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RESEARCH ARTICLE

FORMULATION OF THE CORRELATION ON THE REDUCED LINEAR BIQUATERNION CANONIC TRANSFORMATION

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Abstract

This research aims to formulate a correlation theorem on the canonical transformation of reduced linear biquaternion (RBLCT).This research is a literature review conducted at the Department of Industrial Technology, Makassar PSDKU Graphics Engineering Study Programme, Politeknik Negeri Media Kreatif.The definition of reduced biquaternion linear canonical transformation (RBLCT) is obtained by first constructing the definition of reduced biquaternion Fourier transform (FTRB) and investigating its properties by replacing the kernel of Fourier Transform with the kernel of FTRB in the definition of Linear Canonical Transformation (LCT). The proposed Correlation Theorem for RBLCT is an extension of the correlation theorem of linear canonical transform to the domain of RBLCT. A different proof of the Parseval formula for RBLCT is also given whose proof is much simpler by using the conjugacy property of the RBLCT kernel. As an application of the obtained results, the RBLCT correlation theorem is briefly discussed to study frequency-swift filters in general.

Introduction:

Digital image processing and signal processing are objects of current discussion which are activities that are closely related to mathematical processes. The Fourier transform was first discovered by a mathematician Joseph Fourier. The Fourier transform is an extension of the Fourier series. The development of the Fourier series to the Fourier transform is because non-periodic functions are easier to analyze with the Fourier transform.

In reality, apart from being in contact with real space, there is also a complex space, so the Fourier transform is being developed in a complex space, namely the Fourier Quaternion transform (TFQ). TFQ is a generalization of the real and complex Fourier transform using quaternion algebra. Quaternion is an expansion of complex numbers which was first discovered by Hamilton. The discussion of quaternions has been widely developed on the problem of signal processing, image processing, aircraft radar and so on. However, since it is known that the multiplication rule of quaternions is not commutative, this limits the application of quaternions in signal and image processing. Moreover, in general, the convolution of the two signal quaternions f(x,y) and g(x,y) cannot be calculated by the product of the Fourier transforms F(u,v) and G(u,v) in the frequency domain.

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With these weaknesses, a reduced biquatenion has been proposed which has a commutative multiplication rule [1]. This commutative property is an advantage over quaternions. A biquaternion is an eight-dimensional hypercomplex number with addition and multiplication operations similar to that of a quaternion. The collection of biquaternions forms four-dimensional (4D) algebra on complex numbers.

Linear canonical transformation (LCT) which is a generalization of several transformations, including Fourier transform, Laplace transform, fractional Fourier transform, Fresnel transform and other transformations has an important role in many fields of optics [2] as well as processing signal [3]. LCT is more attractive in various applications due to the accuracy and efficiency of its transformation calculations [4], and many of the basic properties of these transformations are known, including shift, modulation, convolution, correlation and the uncertainty principle [5].

In previous studies, the convolution theorem for linear canonical transformation (LCT) has been introduced which is based on the properties of the convolution theorem for the Fourier Transform which is explicitly shown by the authors some important properties of the relation between LCT and convolution, and provides an alternative form of the LCT correlation theorem [6]. Likewise, the convolution for one-sided LCTQ and its important properties such as linearity, shift, modulation and so on [7].

This research was conducted using a literature review method, by first introducing the definition of the reduced Biquaternion Fourier transform (RBFT) and its important properties such as linearity, shift or dilation, scale, modulation, parseval, and plansherel. Furthermore, based on the definition of the reduced biquaternion Fourier transform (RBFT), we obtain the definition of the reduced linear biquaternion canonical transform (RBLCT) by replacing the Kernel of TF with the kernel of RBFT in the definition of LCT. And in the end it will be obtained an alternative form of the correlation for the reduced biquaternion linear canonical transformation.

Research Methods:

Problem identification

Problem identification is the initial stage of research to determine the focus of the research problem.

