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Riteria For Plurigarmonicity Of M-Harmonic Functions

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Abstract: *It is proved in the paper that if a function* $f(z)$ *is M-harmonic in a polydisk* U^n*, then the function* $f^2(z)$ *is M-subharmonic. In addition, the case is* $\tilde{\Delta} f = \tilde{\Delta} f^{\alpha} = 0 (\alpha \neq 0, \alpha \neq 1, \alpha \in \mathbb{R})$ *proved to* $f(z)$ *be a* n *-harmonic function. Moreover, if the function is harmonic on the unit ball and* $f^m(z)$ *is M-harmonic, then it is proved that* $f^m(z)$ *is pluriharmonic. At the same time, if the function is harmonic and M-harmonic in the polydisk* $U^2 \subset C^2$*, then it is proved that it is n-harmonic..*

Keywords — holomorphic function, harmonic function, subharmonic function, -harmonic function, -subharmonic function, pluriharmonic function, plurisubharmonic function, M-harmonic function, M-subharmonic function.

INTRODUCTION.

Suppose an B open unit ball in C^n with center at the origin B, a $\varphi_a(z)-$ linear fractional biholomoric mapping of the ball $(B = \{z \in C^n : |z| < 1\},\ \partial B = \{\omega \in C^n : |\omega| = 1\})$ onto itself of the following form:

$$\varphi_a(z) = \frac{a - P_a(z) - \left(1-|a|^2\right)^{\frac{1}{2}}\left(z - P_a(z)\right)}{1 - \langle z, a\rangle}.$$

At $a \in B$, $a \neq 0$, $\varphi_a(0) = a$, $\varphi_a(a) = 0$, $\varphi_0(z) = -z$, where the angle brackets denote the Hermitian dot product of $\langle z, w\rangle = \sum_{i=1}^{n} z_i \bar{w}_i,$ $|z| = \langle z, z\rangle^{\frac{1}{2}}$, and $P_a(z) = \frac{\langle z, a\rangle}{\langle a, a\rangle} a$, $a \neq 0$, $P_0(z) = 0$. Obviously $\varphi_a(0) = a$, $\varphi_a(a) = 0$.

If $G = \{z \in C^n : \langle z, a\rangle \neq 1\}$, then φ_a holomorphically maps G to C^n. It is clear that $G \supset \bar{B}$, since $|a| < 1$.

The invariant Laplace operator $\tilde{\Delta}$ of functions on doubly smooth functions in B is defined as follows:

$$\tilde{\Delta} f(z) = \Delta(f \circ \varphi_z)(0)$$

where $z \in B$, $\Delta -$ are the usual Laplacian [3: 54].

The operator $\tilde{\Delta}$ is called invariant because it commutes with automorphisms of the ball B in the following sense: $\tilde{\Delta}(f \circ \varphi) = (\tilde{\Delta} f) \circ \varphi$, where $f \in C^2(G)$ ($G \subset B$), and $\varphi -$ is any biholomorphic automorphism of the ball B.

For an arbitrary $\lambda \in C$, let us denote by X_λ the space of all functions $f \in C^2(B)$ ($f \in C^2(U^n)$), satisfying the equation

$$\tilde{\Delta} f(z) = \lambda \cdot f(z).$$

Case $\lambda = 0$ is the most interesting. We will call elements of the space X_0 M -harmonic functions.

Definition 1.1. Let B be the unit ball in C^n. The function $f \in C^2(B)$ is called M -harmonic (M -subharmonic) if $\tilde{\Delta} f = 0$ $(\tilde{\Delta} f \geq 0)$, where

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$\Delta f = (1-|z|^2)(\Delta f - 4\sum_{i,j=1}^{n} z_i \bar{z}_j \frac{\partial^2 f}{\partial z_i \partial \bar{z}_j})$ is the invariant Laplacian in B [3: 55].

For the polycircle $U^n \subset \mathbb{C}^n$ a similar definition holds.

Definition 1.2. Let $U^n -$ be the unit polydisk in C^n. The function $f \in C^2(U^n)$ is called M-harmonic (M-subharmonic) in U^n if $\tilde{\Delta} f = 0$ $(\tilde{\Delta} f \geq 0)$, where $\Delta f = 2\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j}$ is the invariant Laplacian in U^n [4: 24].

U. Rudin in [3] obtained the following result: if the function $f(z)$ is harmonic and M-harmonic in B, then $f(z)$ is pluriharmonic in B.

R.M. Madrakhimov and M.D. Vaisov in [7] proved the following theorem: if the function $f(z)$ is harmonic and M-subharmonic in B, then $f(z)$ is pluriharmonic in B.

