x_2, x_1) \vee_3 (y_3, 1 - y_2, y_1)$$
$$= (\max(x_3, y_3), 0, \min(x_1, y_1)).$$

Hence $(x \wedge_3 y)^c = x^c \vee_3 y^c$.
By (1) and Proposition 5.3 (5), we can get that $(x \vee_3 y)^c = x^c \wedge_3 y^c$.

Theorem 5.1 Denote $1_{D^*} = (1, 1, 0)$ and $0_{D^*} = (0, 0, 1)$. Then $(D^*, \vee_3, \wedge_3, {}^c, 0_{D*}, 1_{D*})$ is a generalized De Morgan algebra.

Proof By Proposition 5.1–5.4 and Definition 2.10, we can get that $(D^*, \vee_3, \wedge_3, {}^c, 0_{D*}, 1_{D*})$ is a generalized De Morgan algebra.

The following example shows that $(D^*, \vee_3, \wedge_3, {}^c, 0_{D*}, 1_{D*})$ is not a De Morgan algebra, since distributive law is not true.

Example 5.1 Let

$x = (0.2, 0.8, 0.3)$,
$y = (0.6, 0.3, 0.4)$,
$z = (0.5, 0.9, 0.38)$.

Then

$(x \wedge_3 y) \vee_3 z = (0.2, 1, 0.4) \vee_3 (0.5, 0.9, 0.38) = (0.5, 0.9, 0.38)$;

$(x \vee_3 z) \wedge_3 (y \vee_3 z) = (0.5, 0, 0.3) \wedge_3 (0.6, 0, 0.38) = (0.5, 1, 0.38)$;

$(x \vee_3 y) \wedge_3 z = (0.6, 0, 0.3) \wedge_3 (0.5, 0.9, 0.38) = (0.5, 0.9, 0.38)$;

$(x \wedge_3 z) \vee_3 (y \wedge_3 z) = (0.2, 1, 0.3) \vee_3 (0.5, 1, 0.4) = (0.5, 0, 0.3)$.

Therefore,

$(x \wedge_3 y) \vee_3 z \neq (x \vee_3 z) \wedge_3 (y \vee_3 z)$,

$(x \vee_3 y) \wedge_3 z \neq (x \wedge_3 z) \vee_3 (y \wedge_3 z)$.

This means that the distributive law is not satisfied in the generalized De Morgan algebra $(D^*, \vee_3, \wedge_3, {}^c, 0_{D*}, 1_{D*})$.
By Definition 4.1 and Proposition 3.2, we have
For any neutrosophic sets A and B in X, by Definition 3.1 we have

$A \subseteq_3 B$ if and only if
$(T_A(x), I_A(x), F_A(x)) \leq_3 (T_B(x), I_B(x), F_B(x)), \forall x \in X.$

From this, using Proposition 5.1, we can easy to prove Proposition 3.1.

Moreover, for any neutrosophic sets A and B in X, by Definition 3.2 and 5.1 we have

$$A \cup_3 B = \{ \left(T_A(x), I_A(x), F_A(x) \right) \vee_3 \left(T_B(x), I_B(x), F_B(x) \right) | x \in X \};$$
$$A \cap_3 B = \{ \left(T_A(x), I_A(x), F_A(x) \right) \wedge_3 \left(T_B(x), I_B(x), F_B(x) \right) | x \in X \}.$$

From this, using Proposition 5.2–5.4, we can easy to prove Proposition 3.2–3.4. It follows that Theorem 3.1 is a corollary of Theorem 5.1.

Finally, from Example 5.1 we can get that $(NS(X), \cup_3, \cap_3)$ is not a distribute lattice, that is, there exits A, B, C in $NS(X)$ such that

$$(A \cap_3 B) \cup_3 C \neq (A \cup_3 C) \cap_3 (B \cup_3 C),$$
$$(A \cup_3 C) \cap_3 C \neq (A \cap_3 C) \cup_3 (B \cap_3 C).$$

This result shows that neutrosophic set is different from the fuzzy set and intuitionistic fuzzy set from the algebraic structure.

6 Conclusion

In this paper, the inclusion relation between neutrosophic sets is deeply analyzed and studied. Two basic definitions of inclusion relation (type-1 and type-2 inclusion relations) are summarized, and their shortcomings are pointed out. The definition of type-3 inclusion relation of neutrosophic sets is proposed. The difference between type-3 and type-1 (type-2) inclusion relation is mainly reflected in the processing method of second component of neutrosophic set. For the original definitions (type-1 and type-2), the second component of neutrosophic set is rigidly assigned to first or third components, which does not really reflect the original meaning of second component. The type-3 inclusion relation proposed in this paper overcomes this shortcoming.

Moreover, applying the new inclusion relation and its basic idea, this paper also proposes a new ranking method of neutrosophic sets, and illustrates its application in multi-attribute decision making by some examples.

Finally, based on type-3 inclusion relation, the lattice structure of neutrosophic set is analyzed, the type-3 union and intersection operations are established, and the basic algebraic properties are proved. In particular, we get the following important conclusions: all of neutrosophic sets in a certainty universe along with type-3 union, type-3 intersection and the complement constitute a generalized De Morgan algebra (non-distribute), which in theory clearly distinguishes the essential difference between neutrosophic set and fuzzy set (intuitionistic fuzzy set), since one can obtain distributive De Morgan algebras for fuzzy set, intuitionistic

fuzzy set and so on. This can be seen as a meaningful result of this paper.

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