Literature Study

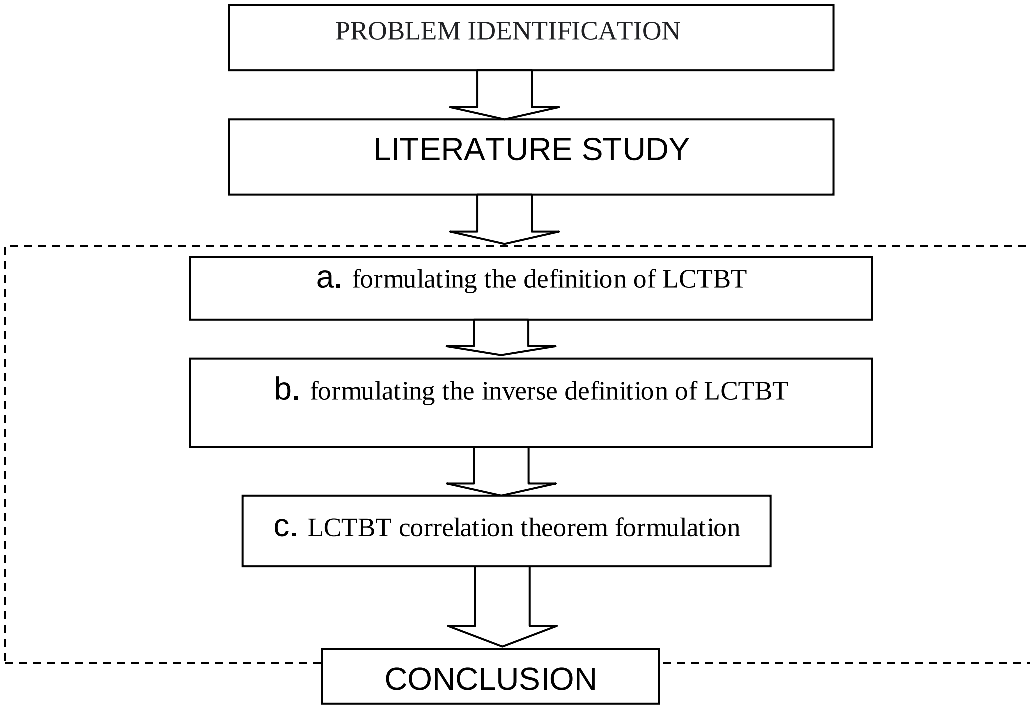
Literature studies were conducted on research journals related to the field of research as a stage to complete the basic knowledge of researchers for the purposes of conducting research.

a) Activities for formulating the definition of RBLCT.
b) The activity of formulating the inverse definition of RBLCT.
c) RBLCT correlation theorem formulation activities.

Research framework

After compiling the definition of RBLCT, then a framework is made based on the formulation of the problem and research objectives.

The order of the framework of thought in this research activity can be described in a flow chart below.



Result and Discussion:

RBLCT Definition

Based on the definition of the reduced Biquaternion Fourier transform (RBFT), the definition of RBLCT is obtained by replacing the kernel from TF with the kernel from RBFT in the definition of LCT.

Denoted by $SL(2,\mathbb{R})$,a special linear group of degree 2 on $\mathbb{R}$, is a group of a matrix of the order 2×2 with a determinant of one. Suppose

Definition 3. The reduced biquaternion linear canonical transformation (RBLCT) of the reduced biquaternion signal f is defined by

$$\mathbf{L}^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = \begin{cases} \int_{R^2} K_{A_1}(x_1, \omega_1) f(\mathbf{x}) K_{A_2}(x_{2,}, \omega_2) d\mathbf{x}, b_n \neq 0, n = 1,2 \\ \sqrt{d_1 d_2}\, e^{\mathbf{i}\left(\frac{c_1 d_1}{2}\right)\omega_1^2} f(d_1\omega_1, d_2\omega_2) e^{\mathbf{k}\left(\frac{c_2 d_2}{2}\right)\omega_2^2}, b_1 = 0 \text{ atau} b_2 = 0 \end{cases}, \quad (1)$$

where the kernel of RBLCT is given by each.

$$K_{A_1}(x_1, \omega_1) = \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\,\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}, \quad (2)$$

and

$$K_{A_2}(x_2, \omega_2) = \frac{1}{\sqrt{2\pi b_2 \mathbf{k}}} e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\,\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}. \quad (3)$$

With $e^{\mathbf{i}\left(\frac{c_1 d_1}{2}\right)\omega_1^2}$ dan $e^{\mathbf{k}\left(\frac{c_1 d_1}{2}\right)\omega_1^2}$called Chirp signal in signal processing. Because $\mathbf{L}^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})$is trivial for $b_1 = 0$ or $b_2 = 0$, in this study it is always assumed that $b_n \neq 0$ for$n = 1,2$. As a special case, when $A_1 = A_2 = (a_s, b_s, c_s, d_s) = (0, 1, -1, 0)fors = 1, 2$, the definition of RBLCT in equation (1) is reduced to the definition of RBFT, namely

$$L^{i,k}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = \int_{R^2} \frac{1}{\sqrt{2\pi\mathbf{i}}} e^{-\mathbf{i}\omega_1 x_1} f(\mathbf{x}) \frac{1}{\sqrt{2\pi\mathbf{k}}} e^{-\mathbf{k}\omega_2 x_2}\, d\mathbf{x}$$
$$= \frac{1}{\sqrt{2\pi\mathbf{i}}} \mathcal{F}_{RB}\{f\}(\boldsymbol{\omega}) \frac{1}{\sqrt{2\pi\mathbf{k}}}. \quad (4)$$

