Systems of Linear Ordinary Differential Equations

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1 Introduction

Ordinary Differential Equations (ODEs) involve functions of a single variable and their derivatives. They are used to describe the relationship between a function and its derivatives with respect to one independent variable. In contrast, Partial Differential Equations (PDEs) involve functions of multiple variables and their partial derivatives. PDEs are used to describe phenomena where the function depends on several independent variables, such as in heat conduction, wave propagation, and fluid dynamics.

This document covers the solution of systems of linear ordinary differential equations (ODEs) using exponential methods in algebra. We will discuss both homogeneous and nonhomogeneous equations.

2 Homogeneous Linear ODEs

2.1 Solving the Homogeneous linear ODE: $u' = a \cdot u$

To solve $u' = a \cdot u$, we first rewrite the expression as:

$$\frac{du}{dt} = a \cdot u$$

Then we separate the variables u and t and integrate both sides:

$$\frac{1}{u}\, du = a\, dt$$

$$\int \frac{1}{u}\, du = \int a\, dt$$

After solving both integrals separately using C as the constant of integration:

$$\ln |u| = at + C$$

We can then exponentiate both sides to solve for u:

$$|u| = e^{at+C}$$

Expanding the last expression, and taking $e^C = C_1$ (where C_1 as a new constant, which can be positive or negative):

$$|u| = e^{at} \cdot e^C = e^{at} \cdot C_1$$

If we remove the absolute value (considering C_1 can absorb the sign):

$$u = C_1 e^{at}$$

The general solution to $u' = a \cdot u$ is:

$$u(t) = C_1 e^{at}$$

where C_1 is an arbitrary constant determined by initial conditions.

If we have an initial condition, such as $u(0) = u_0$, we can find C_1 by substituting $t = 0$ and $u = u_0$:

$$u_0 = C_1 e^{a \cdot 0} \implies u_0 = C_1$$

Thus, $C_1 = u_0$, and the specific solution is:

$$u(t) = u_0 e^{at}$$

2.2 System of Homogeneous Linear ODEs

Now, considering a system of homogeneous linear ODEs in matrix form with where A as a constant matrix:

$$\mathbf{X}'(t) = A\mathbf{X}(t)$$

The general solution can be expressed as:

$$\mathbf{X}(t) = e^{At}\mathbf{u_0}$$

where e^{At} is the matrix exponential and $\mathbf{u_0}$ is a vector of constants determined by initial conditions.

Analogous to the Taylor series for the exponential function e^x, the matrix exponential e^{At} is defined as:

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

It has the important properties:

$$e^{A \cdot 0} = I$$

and its derivative is:

$$\frac{d}{dt} e^{At} = A e^{At}$$

Then, the relationship $e^{At} = Pe^{Dt}P^{-1}$ comes from the process of diagonalizing the matrix A, i.e., if a matrix A can be diagonalized, it means there exists an invertible matrix P and a diagonal matrix D such that:

$$A = PDP^{-1}$$

where D is a diagonal matrix whose diagonal elements are the eigenvalues of A, and the columns of P are the corresponding eigenvectors. Then, we can write:

$$e^{At} = e^{(PDP^{-1})t} = Pe^{Dt}P^{-1}$$

Since D is a diagonal matrix, e^{Dt} is simply the exponential of each diagonal element:

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

If a matrix A is not diagonalizable, we can use its Jordan canonical form. For a matrix A that is not diagonalizable, there exists an invertible matrix P and a Jordan matrix J such that:

$$A = PJP^{-1}$$

where J is a block diagonal matrix with Jordan blocks on the diagonal. Each Jordan block corresponds to an eigenvalue of A and has the form:

$$J_k = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

The matrix exponential e^{At} can be computed using the Jordan form:

$$e^{At} = e^{(PJP^{-1})t} = Pe^{Jt}P^{-1}$$

The exponential of a Jordan block J_k can be computed as:

$$e^{J_k t} = e^{\lambda t} \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & t \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Given the system $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$, the solution can be expressed as:

$$\mathbf{u}(t) = e^{At}\mathbf{u}(0) = Pe^{Jt}P^{-1}\mathbf{u}(0)$$

This approach allows you to solve the system even when A is not diagonalizable by leveraging the Jordan canonical form.

