A New Lower Bound on the Real Grothendieck Constant

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ABSTRACT

A new method shows that the Grothendieck constant $K_G \geq 1.6769...$. The previous best lower bound was $K_G \geq \pi/2 \approx 1.57...$.

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1 Introduction

One form of Grothendieck's inequality [G] states that there is a best constant $K_G < \infty$ such that for any bounded linear operator $T : L^\infty(X,\mu) \to L^1(X,\mu)$ and for any real bounded functions $f_1, ..., f_n \in L^\infty$,

$$\left\| \sqrt{\sum (Tf_i)^2} \right\|_1 \le K_G \, \|T\| \cdot \left\| \sqrt{\sum f_i^2} \right\|_\infty ,$$

where $\|T\|$ is the operator norm. The constant K_G is universal: it does not depend on T, or on the functions f_i, or even on the underlying measure space (X,μ). For general information about Grothendieck's inequality, see [K], [L] and especially [P].

The numerical value of K_G has been a minor mystery from the beginning. Grothendieck [G] showed that $\pi/2 \le K_G \le \sinh(\pi/2) = 2.301...$. Krivine [K] showed that $K_G \le \pi/2\log(1+\sqrt{2}) = 1.782$, and stated that K_G was probably equal to this last value.

This paper proves that $K_G > \pi/2$, and by a numerical computation, that $K_G \ge 1.67$. The general method is to construct a particular family of operators $T_\lambda : L^\infty \to L^1$, for $\lambda \in \mathbf{R}$ and show that for all such λ, $K_G \ge |1-\lambda|/\|T_\lambda\|$. Optimization with respect to λ yields the new bound.

The main idea for constructing T_λ comes from [F], where it was observed (in a matrix case) that subtracting a small multiple of the identity operator from a projection operator sometimes increases the value of the "Grothendieck ratio": $r(T_\lambda) = r(T - \lambda I) > r(T)$, where

$$r(T) = \sup \left\{ \frac{\left\| \sqrt{\sum (Tf_i)^2} \right\|_1}{\|T\| \left\| \sqrt{\sum f_i^2} \right\|_\infty} \; : \; f_i \in L^\infty \right\} .$$

Originally, [F] started with a matrix T, whose entries are inner products of root vectors [Hu] in E_8, A_n and D_n. In the current paper we modify an operator T for which $r(T) = \pi/2$ which is (in the notation of [P], section 5.e, $T = \gamma\gamma^*$) essentially Grothendieck's original example implying $K_G \geq \pi/2$.

The Operators T and T_λ

Let (Ω, P) be a probability space on which $Z_1, Z_2, ...$ is a sequence of independent identically distributed Gaussian random variables, $P(Z_i \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2} dz = \int_{-\infty}^{x} \phi(z) dz$. Let Ef denote the expectation or integral of f with respect to P, that is, $Ef = \int_\Omega f dP$. Let $T : L^\infty(\Omega, P) \to L^1(\Omega, P)$ be the restriction to $L^\infty(\Omega, P) \subseteq L^2(\Omega, P)$ of the orthogonal projection onto the closed linear span of the Z_n. If f is bounded, Bessel's inequality implies $\sum_{n\geq 1}(EfZ_n)^2 \leq Ef^2 < \infty$ and so the series

$$Tf = \sum_{n\geq 1} Z_n EfZ_n$$

converges in L^2 and hence in L^1. For $\lambda \geq 0$, define $T_\lambda : L^\infty(\Omega, P) \to L^1(\Omega, P)$ by $T_\lambda f = Tf - \lambda f$.

Our first result is an estimate on $r(T_\lambda)$:

Theorem 1 $r(T_\lambda) \geq |1 - \lambda| / \|T_\lambda\|$.