Theorem 1. The inverse of the reduced biquaternion linear canonical transformation is given by

$$f(\mathbf{x}) = \begin{cases} \int\int_{R^2} K_{A_1^{-1}}(x_1, \omega_1) L^{\mathbf{i},\mathbf{k}}_{A_1,A_2} f(\boldsymbol{\omega}) K_{A_2^{-1}}(x_{2,}, \omega_2) d\boldsymbol{\omega}, b_n \neq 0, n = 1,2 \\ \sqrt{a_1 a_2} e^{-\mathbf{i}\left(\frac{c_1 a_1}{2}\right)x_1^2} f(a_1 x_1, a_2 x_2) e^{-\mathbf{k}\left(\frac{c_2 a_2}{2}\right)x_2^2}, b_1 = 0 \text{ ataub}_2 = 0 \end{cases}, \quad (5)$$

when$A_1^{-1} = (d_1, -b_1, -c_1, a_1)andA_2^{-1} = (d_2, -b_2, -c_2, a_2)$.

Characteristics of RBLCT

The following proposition presents some useful properties of the kernel functions $K_{A_1}(x_1, \omega_1)$ and $K_{A_2}(x_2, \omega_2)$RBLCT, which will be used to derive the Parseval formula

Proposition 1. Given the kernels of the functions $K_{A_1}(x_1, \omega_1)$ and $K_{A_2}(x_2, \omega_2)$defined by (2) and (3). next we get:

I. $K_{A_1}(-x_1, \omega_1) = K_{A_1}(x_1, -\omega_1)$ and $K_{A_2}(-x_2, \omega_2) = K_{A_2}(x_2, -\omega_2)$;

II. $K_{A_1}(-x_1, -\omega_1) = K_{A_1}(x_1, \omega_1)$ dand $K_{A_2}(-x_2, -\omega_2) = K_{A_2}(x_2, \omega_2)$;

III. $\overline{K_{A_1}(x_1, \omega_1) K_{A_2}(x_2, \omega_2)} = K_{A_1^{-1}}(x_1, \omega_1) K_{A_2^{-1}}(x_2, \omega_2)$.

Proof 1. By using equation (2) the proof of the proposition $K_{A_1}(-x_1, \omega_1) = K_{A_1}(x_1, -\omega_1)$ is as follows

$$K_{A_1}(-x_1, \omega_1) = \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}(-x_1)^2 - \frac{2}{b_1}(-x_1)\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 + \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)},$$

Similar to

$$K_{A_1}(x_1, -\omega_1) = \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1(-\omega_1) + \frac{d_1}{b_1}(-\omega_1)^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 + \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}$$
$$= K_{A_1}(-x_1, \omega_1).$$

If the same operation is performed in equation (3), then we get

$$K_{A_2}(-x_2, \omega_2) = \frac{1}{\sqrt{2\pi b_2 \mathbf{k}}} e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}(-x_2)^2 - \frac{2}{b_2}(-x_2)\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_2 \mathbf{k}}} e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 + \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)},$$

Similar to

$$K_{A_2}(x_2, -\omega_2) = \frac{1}{\sqrt{2\pi b_2 \mathbf{k}}} e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2(-\omega_2) + \frac{d_2}{b_2}(-\omega_2)^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_2 \mathbf{k}}} e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 + \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}$$
$$= K_{A_2}(-x_2, \omega_2).$$

Proof 2. By using equation (2) the proof of the proposition $K_{A_1}(-x_1, -\omega_1) = K_{A_1}(x_1, \omega_1)$ is as follows

$$K_{A_1}(-x_1, -\omega_1) = \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}(-x_1)^2 - \frac{2}{b_1}(-x_1)(-\omega_1) + \frac{d_1}{b_1}(-\omega_1)^2\right)}$$

$$= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\,\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}$$
$$= K_{A_1}(x_1, \omega_1).$$

while $K_{A_2}(-x_2, -\omega_2) = K_{A_2}(x_2, \omega_2)$can be proven by using equation (3), as follows

$$K_{A_2}(-x_2, -\omega_2) = \frac{1}{\sqrt{2\pi b_2 \mathbf{k}}} e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}(-x_2)^2 - \frac{2}{b_2}(-x_2\,)(-\omega_2) + \frac{d_2}{b_2}(-\omega_2)^2\right)}$$
$$= \frac{1}{\sqrt{2\pi b_2 \mathbf{k}}} e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\,\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}$$
$$= K_{A_2}(x_2, \omega_2).$$