Let's suppose a matrix with eigenvalues $\{2, 2, 2, 4\}$. Its Jordan form depends on the geometric multiplicity of each eigenvalue and the structure of the corresponding Jordan blocks. For example, if the geometric multiplicity of eigenvalue 2 is 2 (two linearly independent eigenvectors), the Jordan form is:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

and the matrix exponential e^{Jt} for the entire Jordan form J is:

$$e^{Jt} = \begin{bmatrix} e^{J_2 t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{4t} \end{bmatrix}$$

If the geometric multiplicity of eigenvalue 2 is 3 (only one eigenvector), the Jordan form is:

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

and the matrix exponential e^{Jt} for the entire Jordan form J is:

$$e^{Jt} = \begin{bmatrix} e^{J_3 t} & 0 \\ 0 & e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t}\begin{bmatrix} 1 & t & \frac{t^2}{2!} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} & 0 \\ 0 & e^{4t} \end{bmatrix} = \begin{bmatrix} e^{2t} & te^{2t} & \frac{t^2}{2!}e^{2t} & 0 \\ 0 & e^{2t} & te^{2t} & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{4t} \end{bmatrix}$$

Example 1

Consider the initial value problem:

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}, \quad \mathbf{u}(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}.$$

To solve this system, we use the diagonalization method. First, we find the eigenvalues and eigenvectors of A. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$, and the corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$.

We can write A as:

$$A = PDP^{-1}$$

where

$$P = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The solution to the system is given by:

$$\mathbf{u}(t) = e^{At}\mathbf{u}(0) = Pe^{Dt}P^{-1}\mathbf{u}(0)$$

First, compute P^{-1}:

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 2 & -5 \\ -1 & 1 \end{bmatrix} = \frac{1}{2-5} \begin{bmatrix} 2 & -5 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{5}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Next, compute e^{Dt}:

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

Now, combine these results:

$$\mathbf{u}(t) = Pe^{Dt}P^{-1}\mathbf{u}(0) = \begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & \frac{5}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

Simplify the expression step-by-step:

$$\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} = \begin{bmatrix} e^{-t} & 5e^{2t} \\ e^{-t} & 2e^{2t} \end{bmatrix}$$

Then,

$$\begin{bmatrix} e^{-t} & 5e^{2t} \\ e^{-t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & \frac{5}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} e^{-t}\left(-\frac{2}{3}\right) + 5e^{2t}\left(\frac{1}{3}\right) & e^{-t}\left(\frac{5}{3}\right) + 5e^{2t}\left(-\frac{1}{3}\right) \\ e^{-t}\left(-\frac{2}{3}\right) + 2e^{2t}\left(\frac{1}{3}\right) & e^{-t}\left(\frac{5}{3}\right) + 2e^{2t}\left(-\frac{1}{3}\right) \end{bmatrix}$$

Finally, multiply by the initial condition:

$$\mathbf{u}(t) = \begin{bmatrix} 3e^{-t} & 5e^{2t} \\ 3e^{-t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} 8 \\ 5 \end{bmatrix} = 3e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Thus, the solution to the initial value problem is:

$$\mathbf{u}(t) = 3e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Example 2

Consider the initial value problem:

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}, \quad \mathbf{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}.$$

To solve this system, we first find the eigenvalues and eigenvectors of A:

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - (-1)(1) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$