2. Main results. Theorem 2.1. If a function f is M-harmonic in U^n, then f^2 is M-subharmonic in U^n.

Proof. For the function f^2, the invariant Laplacian has the following form:

$$\tilde{\Delta} f^2 = 2\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial^2 f^2}{\partial z_j \partial \bar{z}_j} = 2\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \left(2\frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} + 2f\frac{\partial^2 f}{\partial z_j \partial \bar{z}_j}\right) =$$

$$= 4\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} + 4f\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = 4\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j}.$$

By the conditions of the theorem, $\Delta f = 2\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = 0$ and if f is a real function, then $\frac{\partial f}{\partial \bar{z}_j} = \overline{\frac{\partial f}{\partial z_j}}$. From the following two equalities

$$\tilde{\Delta} f^2 = 4\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} = 4\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \left|\frac{\partial f}{\partial z_j}\right|^2$$

it follows that for $\forall z_j \in B$ $(j = 1, \ldots, n)$ with $\left(1-|z_j|^2\right)^2 > 0$ and $\left|\frac{\partial f}{\partial z_j}\right|^2 \geq 0$, then $\tilde{\Delta} f^2 \geq 0$. This means that the function f^2 is a M-subharmonic function.

Theorem 2.2. Suppose that $\tilde{\Delta} f = \tilde{\Delta} f^{\alpha} = 0$ $(\alpha \neq 0, \alpha \neq 1, \alpha \in R)$ is in $U^n \subset C^n$, then $f(z)$ is n-harmonic in U^n.

Proof. For the function f^{α}, the invariant of the Laplace operator has the form:

$$\tilde{\Delta} f^{\alpha} = 2\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial^2 f^{\alpha}}{\partial z_j \partial \bar{z}_j} = 2\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\left(\alpha(\alpha-1)f^{\alpha-2}\frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} + \alpha f^{\alpha-1}\frac{\partial^2 f}{\partial z_j \partial \bar{z}_j}\right) = 2\alpha(\alpha-1)f^{\alpha-2}\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} + 2\alpha f^{\alpha-1}\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2\frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = 0.$$

From the conditions of the theorem $\Delta f = 2\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = 0$ and $\alpha \neq 0$, $\alpha \neq 1$, $\alpha \in R$ it follows that $\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} = 0$. It is known that if f is a real function, then $\frac{\partial f}{\partial \bar{z}_j} = \overline{\frac{\partial f}{\partial z_j}}$. So,

$$\sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} = \sum_{j=1}^{n}\left(1-|z_j|^2\right)^2 \frac{\partial f}{\partial z_j}\overline{\frac{\partial f}{\partial z_j}} = \sum_{j=1}^{n}$$

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For $\forall z_j \in B\,(j=1,\ldots,n)$. Then $\left|\frac{\partial f}{\partial z_j}\right|^2 = \frac{\partial f}{\partial z_j}\frac{\partial f}{\partial \bar{z}_j} = 0$. It follows that $\frac{\partial f}{\partial z_j} = 0$ or $\frac{\partial f}{\partial \bar{z}_j} = 0$ Then, $\frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = 0$. This means that the function f is n-harmonic.

Theorem 2.3. Let B be the unit ball in C^n and $u(z)$ satisfy the following conditions:

a) Function $u(z)$ is harmonic in B;

b) Function $u^m(z)$ is M-harmonic in B.

Then $u^m(z)$ is pluriharmonic in B.

Proof. Since the $u^m(z)$ M-harmonic function is real-analytic in B, it can be expanded into a series of homogeneous polynomials:

(2.1) $$u^m(z) = \sum_{k=1}^{\infty} P_k(z)$$

where $P_k(z)$ is a homogeneous polynomial in $z_1,\ldots,z_n,\bar{z}_1,\ldots,\bar{z}_n$ of degree k. The series converges uniformly on compact subsets of B, and successive derivatives of $u^m(z)$ can be obtained by differentiation along the series.

Each polynomial in (2.1) can be written as

(2.2) $$P_k(z) = \sum_{p+q=k} f_{p,q}$$

where $f_{p,q}$ are polynomials in which the total order in $z_1,\ldots,z_n$ is p and the total order in $\bar{z}_1,\ldots,\bar{z}_n$ is q. Each term of the polynomial $f_{p,q}$ has the form $M(z) = Cz_1^{\alpha_1}\ldots z_n^{\alpha_n}\bar{z}_1^{\beta_n}\ldots\bar{z}_n^{\beta_n}$, $\alpha_1+\ldots+\alpha_n = p, \beta_1+\ldots+\beta_n = q$. From the conditions of the theorem,

$$\tilde{\Delta}u^m(z) = (1-|z|^2)\Big(\Delta u^m(z) - 4\sum_{i,j=1}^{n} z_i\bar{z}_j \frac{\partial^2 u^m(z)}{\partial z_i \partial \bar{z}_j}\Big) = 0$$