Proof 3. From equations (2) and (3) it can be written as follows

$$\overline{K_{A_1}(x_1, \omega_1) K_{A_2}(x_2, \omega_2)} = \overline{\frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\,\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)}}$$
$$\overline{\frac{1}{\sqrt{2\pi b_2 \mathbf{k}}} e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\,\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}}$$
$$= \frac{1}{\sqrt{2\pi b_1 (-\mathbf{i})}} e^{-\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\,\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} \frac{1}{\sqrt{2\pi b_2 (-\mathbf{k})}} e^{-\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\,\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}$$
$$= \frac{1}{\sqrt{-2\pi b_1 \mathbf{i}}} e^{-\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1} - \frac{2}{b_1}x_1\,\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} \frac{1}{\sqrt{-2\pi b_2 \mathbf{k}}} e^{-\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\,\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)},$$

by using the inverse RBLCT in equation (5), it can be written as follows

$$K_{A_1^{-1}}(x_1, \omega_1) K_{A_2^{-1}}(x_2, \omega_2) = \frac{1}{\sqrt{2\pi(-b_1)\mathbf{i}}} e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{(-b_1)}x_1^2 - \frac{2}{(-b_1)}x_1\,\omega_1 + \frac{d_1}{(-b_1)}\omega_1^2\right)}$$
$$\frac{1}{\sqrt{2\pi(-b_2)\mathbf{k}}} e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{(-b_2)}x_2^2 - \frac{2}{(-b_2)}x_2\,\omega_2 + \frac{d_2}{(-b_2)}\omega_2^2\right)}$$
$$= \frac{1}{\sqrt{-2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{1}{2}\left(-\frac{a_1}{b_1}x_1^2 + \frac{2}{b_1}x_1\,\omega_1 - \frac{d_1}{b_1}\omega_1^2\right)} \frac{1}{\sqrt{-2\pi b_2 \mathbf{k}}} e^{\mathbf{k}\frac{1}{2}\left(-\frac{a_2}{b_2}x_2^2 + \frac{2}{b_2}x_2\,\omega_2 - \frac{d_2}{b_2}\omega_2^2\right)}$$
$$= \frac{1}{\sqrt{-2\pi b_1 \mathbf{i}}} e^{-\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\,\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} \frac{1}{\sqrt{-2\pi b_2 \mathbf{k}}} e^{-\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\,\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)}$$
$$= \overline{K_{A_1}(x_1, \omega_1) K_{A_2}(x_2, \omega_2)}.$$

The following lemma describes in general the relationship between RBLCT and RBFT of a signal f.

Lemma 1. RBLCT of signal f with parameter matrix $A_1 = (a_1, b_1, c_1, d_1)$dan $A_2 = (a_2, b_2, c_2, d_2)$ can be written as signal f of RBFT which is written in the form

$$L_{A_1,A_2}^{i,k}\{f\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} \frac{1}{\sqrt{2\pi b_1 \mathbf{k}}} e^{\frac{\mathbf{i}d_1}{2b_1}\omega_1^2} e^{\frac{\mathbf{k}d_2}{2b_2}\omega_2^2} \mathcal{F}_{RB}\left\{e^{\frac{\mathbf{i}a_1}{2b_1}x_1^2} e^{\frac{\mathbf{k}a_2}{2b_2}x_2^2} f(\mathbf{x})\right\}$$
$$\left(\frac{\omega_1}{b_1}, \frac{\omega_2}{b_2}\right). \qquad (6)$$

Proof 4. A simple calculation using definition 1 shows that

$$L_{A_1,A_2}^{i,k}\{f\}(\boldsymbol{\omega}) = \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} \int_{\mathbb{R}^2} e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2 - \frac{2}{b_1}x_1\omega_1 + \frac{d_1}{b_1}\omega_1^2\right)} f(\mathbf{x}) \frac{1}{\sqrt{2\pi b_1 k}}$$
$$e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2 - \frac{2}{b_2}x_2\omega_2 + \frac{d_2}{b_2}\omega_2^2\right)} d\mathbf{x}$$
$$= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} \int_{\mathbb{R}^2} e^{\mathbf{i}\frac{a_1}{2b_1}x_1^2 - \mathbf{i}\frac{\omega_1}{b_1}x_1 + \mathbf{i}\frac{d_1}{2b_1}\omega_1^2} f(\mathbf{x}) \frac{1}{\sqrt{2\pi b_1 \mathbf{k}}} e^{\mathbf{k}\frac{a_2}{2b_2}x_2^2 - \mathbf{k}\frac{\omega_2}{b_2}x_2 + k\frac{d_2}{2b_2}\omega_2^2} d\mathbf{x}$$
$$= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{d_1}{2b_1}\omega_1^2} \int_{\mathbb{R}^2} e^{-\mathbf{i}\frac{\omega_1}{b_1}x_1} \left(e^{\mathbf{i}\frac{a_1}{2b_1}x_1^2} e^{\mathbf{k}\frac{a_2}{2b_2}x_2^2} f(\mathbf{x})\right) \frac{1}{\sqrt{2\pi b_1 \mathbf{k}}} e^{-\mathbf{k}x_2\frac{\omega_2}{b_2}} e^{\mathbf{k}\frac{d_2}{2b_2}\omega_2^2} d\mathbf{x}$$