Proof

For fixed n, let $R = \sqrt{\sum_1^n Z_i^2}$, and let $X_i = Z_i/R$ for $1 \le i \le n$. Then the n-tuple $(X_1, ..., X_n)$ is uniformly distributed over the unit sphere in $\mathbf{R}^n$. R^2 is a chi-squared random variable with n degrees of freedom and is independent of $(X_1, ..., X_n)$. Since $EX_iZ_j = 0$ unless $i = j$, it is easy to check that $TX_i = Z_iE(X_iZ_i) = Z_iE(X_i^2R) = Z_i(ER)EX_i^2 = Z_iER/n$. Thus

$$T_\lambda X_i = Z_iER/n - \lambda X_i = X_i\left(\frac{RER}{n} - \lambda\right) ,$$

and

$$\sqrt{\sum (T_\lambda X_i)^2} = \left|\frac{RER}{n} - \lambda\right| .$$

Thus, taking $f_i = X_i$ for $i = 1, ..., n$, we see that

$$\begin{aligned} r(T_\lambda) &\ge \frac{\left\|\sqrt{\sum (T_\lambda f_i)^2}\right\|_1}{\left\|\sqrt{\sum f_i^2}\right\|_\infty} \frac{1}{\|T_\lambda\|} \\ &= \frac{E\left|\frac{RER}{n} - \lambda\right|}{\|T_\lambda\|} , \end{aligned}$$

since $\sqrt{\sum f_i^2} = \sqrt{\sum X_i^2} = 1$ with probability one.

Now, as $n \to \infty$, $R/\sqrt{n}$ converges almost surely to 1, $ER/\sqrt{n} \to 1$ and $E\left|\frac{R}{\sqrt{n}}\left(\frac{ER}{\sqrt{n}}\right) - \lambda\right|$ converges to $|1-\lambda|$. Thus, $r(T_\lambda) \ge |1-\lambda|/\|T_\lambda\|$. ■

Norm of T_λ

To use the estimate $K_G \ge r(T_\lambda) \ge |1-\lambda|/\|T_\lambda\|$ we need to calculate $\|T_\lambda\|$, which is of course the supremum of $E|T_\lambda f|$ over all random variables f for which $P(|f| \le 1) = 1$. We show that this supremum is attained by random variables of a very special form:

Theorem 2 *There exists a real $h \geq 0$ such that $\|T_\lambda\| = \|T_\lambda g_h\|_1$, where*

$$\begin{aligned} g_h = 1 \quad &\textit{if} \quad Z_1 \in (-h, 0) \cup (h, \infty) \\ -1 \quad & \quad\;\; Z_1 \in (-\infty, -h) \cup (0, h) \ . \end{aligned}$$

Once we have this result we can write our lower bound for K_G:

$$\begin{aligned} K_G \quad &\geq \quad \sup_\lambda \left(|1-\lambda| / \sup_h \|T_\lambda g_h\|_1 \right) \\ &= \quad \sup_\lambda \inf_h \frac{|1-\lambda|}{\|T_\lambda g_h\|_1} \ . \end{aligned}$$

The task of actually computing this bound is straightforward but tedious.

The rest of this section gives the proof of Theorem 2.

Let $\mathcal{G}$ be the extreme points of the unit ball in L^∞, so $\mathcal{G} = \{g : |g| = 1 \text{ a.e.}\}$. For $\alpha \geq 0$ let $\mathcal{G}_\alpha = \{g \in \mathcal{G} : \ EgZ_1 = \alpha, \ EgZ_n = 0, \text{ for } n > 1\}$. Since $\|T_\lambda g\|_1$ is a convex function of g, $\|T_\lambda\| = \sup\{\|T_\lambda g\|_1 : g \in \mathcal{G}\}$, and a simple change of basis argument (in G, the L^2 closed span of the Z_i) shows that $\|T_\lambda\| = \sup\{\|T_\lambda g\|_1 : g \in \cup_{\alpha \geq 0} \mathcal{G}_\alpha\}$. So our strategy is to maximize $\|T_\lambda g\|_1$ over $g \in \mathcal{G}_\alpha$ for fixed α, and then to maximize over α:

$$\|T_\lambda\| = \sup_\alpha \sup\{\|T_\lambda g\|_1 : \ g \in \mathcal{G}_\alpha\} \ .$$

For any function $g \in \mathcal{G}_\alpha$ let $\theta_g(z)$ be the conditional expectation of g given Z_1, $\theta_g(z) = E(g|Z_1 = z)$ (so that for any h, $Egh(Z_1) = E\theta_g(Z_1)h(Z_1)$). Then clearly $|\theta_g(z)| \leq 1$ and $\alpha = \int_{\mathbf{R}} \phi(z) z \theta_g(z) dz$. Conversely, if $\theta(z)$ is such that $|\theta(z)| \leq 1$ a.e. and $\alpha = \int_{\mathbf{R}} \phi(z) z \theta(z) dz$, we construct an element $g \in \mathcal{G}_\alpha$ with $\theta = \theta_g$, as follows: let U be a random variable independent of all Z_i, uniformly distributed on $[0, 1]$. Let $g = 1$ if $U \leq \frac{1+\theta(Z_1)}{2}$ and let $g = -1$ if $U > \frac{1+\theta(Z_1)}{2}$.