and

$$\Delta u^m = 4\sum_{i=1}^{n}\frac{\partial^2 u^m}{\partial z_i\partial\bar{z}_i} = 4\sum_{i=1}^{n} m(m-1)u^{m-2}\frac{\partial u}{\partial z_i}\frac{\partial u}{\partial \bar{z}_i} + 4\sum_{i=1}^{n} m u^{m-1}\frac{\partial^2 u}{\partial z_i\partial\bar{z}_i} = 4m(m-1)u^{m-2}\sum_{i=1}^{n}\frac{\partial u}{\partial z_i}\overline{\frac{\partial u}{\partial z_\imath}} = 4m(m-1)u^{m-2}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial z_i}\right|^2.$$

Let us prove the theorem for the following cases:

a) Let m be an even number. Then $u^{m-2}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial z_i}\right|^2 \ge 0$. It is easy to see that $\Delta u^m = 4\sum_{i,j=1}^{n} z_i\bar{z}_j\frac{\partial^2 u^m}{\partial z_i\partial\bar{z}_j} \ge 0$. From this we obtain the inequality

$$\sum_{i,j=1}^{n} z_i\bar{z}_j\frac{\partial^2 u^m}{\partial z_i\partial\bar{z}_j} = \sum_{k=2}^{\infty}\sum_{p+q=k} pqf_{p,q} \ge 0.$$

Consequently, at $pq \neq 0$ all polynomials $f_{p,q}$ vanish, since in this case $\sum_{p+q=k} pqf_{p,q}(0) = 0$, and from the minimum principle we obtain that $\sum_{p+q=k} pqf_{p,q} \equiv 0$, it follows that $pqf_{p,q} = 0$ for all p and q. Hence $f_{pq} = 0$ or $p = 0$ or $q = 0$. This means that expansion (2.8) consists only of terms of the form $f_{p,0}$ and $f_{0,q}$, i.e. $P_k(z)$ is a pluriharmonic polynomial, and the sum of series (2.1), i.e. $u(z)$ is a pluriharmonic function and the theorem is proven.

b) Let m be an odd number, then $u^{m-2}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial z_i}\right|^2 \ge 0$ or $u^{m-2}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial z_i}\right|^2 \le 0$. We have proved the case of $u^{m-2}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial z_i}\right|^2 \ge 0$ in a). Let us prove it for the case of $u^{m-2}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial z_i}\right|^2 \le 0$. It was proved above that

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$$\Delta u^m = 4m(m-1)u^{m-2}\sum_{i=1}^{n}\left|\frac{\partial u}{\partial z_i}\right|^2.$$ Therefore,

$$\Delta u^m = 4\sum_{i,j=1}^{n} z_i\overline{z}_j \frac{\partial^2 u^m}{\partial z_i \partial \overline{z}_j} \le 0$$. From this we obtain that

$$\sum_{i,j=1}^{n} z_i\overline{z}_j \frac{\partial^2 u^m}{\partial z_i \partial \overline{z}_j} = \sum_{k=2}^{\infty}\sum_{p+q=k} pqf_{p,q} \le 0.$$

Consequently, at $pq \neq 0$ all polynomials $f_{p,q}$ vanish, since in this case $\sum_{p+q=k} pqf_{p,q}(0) = 0$, and from the maximum principle we obtain that $\sum_{p+q=k} pqf_{p,q} \equiv 0$, it follows that $pqf_{p,q} = 0$ for all p and q. Hence $f_{pq} = 0$ or $p = 0$ or $q = 0$. This means that expansion (2.2) consists only of terms of the form $f_{p,0}$ and $f_{0,q}$, i.e. $P_k(z)$ is a pluriharmonic polynomial, and the sum of series (2.1), i.e. $u(z)$ is a pluriharmonic function and the theorem is proven.

Note. If the function $u(z)$ is harmonic in B, then is $u^2(z)$ harmonic.

The answer to this question is negative, as the following example shows function

$$u(z) = z_1 z_2 + \bar{z}_1 \bar{z}_2 \qquad z \in B \subset C^2$$

Since $\Delta u = 0$ in B, and $\Delta u^2 = 8(z_1\overline{z}_1 + z_2\overline{z}_2)$, if $z_1\overline{z}_1 \neq 0,\ z_2\overline{z}_2 \neq 0$, the function $u^2(z)$ is not harmonic in $B \cap \{z_1 \neq 0,\ z_2 \neq 0\}$.

It is known that the sphere and the polycircle are biholomorphically inequivalent. If we consider the question in the polycircle in a similar way, then we get a weaker result.