$$= \frac{1}{\sqrt{2\pi b_1 \mathbf{i}}} e^{\mathbf{i}\frac{d_1}{2b_1}\omega_1^2} \mathcal{F}_{RB}\left\{e^{\mathbf{i}\frac{a_1}{2b_1}x_1^2} e^{\mathbf{k}\frac{a_2}{2b_2}x_2^2} f(\mathbf{x})\right\}\left(\frac{\omega_1}{b_1}, \frac{\omega_2}{b_2}\right) \frac{1}{\sqrt{2\pi b_1 \mathbf{k}}} e^{\frac{\mathbf{k}d_2}{2b_2}\omega_2^2}. \quad (7)$$

Where the last line is obtained from the definition of RBFT in equation (4).

Furthermore, an alternative proof of the Plancherel formula for RBLCT is provided.

Theorem 2. (Plancherel RBLCT). The two reduced biquaternion functions f and g of RBLCT have the Plancherel formula, which is given as

$$\int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x} = \int_{\mathbb{R}^2} L^{i,k}_{A_1,A_2}\{f\}(\omega)\overline{L^{i,k}_{A_1,A_2}\{g\}(\omega)}d\boldsymbol{\omega}. \quad (8)$$

Proof 5. By using the RBLCT inverse in equation (5), it is given that

$$\int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} L^{i,k}_{A_1,A_2}\{f\}(\boldsymbol{\omega})K_{A_1^{-1}}(x_1, \omega_1)$$
$$K_{A_2^{-1}}(x_2, \omega_2)d\boldsymbol{\omega}\overline{g(\mathbf{x})}d\mathbf{x}$$

By using proposition 1 part (iii), then the above equation can be written as

$$= \int_{\mathbb{R}^2} L^{i,k}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \int_{\mathbb{R}^2} K_{A_1^{-1}}(x_1, \omega_1)K_{A_2^{-1}}(x_2, \omega_2)\overline{g(\mathbf{x})}d\mathbf{x}d\boldsymbol{\omega}.$$
$$= \int_{\mathbb{R}^2} L^{i,k}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) \overline{\int_{\mathbb{R}^2} K_{A_1}(x_1, \omega_1)g(\mathbf{x})K_{A_2}(x_2, \omega_2)d\mathbf{x}}\, d\boldsymbol{\omega}$$
$$= \int_{\mathbb{R}^2} L^{i,k}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\overline{L^{i,k}_{A_1,A_2}\{g\}(\boldsymbol{\omega})}d\boldsymbol{\omega}. \quad (9)$$

In the first equation of equation (9) the reduced biquaternion function f has been replaced by the inverse RBLCT (7). In the second equation, the order of integration has been exchanged and in the third equation part (iii) in Proposition (1) has been applied, while in the last equation, the definition of RBLCT in equation (1) is used to complete the proof of the theorem.

A special case of the Plancherel formula for RBLCT obtained the Plancherel formula from RBLCT as follows.
Corollaries 1. (Parseval of RBLCT). If the reduced biquaternion function is f(x) = g(x), then the Plancherel formula from RBLCT will be reduced to the Parseval formula from RBLCT, which states that

$$\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbb{R}^2} \left|L^{i,k}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}. \quad (10)$$

Equation (10) is a statement about the energy content in the reduced biquaternion signal. It states that the total energy signal calculated in the spatial domain is the same as the total energy calculated in the RBLCT domain. Parseval's formula allows the reduced value biquaternion energy signal in either the spatial domain or the RBLCT domain and the domains can be interchanged for easy computation.

Correlation Theorem for RBLCT

Cross-correlation of reduced biquaternions and convolution of reduced biquaternions are actually related in that the correlation of reduced biquaternions can be considered as the conjugate of convolution of reduced biquaternions. In this section, the correlation of two reduced biquaternion signals in the RBLCT domain will be defined. This definition is similar to the definition of correlation in RBFT domain. Furthermore, a theorem will be constructed, which describes the relationship between RBLCT and the correlation of two reduced biquaternion signals.