Thus, the eigenvalue is $\lambda = 2$ (with algebraic multiplicity 2). Next, we find the eigenvectors corresponding to $\lambda = 2$:

$$(A - 2I)\mathbf{v} = 0$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives $v_1 = v_2$, so an eigenvector is $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Since A has a single eigenvalue with algebraic multiplicity 2 but only one linearly independent eigenvector, the matrix A does not diagonalize, and we need to find a generalized eigenvector $\mathbf{w}$ such that:

$$(A - 2I)\mathbf{w} = \mathbf{v}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving this, we get $w_1 = 0$ and $w_2 = 1$, so a generalized eigenvector is $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Therefore, the Jordan form J of A is:

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

and the matrix P of eigenvectors and generalized eigenvectors is:

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Which verifies $A = PJP^{-1}$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

Then, the matrix exponential e^{At} is given by:

$$e^{At} = Pe^{Jt}P^{-1}$$

where

$$e^{Jt} = e^{2t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$$

So that:

$$e^{At} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} e^{2t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix} = e^{2t}\begin{bmatrix}1-t & t\\ -t & 1+t\end{bmatrix}$$

Using the initial condition $\mathbf{u}(0) = \begin{bmatrix}1\\1\end{bmatrix}$:

$$\mathbf{u}(t) = e^{At}\mathbf{u}(0) = e^{2t}\begin{bmatrix}1-t & t\\ -t & 1+t\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$$

Thus, the solution is:

$$\mathbf{u}(t) = e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$$

2.3 Solving Homogeneous Linear ODEs of Higher Order

To solve a higher-order homogeneous linear ODE using matrix methods, we convert the ODE into a system of first-order linear differential equations. Consider a general n-th order homogeneous linear ODE:

$$a_n\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0y = 0$$

1. Convert to a System of First-Order Equations. Define a vector $\mathbf{u}$ such that:

$$\mathbf{u} = \begin{bmatrix}y\\ y'\\ y''\\ \vdots\\ y^{(n-1)}\end{bmatrix}$$

   Then, the system can be written as:

$$\mathbf{u}' = A\mathbf{u}$$

   where A is an $n \times n$ matrix constructed from the coefficients of the ODE.

2. Find Eigenvalues and Eigenvectors. Solve the characteristic equation $\det(A - \lambda I) = 0$ to find the eigenvalues λ and corresponding eigenvectors $\mathbf{v}$.

3. The general solution to the system is:

$$\mathbf{u}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + \cdots + C_n e^{\lambda_n t}\mathbf{v}_n$$

   where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues, and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are the corresponding eigenvectors.

4. Use the initial conditions to solve for the constants $C_1, C_2, \ldots, C_n$.

Example 3

Consider the second-order homogeneous linear ODE:

$$y'' + 4y' + 3y = 0$$

with initial conditions $y(0) = 3$ and $y'(0) = -7$.

First we create the System of First-Order Equations:

$$\mathbf{u} = \begin{bmatrix} y \\ y' \end{bmatrix}, \quad \mathbf{u}' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$$

Thus,

$$\mathbf{u}' = A\mathbf{u}, \quad A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$$

The characteristic equation is:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -3 & -4 - \lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$$

Solving for λ:

$$\lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0 \implies \lambda_1 = -1, \quad \lambda_2 = -3$$

For $\lambda_1 = -1$:

$$(A + I)\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0 \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda_2 = -3$:

$$(A + 3I)\mathbf{v}_2 = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0 \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

The General Solution is:

$$\mathbf{u}(t) = C_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

We apply the initial conditions $\mathbf{u}(0) = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$:

$$\begin{bmatrix} 3 \\ -7 \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Solving the system:

$$\begin{cases} C_1 + C_2 = 3 \\ -C_1 - 3C_2 = -7 \end{cases}$$

We obtain $C_1 = 1$ and $C_2 = 2$, and the solution is:

$$\mathbf{u}(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2e^{-3t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Thus:

$$y(t) = e^{-t} + 2e^{-3t}$$

3 Nonhomogeneous Linear ODEs

3.1 Solving $u' = a \cdot u + f(t)$

To solve the nonhomogeneous linear ODE $u' = a \cdot u + f(t)$, we use the method of integrating factors as follows.