It is known that the ball and the polydisk are biholomorphically nonequivalent. If we consider a similar question in the polydisk, we obtain a weaker result.

Note that if a function $u(z)$ is $\tilde{\Delta} u_j = 0$ (for all $j = 1,\dots,n$), then it is $n-$ harmonic in U^n. Now we give an example of a function satisfying $\tilde{\Delta}_{U^n} u = 0$, but not $n-$ harmonic on U^n, for U^n, $\omega \in T$. By

$$P(z,\omega) = \frac{\left(1-|z|^2\right)}{|1-z\overline{\omega}|^2}$$

we denote the Poisson kernel on U^2.

As we see, for a fixed $\omega \in T$, $\alpha > 0$

$$\tilde{\Delta} P^{\alpha}(z,\omega) = 2\alpha(\alpha-1)P^{\alpha}(z,\omega)$$

If is $\alpha_i = \lambda_i + \frac{1}{2}$, $i = 1,2$. Then

$$\tilde{\Delta}_i P^{\lambda_i+\frac{1}{2}}(z_i,\omega_i) = 2\left(\lambda_i^2 - \frac{1}{4}\right)P^{\lambda_i+\frac{1}{2}}(z_i,\omega_i).$$

Thus, if $F(z_1,z_2) = P^{\lambda_1+\frac{1}{2}}(z_1,\omega_1)P^{\lambda_2+\frac{1}{2}}(z_2,\omega_2)$,

$$\tilde{\Delta}_{U^2} F(z_1,z_2) = 2\left(\lambda_1^2 + \lambda_2^2 - \frac{1}{2}\right)F(z_1,z_2).$$

Therefore, for any λ_1, λ_2 satisfying $\lambda_1^2 + \lambda_2^2 = \frac{1}{2}$, $\tilde{\Delta}_{U^2} F(z_1,z_2) = 0$. If λ_1 and λ_2 are not equal to $\frac{1}{2}$, then the function $F(z_1,z_2)$ is not $n-$ harmonic in U^2.

It is quite clear that every $n-$ harmonic function $u(z)$ in U^n satisfies $\Delta u = \tilde{\Delta} u = 0$. For $n = 2$ we will prove the opposite of this proposition.

Theorem 2.4. Let $U^2 \subset C^2$ and function $u(z)$ satisfy the following conditions:

a) function $u(z)$ is harmonic in U^2;

b) function $u(z)$ is harmonic in U^2.

Then $u(z)$ is 2 $-$harmonic in U^2.

Proof. If $u(z)$ is a harmonic and M-harmonic function in U^2, we obtain equalities $\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + \frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} = 0$ and $\left(1-|z_1|^2\right)^2 \frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} + \left(1-|z_2|^2\right)^2 \frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} = 0$. From the obtained equalities it follows that $\frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2} = -\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1}$ and we arrive at the equation:

(2.3) $(1-|z_1|^2)^2 \frac{\partial u}{\partial z_1 \partial \bar{z}_1} - (1-|z_2|^2)^2 \frac{\partial u}{\partial z_1 \partial \bar{z}_1} = \frac{\partial u}{\partial z_1 \partial \bar{z}_1}(|z_2|^2 - |z_1|^2)(2-|z_1|^2-|z_2|^2) = 0.$

From equation (2.3) follow the relations $\frac{\partial u}{\partial z_1 \partial \overline{z}_1} = 0$ or $|z_1| = |z_2|$.

Similarly, expressing the first term through the second, i.e.

$$\frac{\partial^2 u}{\partial z_1 \partial \overline{z}_1} = -\frac{\partial^2 u}{\partial z_2 \partial \overline{z}_2},$$

we arrive at the equation

(2.4) $(1-|z_2|^2)^2 \frac{\partial u}{\partial z_2 \partial \bar{z}_2} - (1-|z_1|^2)^2 \frac{\partial u}{\partial z_2 \partial \bar{z}_2} = \frac{\partial u}{\partial z_2 \partial \bar{z}_2}(|z_1|^2 - |z_2|^2)(2-|z_1|^2-|z_2|^2) = 0$ (2).

From equation (2.4) follow the relations $\frac{\partial u}{\partial z_2 \partial \overline{z}_2} = 0$ or $|z_1| = |z_2|$. This means that the function $u(z)$ satisfies the conditions $\frac{\partial u}{\partial z_2 \partial \overline{z}_2} = 0$ and $\frac{\partial u}{\partial z_2 \partial \overline{z}_2} = 0$ outside the line $|z_1| = |z_2|$ and $u \in C^2(U^2)$, i.e. it is an 2-harmonic function in U^2.

If in the previous theorem we consider U^n (n>2) instead of U^2, then the question remains open..

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