Definition 4. For each reduced biquaternion signal f and g, the correlation in the RBLCT domain is defined as

$$(f \oslash g)(\mathbf{x}) = \int_{\mathbb{R}^2} \overline{\frac{1}{\sqrt{-2\pi b_1 \mathbf{i}}}\frac{1}{\sqrt{-2\pi b_2 \mathbf{k}}} f(\mathbf{t})} g(\mathbf{x}+\mathbf{t}) e^{\mathbf{i}\frac{a_1}{b_1}t_1(t_1+x_1)}$$
$$\times e^{\mathbf{k}\frac{a_2}{b_2}t_2(t_2+x_2)} d\mathbf{t}. \quad (11)$$

Theorem4. For two reduced biquaternion signals f and g, the correlation for RBLCT is obtained as

$$L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f \oslash g\}(\boldsymbol{\omega}) = L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\overline{L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f\}(-\boldsymbol{\omega})}e^{-\mathbf{i}\frac{d_1\omega_1^2}{2b_1}}e^{-\mathbf{k}\frac{d_2\omega_2^2}{2b_2}}. \quad (12)$$

Where is the parameter matrix $A_1^* = (a_1, -b_1, c_1, d_1)$ and $A_2^* = (a_2, -b_2, c_2, d_2)$Noting that when $A_1^* = A_2^* = (a_1, -b_1, c_1, d_1) = (a_2, -b_2, c_2, d_2) = (0, 1, -1, 0)$the above equation is simplified to

$$L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f \oslash g\}(\boldsymbol{\omega}) = \overline{L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f\}(-\boldsymbol{\omega})}L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega}). \qquad (13)$$

Proof 6. From the definition of RBLCT in equation (4.1) and the definition of the correlation of two reduced biquaternion signals in the RBLCT domain in equation (4.21), we get

$$L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f \oslash g\}(\boldsymbol{\omega})$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\frac{1}{\sqrt{-2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{-2\pi b_2\mathbf{k}}}f(\mathbf{t})}g(\mathbf{x}+\mathbf{t})\, e^{\mathbf{i}\frac{a_1}{b_1}t_1(t_1+x_1)}e^{\mathbf{k}\frac{a_2}{b_2}t_2(t_2+x_2)}$$
$$\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{2\pi b_2\mathbf{k}}}e^{\frac{1}{2}\mathbf{i}\left(\frac{a_1}{b_1}x_1^2-\frac{2}{b_1}x_1\omega_1+\frac{d_1}{b_1}\omega_1^2\right)}e^{\frac{1}{2}\mathbf{k}\left(\frac{a_2}{b_2}x_2^2-\frac{2}{b_2}x_2\omega_2+\frac{d_2}{b_2}\omega_2^2\right)}d\mathbf{t}d\mathbf{x}.$$

By substituting $\mathbf{v} = \mathbf{x} + \mathbf{t}$,$v_1 = x_1 + t_1$, and $v_2 = x_2 + t_2$,which can be written as$\mathbf{x} = \mathbf{v} - \mathbf{t}$,$x_1 = v_1 - t_1$, and $x_2 = v_2 - t_2$,then we obtain