It is useful below to write

$$\theta_g(z) = P(g = 1|Z_1 = z) - P(g = -1|Z_1 = z) \ .$$

For any $g \in \mathcal{G}_\alpha$ we can express $\|T_\lambda g\|_1$ in terms of θ_g, as follows. Since $EgZ_1 = \alpha$ and $EgZ_n = 0$ for $n > 1$, we have $T_\lambda g = \alpha Z_1 - \lambda g$, and

$$\begin{aligned}
\|T_\lambda g\|_1 &= E|\alpha Z_1 - \lambda g| \\
&= \int_{\mathbf{R}} \phi(z) \left(P(g = 1|Z_1 = z)|\alpha z - \lambda| + P(g = -1|Z_1 = z)|\alpha z + \lambda| \right) dz \\
&= \int_{\mathbf{R}} \phi(z) \frac{|\alpha z - \lambda| + |\alpha z + \lambda|}{2} dz + \int_{\mathbf{R}} \phi(z) \frac{|\alpha z - \lambda| - |\alpha z + \lambda|}{2} \theta_g(z) dz \ .
\end{aligned}$$

So the problem of maximizing $\|T_\lambda g\|_1$ for $g \in \mathcal{G}_\alpha$ is the same as that of maximizing

$$\int_{\mathbf{R}} \phi(z) \frac{|\alpha z - \lambda| + |\alpha z + \lambda|}{2} dz + \int_{\mathbf{R}} \phi(z) \frac{|\alpha z - \lambda| - |\alpha z + \lambda|}{2} \theta(z) dz \qquad (1)$$

subject to $|\theta(z)| \leq 1$ a.e. and $\int_{\mathbf{R}} \phi(z) z \theta(z) = \alpha$.

Let $\psi(z) = (|\alpha z - \lambda| - |\alpha z + \lambda|)/2$. Both the objective functional $\int_{\mathbf{R}} \phi(z)\psi(z)\theta(z)dz$ and the constraint functional $\int_{\mathbf{R}} \phi(z) z \theta(z) dz$ involve integrals of $\theta(z)$ against odd functions, so the optimizing θ might as well be odd. Indeed, if $\theta(z)$ solves the optimization problem, so does the odd function $\tilde{\theta}(z) = 1/2(\theta(z) - \theta(-z))$. So an equivalent optimization problem is, find $|\theta(z)| \leq 1$ for $z > 0$, to maximize

$$\int_0^\infty 2\phi(z)\psi(z)\theta(z)dz \quad \text{subject to} \quad \int_0^\infty 2\phi(z) z \theta(z) dz = \alpha \ .$$

This optimization problem can be solved by the Neyman-Pearson or "water pouring" monotone greedy assignment method. Since $\psi(z)/z$ is monotone increasing in $z \geq 0$, larger values of z are more "profitable," and the optimizing $\theta(z)$ will be of the form $\theta(z) = 1$ for all $z \geq h$ and $\theta(z) = -1$ for all $z < h$, for some $h \geq 0$. To see this, first let $h \geq 0$ solve the equation

$$\alpha \;=\; \left(\int_h^\infty - \int_0^h \right) 2\phi(z) z dz$$

$$= \sqrt{\frac{2}{\pi}}(2e^{-h^2/2} - 1) \; .$$

Suppose $\theta^*(z)$ is any other function satisfying the equality constraint $\int_0^\infty 2\phi(z) z\theta^*(z)dz = \alpha$ and the inequality constraint $|\theta^*(z)| \leq 1$. Let $\tau = \psi(h)/h$. Then for all $z \geq 0$,