First, rewrite the equation in standard form:

$$u' - a \cdot u = f(t)$$

Next, we find the integrating factor $\mu(t)$, a function that, when multiplied by the entire differential equation, allows it to be written in a form where the left-hand side is the derivative of a product of functions:

$$\mu(t) = e^{\int -a \, dt} = e^{-at}$$

Multiply both sides of the ODE by the integrating factor:

$$e^{-at}u' - ae^{-at}u = e^{-at}f(t)$$

The left-hand side becomes the derivative of $e^{-at}u$:

$$\frac{d}{dt}\left(e^{-at}u\right) = e^{-at}f(t)$$

Integrate both sides with respect to t:

$$e^{-at}u = \int e^{-at}f(t)\,dt + C$$

Finally, solve for u:

$$u = e^{at}\left(\int e^{-at}f(t)\,dt + C\right)$$

Example 4

Consider the nonhomogeneous linear ODE:

$$u' = 3u + e^{2t}$$

Rewrite in standard form:

$$u' - 3u = e^{2t}$$

The integrating factor is:

$$\mu(t) = e^{\int -3\,dt} = e^{-3t}$$

Multiply both sides by the integrating factor:

$$e^{-3t}u' - 3e^{-3t}u = e^{-3t}e^{2t}$$

Simplify and integrate:

$$\frac{d}{dt}\left(e^{-3t}u\right) = e^{-t}$$

$$e^{-3t}u = \int e^{-t}\,dt = -e^{-t} + C$$

Solve for u:

$$u = e^{3t}\left(-e^{-t} + C\right) = -e^{2t} + Ce^{3t}$$

The general solution is:

$$u(t) = Ce^{3t} - e^{2t}$$

3.2 System of Nonhomogeneous Linear ODEs

Consider a system of nonhomogeneous linear ODEs in matrix form:

$$\mathbf{X}'(t) = A\mathbf{X}(t) + \mathbf{F}(t)$$

The general solution can be expressed as the sum of the homogeneous solution and a particular solution:

$$\mathbf{X}(t) = \mathbf{X}_h(t) + \mathbf{X}_p(t)$$

Where $\mathbf{X}_h(t)$ is the solution to the homogeneous system $\mathbf{X}'(t) = A\mathbf{X}(t)$, and $\mathbf{X}_p(t)$ is a particular solution to the nonhomogeneous system, which can be solved using the method of undetermined coefficients or variation of parameters.

Differentiating $\mathbf{X}_p(t) = \mathbf{U}(t)e^{At}$, where $\mathbf{U}(t)$ is a matrix of functions to be determined:

$$\mathbf{X}_p'(t) = \mathbf{U}'(t)e^{At} + \mathbf{U}(t)Ae^{At}$$

If we substitute into the original equation $\mathbf{X}'(t) = A\mathbf{X}(t) + \mathbf{F}(t)$:

$$\mathbf{U}'(t)e^{At} + \mathbf{U}(t)Ae^{At} = A\mathbf{U}(t)e^{At} + \mathbf{F}(t)$$

Simplifying:

$$\mathbf{U}'(t)e^{At} = \mathbf{F}(t)$$

and solving for $\mathbf{U}(t)$ by integrating both sides:

$$\mathbf{U}'(t) = \mathbf{F}(t)e^{-At}$$

$$\mathbf{U}(t) = \int \mathbf{F}(t)e^{-At}\,dt$$

Thus, the particular solution is:

$$\mathbf{X}_p(t) = \mathbf{U}(t)e^{At} = e^{At}\int \mathbf{F}(t)e^{-At}\,dt$$

and the general solution to the nonhomogeneous system is:

$$\mathbf{X}(t) = e^{At}\mathbf{X}_0 + e^{At}\int \mathbf{F}(t)e^{-At}\,dt$$

where $\mathbf{X}_0$ is the vector of initial conditions.