$$L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f \oslash g\}(\boldsymbol{\omega})$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\frac{1}{\sqrt{-2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{-2\pi b_2\mathbf{k}}}f(\mathbf{t})}g(\mathbf{v})\, e^{\mathbf{i}\frac{a_1}{b_1}(t_1v_1)}e^{\mathbf{k}\frac{a_2}{b_2}t_2(t_2v_2)}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{2\pi b_2\mathbf{k}}}$$
$$e^{\frac{1}{2}\mathbf{i}\left(\frac{a_1}{b_1}(v_1-t_1)^2-\frac{2}{b_1}(v_1-t_1)\omega_1+\frac{d_1}{b_1}\omega_1^2\right)}e^{\frac{1}{2}\mathbf{k}\left(\frac{a_2}{b_2}(v_2-t_2)^2-\frac{2}{b_2}(v_2-t_2)\omega_2+\frac{d_2}{b_2}\omega_2^2\right)}d\mathbf{t}d\mathbf{v}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\frac{1}{\sqrt{-2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{-2\pi b_2\mathbf{k}}}f(\mathbf{t})}g(\mathbf{v})\, e^{\mathbf{i}\frac{a_1}{b_1}(t_1v_1)}e^{\mathbf{k}\frac{a_2}{b_2}t_2(t_2v_2)}\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{2\pi b_2\mathbf{k}}}$$
$$e^{\frac{1}{2}\mathbf{i}\left(\frac{a_1}{b_1}(v_1^2-2v_1t_1+t_1^2)-\frac{2}{b_1}(v_1-t_1)\omega_1+\frac{d_1}{b_1}\omega_1^2\right)}$$
$$e^{\frac{1}{2}\mathbf{k}\left(\frac{a_2}{b_2}(v_1^2-2v_1t_1+t_1^2)-\frac{2}{b_2}(v_2-t_2)\omega_2+\frac{d_2}{b_2}\omega_2^2\right)}d\mathbf{t}d\mathbf{v}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\frac{1}{\sqrt{-2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{-2\pi b_2\mathbf{k}}}f(\mathbf{t})}g(\mathbf{v})\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{2\pi b_2\mathbf{k}}}e^{\mathbf{i}\frac{a_1}{2b_1}v_1^2}e^{\mathbf{i}\frac{a_1}{2b_1}t_1^2}$$
$$e^{-\mathbf{i}\frac{a_1}{b_1}(v_1t_1)}e^{-\mathbf{i}\frac{1}{b_1}(v_1\omega_1)}e^{\mathbf{i}\frac{1}{b_1}(t_1\omega_1)}e^{\mathbf{k}\frac{a_2}{2b_2}v_2^2}e^{\mathbf{k}\frac{a_2}{2b_2}t_2^2}e^{-\mathbf{k}\frac{a_2}{b_2}(v_2t_2)}e^{-\mathbf{k}\frac{1}{b_2}(v_2\omega_2)}$$
$$e^{\mathbf{k}\frac{1}{b_2}(t_2\omega_2)}e^{\mathbf{i}\frac{d_1}{2b_1}\omega_1^2}e^{\mathbf{k}\frac{d_2}{2b_2}\omega_2^2}e^{\mathbf{i}\frac{a_1}{b_1}(v_1t_1)}e^{\mathbf{k}\frac{a_2}{b_2}(v_2t_2)}d\mathbf{t}d\mathbf{v}$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \overline{\frac{1}{\sqrt{-2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{-2\pi b_2\mathbf{k}}}f(\mathbf{t})}g(\mathbf{v})\frac{1}{\sqrt{2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{2\pi b_2\mathbf{k}}}e^{\mathbf{i}\frac{a_1}{2b_1}v_1^2}e^{-\mathbf{i}\frac{1}{b_1}(v_1\omega_1)}$$
$$e^{\mathbf{i}\frac{d_1}{2b_1}\omega_1^2}e^{\mathbf{i}\frac{a_1}{2b_2}v_2^2}e^{-\mathbf{k}\frac{1}{b_2}(v_2\omega_2)}e^{\mathbf{k}\frac{d_2}{2b_2}\omega_2^2}e^{\mathbf{k}\frac{a_2}{2b_1}t_1^2}e^{\mathbf{i}\frac{1}{b_1}(t_1\omega_1)}e^{\mathbf{k}\frac{a_2}{2b_2}t_2^2}e^{\mathbf{k}\frac{1}{b_2}(t_2\omega_2)}d\mathbf{t}d\mathbf{v}.$$

(14)

Which is based on the definition of RBLCT in equation (1) that has been written previously

$$L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f \oslash g\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2}\overline{\frac{1}{\sqrt{-2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{-2\pi b_2\mathbf{k}}}f(\mathbf{t})}L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})e^{\mathbf{i}\frac{a_1}{2b_1}t_1^2}$$
$$e^{\mathbf{i}\frac{t_1\omega_1}{b_1}}e^{\mathbf{k}\frac{a_2}{2b_2}t_2^2}e^{\mathbf{k}\frac{t_2\omega_2}{b_2}}d\mathbf{t}.$$

Using the conjugate of the reduced biquaternion yields

$$L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f \oslash g\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2}L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})$$
$$\overline{f(\mathbf{t})\frac{1}{\sqrt{-2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{-2\pi b_2\mathbf{k}}}e^{-\mathbf{i}\frac{a_1}{2b_1}t_1^2}e^{-\mathbf{i}\frac{t_1\omega_1}{b_1}}e^{-\mathbf{k}\frac{a_2}{2b_2}t_2^2}e^{-\mathbf{k}\frac{t_2\omega_2}{b_2}}}d\mathbf{t}$$
$$= \int_{\mathbb{R}^2}L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\,\overline{f(\mathbf{t})\frac{1}{\sqrt{-2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{-2\pi b_2\mathbf{k}}}e^{-\mathbf{i}\frac{\mathbf{1}}{\mathbf{2}}\left(\frac{a_1}{b_1}t_1^2+\frac{2}{b_1}t_1\omega_1\right)}e^{-\mathbf{k}\frac{\mathbf{1}}{\mathbf{2}}\left(\frac{a_2}{b_2}t_2^2+\frac{2}{b_2}t_2\omega_2\right)}}d\mathbf{t}.$$

(15)

By multiplying both sides of equation (15) by the identity $e^{\mathbf{i}\frac{d_1\omega_1^2}{2b_1}}$ and $e^{\mathbf{k}\frac{d_2\omega_2^2}{2b_2}}$, easily obtained

$$L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f \oslash g\}(\boldsymbol{\omega})e^{\mathbf{i}\frac{d_1\omega_1^2}{2b_1}}e^{\mathbf{k}\frac{d_2\omega_2^2}{2b_2}} = \int_{\mathbb{R}^2} L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\overline{f(\mathbf{t})\frac{1}{\sqrt{-2\pi b_1\mathbf{i}}}\frac{1}{\sqrt{-2\pi b_2\mathbf{k}}}}$$
$$\overline{e^{\mathbf{i}\frac{1}{2}\left(-\frac{a_1}{b_1}t_1^2-\frac{2}{b_1}t_1\omega_1-\frac{d_1\omega_1^2}{b_1}\right)}e^{\mathbf{k}\frac{1}{2}\left(-\frac{a_2}{b_2}t_2^2-\frac{2}{b_2}t_2\omega_2-\frac{d_2\omega_2^2}{b_2}\right)}}d\mathbf{t}.$$