$$\begin{aligned} \left(\frac{\psi(z)}{z} - \tau\right)\theta(z) &= \left|\frac{\psi(z)}{z} - \tau\right| \\ &\geq \left(\frac{\psi(z)}{z} - \tau\right)\theta^*(z) \end{aligned}$$

and hence the inequality

$$\int_0^\infty 2\phi(z)\left(\frac{\psi(z)}{z} - \tau\right) z\theta^*(z)dz \leq \int_0^\infty 2\phi(z)\left(\frac{\psi(z)}{z} - \tau\right) z\theta(z)dz$$

holds. Since both θ^* and θ satisfy the equality constraint, this already implies that for fixed α, the optimum is attained by a θ of the stated form.

This solves the "inner" optimization problem: given α, find $g \in \mathcal{G}_\alpha$ maximizing $\|T_\lambda g\|_1$. To show that the "outer" supremum over all $\alpha \geq 0$ is attained, note that the condition $\alpha \geq 0$ implies $0 \leq h \leq \sqrt{2\log 2}$, and that the map $h \mapsto \|T_\lambda g_h\|_1$ is continuous, and hence attains its supremum on $[0, \sqrt{2\log 2}]$. ■

Computation of $\|T_\lambda g_h\|_1$

So far we know that $\|T_\lambda\|$ is the maximum value of $\|T_\lambda g_h\|_1$ for $h \geq 0$, which we now proceed to compute. We use the formulae

$$\|T_\lambda g\|_1 = \int_{\mathbf{R}} \phi(z)((P(g=1|Z_1=z)|az-\lambda| + P(g=-1|Z_1=z)|az+\lambda|)dz$$

where

$$a = \int_{\mathbf{R}} \phi(z)(P(g=1|Z_1=z) - P(g=-1|Z_1=z))dz$$

which we met in the proof of Theorem 2. Specializing to the special form of g_h we obtain

$$\|T_\lambda g_h\|_1 = 2\int_h^\infty \phi(z)|az-\lambda|dz + 2\int_0^h \phi(z)|az+\lambda|dz \qquad (2)$$

where

$$a = a(h) = 2\int_h^\infty \phi(z)zdz - 2\int_0^h \phi(z)zdz = \sqrt{\frac{2}{\pi}}(2e^{-h^2/2}-1)\ .$$

To obtain $\|T_\lambda\|$, then, we need only find the maximum value of (2), as $h \geq 0$ varies in such a way that $a \geq 0$, viz, as h ranges over $[0, \sqrt{2\log 2}]$.

Equation (2) simplifies differently according to whether $ah \leq \lambda$ or $ah \geq \lambda$. If $ah \leq \lambda$, then

$$\begin{aligned}
\|T_\lambda g_h\|_1 &= 2\int_h^{\lambda/a} \phi(z)(\lambda - az)dz + 2\int_{\lambda/a}^\infty \phi(z)(az-\lambda)dz \\
&\quad + 2\int_0^h \phi(z)(az+\lambda)dz \\
&= a\left(\int_0^h - \int_h^{\lambda/a} + \int_{\lambda/a}^\infty\right) 2\phi(z)zdz + \lambda\left(\int_0^{\lambda/a} - \int_{\lambda/a}^\infty\right) 2\phi(z)dz \\
&= a(B(\infty) - 2B(\lambda/a) + 2B(h)) - \lambda(1 - 2A(\lambda/a))
\end{aligned}$$

and if $ah \geq \lambda$, then

$$\begin{aligned}
\|T_\lambda g_h\|_1 &= 2\int_h^\infty \phi(z)(az-\lambda)dz + 2\int_0^h \phi(z)(az+\lambda)dz \\
&= a\int_0^\infty 2\phi(z)zdz + \lambda\left(\int_0^h - \int_h^\infty\right) 2\phi(z)dz \\
&= aB(\infty) - \lambda(1-2A(h))\ ,
\end{aligned}$$

where

$$A(u) = \int_0^u 2\phi(z)dz = \sqrt{\frac{2}{\pi}}\int_0^u e^{-z^2/2}dz$$

and

$$B(u) = \int_0^u 2\phi(z)zdz = \sqrt{\frac{2}{\pi}}(1-e^{-u^2/2})\ ,$$

so $a = a(h) = B(\infty) - 2B(h)$.