Example 5

Let's solve the nonhomogeneous initial value problem:

$$\mathbf{u}' = A\mathbf{u} + \mathbf{F}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\mathbf{u} + \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix}, \quad \mathbf{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find the matrix exponential e^{At}, we use the eigenvalues and eigenvectors of A through diagonalization:

$$A = PDP^{-1}$$

First, we find the eigenvalues and eigenvectors of the matrix A:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

The characteristic equation is:

$$\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) = 0$$

Solving for λ, we get the eigenvalues:

$$\lambda_1 = 1, \quad \lambda_2 = 2$$

Next, we find the eigenvectors corresponding to each eigenvalue.

For $\lambda_1 = 1$:

$$(A - I)\mathbf{v}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives v_{12} as a free variable. Let $v_{12} = 1$, then $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

For $\lambda_2 = 2$:

$$(A - 2I)\mathbf{v}_2 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives $v_{21} = v_{22}$. Let $v_{22} = 1$, then $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

giving us:

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The matrix exponential is:

$$e^{At} = Pe^{Dt}P^{-1}$$

Thus,

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^t & (-e^t + e^{2t}) \\ 0 & e^{2t} \end{bmatrix}$$

Then, to find the particular solution $\mathbf{u}_p(t)$, we use variation of parameters. Assuming $\mathbf{u}_p(t) = e^{At}\mathbf{v}(t)$, its derivative is:

$$\mathbf{u}_p'(t) = e^{At}\mathbf{v}'(t) + Ae^{At}\mathbf{v}(t)$$

Substitute into the original equation $\mathbf{u}' = A\mathbf{u} + \mathbf{F}(t)$:

$$e^{At}\mathbf{v}'(t) + Ae^{At}\mathbf{v}(t) = Ae^{At}\mathbf{v}(t) + \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix}$$

Simplifying:

$$e^{At}\mathbf{v}'(t) = \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix}$$

and solving for $\mathbf{v}(t)$ by integrating both sides:

$$\mathbf{v}'(t) = e^{-At} \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix}$$

$$\mathbf{v}(t) = \int e^{-At} \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix} dt$$

Given $e^{-At} = Pe^{-Dt}P^{-1}$:

$$e^{-At} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^{-t} & (-e^{-t} + e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

Thus,

$$\mathbf{v}(t) = \int \begin{bmatrix} e^{-t} & (-e^{-t} + e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix} dt = \int \begin{bmatrix} -e^{t} \\ 0 \end{bmatrix} dt = \begin{bmatrix} 1 - e^{t} \\ 0 \end{bmatrix} + \mathbf{C}$$

Therefore, the general solution is the sum of the homogeneous and particular solutions:

$$\mathbf{u}(t) = e^{At}\mathbf{v}(t) = e^{At} \left(\begin{bmatrix} 1 - e^{t} \\ 0 \end{bmatrix} + \mathbf{C} \right)$$

Given the initial condition $\mathbf{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and solving for $\mathbf{C}$:

$$\mathbf{u}(0) = 1 \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \mathbf{C} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, the general solution is:

$$\mathbf{u}(t) = e^{At} \left(\begin{bmatrix} 1 - e^{t} \\ 0 \end{bmatrix} + \mathbf{C} \right) = e^{At} \left(\begin{bmatrix} 1 - e^{t} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = e^{At} \begin{bmatrix} 2 - e^{t} \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} e^{t} & (-e^{t} + e^{2t}) \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 - e^{t} \\ 1 \end{bmatrix} = \begin{bmatrix} e^{t} \\ e^{2t} \end{bmatrix}$$

and the particular solution is:

$$\mathbf{u}_p(t) = e^{At} \begin{bmatrix} 1 - e^{t} \\ 0 \end{bmatrix} = \begin{bmatrix} e^{t} & (-e^{t} + e^{2t}) \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 - e^{t} \\ 0 \end{bmatrix} = \begin{bmatrix} e^{t} - e^{2t} \\ 0 \end{bmatrix}$$