Finally obtained

$$L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f \oslash g\}(\boldsymbol{\omega}) = L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega})\overline{L^{\mathbf{i},\mathbf{k}}_{A_1^*,A_2^*}\{f\}(-\boldsymbol{\omega})}e^{-\mathbf{i}\frac{d_1\omega_1^2}{2b_1}}e^{-\mathbf{j}\frac{d_2\omega_2^2}{2b_2}}. \qquad (16)$$

This was the expected result.

Conclusion:

Based on the discussion, the reduced biquaternion linear canonical transformation (RBLCT) of the reduced biquaternion signal f is defined by

$$\mathbf{L}^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f\}(\boldsymbol{\omega}) = \begin{cases} \int_{R^2} K_{A_1}(x_1,\omega_1)f(\mathbf{x})K_{A_2}(x_2,\omega_2)d\mathbf{x}, & b_n \neq 0, n = 1,2 \\ \sqrt{d_1 d_2}e^{\mathbf{i}\left(\frac{c_1d_1}{2}\right)\omega_1^2}f(d_1\omega_1, d_2\omega_2)e^{\mathbf{k}\left(\frac{c_2d_2}{2}\right)\omega_2^2}, & b_1 = 0 \text{ atau } b_2 = 0 \end{cases}$$

where

$$K_{A_1}(x_1,\omega_1) = \frac{1}{\sqrt{2\pi b_1\mathbf{i}}}e^{\mathbf{i}\frac{1}{2}\left(\frac{a_1}{b_1}x_1^2-\frac{2}{b_1}x_1\omega_1+\frac{d_1}{b_1}\omega_1^2\right)},$$

and

$$K_{A_2}(x_2,\omega_2) = \frac{1}{\sqrt{2\pi b_2\mathbf{k}}}e^{\mathbf{k}\frac{1}{2}\left(\frac{a_2}{b_2}x_2^2-\frac{2}{b_2}x_2\omega_2+\frac{d_2}{b_2}\omega_2^2\right)}.$$

The inverse of the reduced biquaternion linear canonical transformation is given by

$$f(\mathbf{x}) = \begin{cases} \int_{R^2} K_{A_1^{-1}}(x_1,\omega_1)L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}f(\boldsymbol{\omega})K_{A_2^{-1}}(x_2,\omega_2)d\boldsymbol{\omega}, & b_n \neq 0, n = 1,2 \\ \sqrt{a_1a_2}e^{-\mathbf{i}\left(\frac{c_1a_1}{2}\right)x_1^2}f(a_1x_1, a_2x_2)e^{-\mathbf{k}\left(\frac{c_2a_2}{2}\right)x_2^2}, & b_1 = 0 \text{ atau } b_2 = 0 \end{cases},$$

where $A_1^{-1} = (d_1, -b_1, -c_1, a_1)$ dan $A_2^{-1} = (d_2, -b_2, -c_2, a_2)$

The Plancherel and Parseval properties of the two reduced biquaternion functions f and g from RBLCT are as follows.

a) Plancherel properties

$$\int_{\mathbb{R}^2} f(\mathbf{x})\overline{g(\mathbf{x})}d\mathbf{x} = \int_{\mathbb{R}^2} L^{i,k}_{A_1,A_2}\{f\}(\omega)\overline{L^{i,k}_{A_1,A_2}\{g\}(\omega)}d\boldsymbol{\omega}.$$

b) Parseval Proprties

$$\int_{\mathbb{R}^2} |f(\mathbf{x})|^2 d\mathbf{x} = \int_{\mathbb{R}^2} \left|L^{i,k}_{A_1,A_2}\{f\}(\boldsymbol{\omega})\right|^2 d\boldsymbol{\omega}.$$

The convolution for each function f and g in the reduced biquaternion linear canonical transformation, defined by

$$L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f \odot g\}(\boldsymbol{\omega}) = e^{-\mathbf{i}\frac{d_1\omega_1^2}{2b_1}}e^{-\mathbf{k}\frac{d_2}{2b_2}\omega_2^2}L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{f\}(\boldsymbol{\omega})L^{\mathbf{i},\mathbf{k}}_{A_1,A_2}\{g\}(\boldsymbol{\omega}).$$

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