In short, if we define

$$f_1(h) = a(h)(B(\infty) - 2B(\lambda/a(h)) + 2B(h)) - \lambda(1 - 2A(\lambda/a(h)))$$

and

$$f_2(h) = a(h)B(\infty) - \lambda(1 - 2A(h)) ,$$

then

$$\|T_\lambda g_h\|_1 = \begin{cases} f_1(h) & \text{if } \; a(h)h \leq \lambda \\ f_2(h) & \text{if } \; a(h)h \geq \lambda \; . \end{cases}$$

We will now show that the h in $[0, \sqrt{2\log 2}]$ which maximizes $\|T_\lambda g_h\|_1$ obeys $a(h)h \leq \lambda$. On the one hand, a direct calculation shows that

$$\frac{df_1(h)}{dh} = -2B'(h)(4B(h) - 2B(\lambda/a)) ,$$

so at a critical point of f_1, $B(\lambda/a) = 2B(h) \geq B(h)$, so $\lambda/a \geq h$. And on the other hand, for all $h \in [0, \sqrt{2\log 2}]$, $f_1(h) \geq f_2(h)$, as follows. Note that

$$f_1(h) - f_2(h) = 2a(h)(B(h) - B(\lambda/a(h))) - 2\lambda(A(h) - A(\lambda/a(h))) .$$

If $h > \lambda/a$ the inequality

$$\lambda/a \leq E(Z_1|\lambda/a \leq Z_1 \leq h) = \frac{\int_{\lambda/a}^h \phi(z)zdz}{\int_{\lambda/a}^h \phi(z)dz} = \frac{B(h) - B(\lambda/a)}{A(h) - A(\lambda/a)}$$

holds, which is equivalent to $f_1(h) - f_2(h) \geq 0$. And if $\lambda/\alpha > h$ we similarly have

$$\lambda/a \geq E(Z_1|h \leq Z_1 \leq \lambda/a) = \frac{B(\lambda/a) - B(h)}{A(\lambda/a) - A(h)} ,$$

which also implies $f_1(h) - f_2(h) \geq 0$.

Let h maximize f_1 on $[0, \sqrt{2\log 2}]$. It is easy to check that $f_1(0) > f_1(\sqrt{2\log 2})$ for all values of $\lambda > 0$, so $h < \sqrt{2\log 2}$. Direct calculation shows $f_1'(0) = 0$ and $f_1''(0) > 0$ so $h > 0$, and so h is an interior critical point of f_1, and hence $ha(h) \leq \lambda$, $\|T_\lambda g_h\|_1 = f_1(h)$, and h solves the interior critical point equation $B(\lambda/a(h)) = 2B(h)$. If we introduce the variable $\eta = \lambda/a(h)$, then the condition for a critical point is just $\lambda = \sqrt{\frac{2}{\pi}}\eta e^{-\eta^2/2}$. If $\lambda \in \left(0, \sqrt{\frac{2e}{\pi}}\right)$ there are exactly two positive values, say η_1 and η_2, satisfying the critical point condition, with $0 < \eta_1 < 1 < \eta_2$. The corresponding values of h, which are the only positive roots of $f_1'(h)$, are obtained from the η_i from $2B(h_i) = B(\eta_i)$, which results in

$$h_i = \sqrt{-2\log\frac{1+e^{-\eta_i^2/2}}{2}} \ .$$

It is easy to check that $f_1''(h_1) < 0$ and $f_1''(h_2) > 0$, so at last we obtain, after some manipulation, our formula for $\|T_\lambda g_h\|_1$, valid for $\lambda \in \left(0, \sqrt{\frac{\pi e}{2}}\right)$:

$$\|T_\lambda g_h\|_1 = f_1(h_1) = (\lambda/\eta_1)^2 - \lambda(1 - 2A(\eta_1)) \ .$$

Substituting $\lambda = \sqrt{\frac{2}{\pi}}\eta e^{-\eta^2/2}$, and regarding $0 < \eta < 1$ as the independent variable, our bound for K_G is thus

$$\begin{aligned} K_G &\geq \sup_{\eta\in[0,1]} \frac{1-\lambda}{(\lambda/\eta)^2 - \lambda(1-2A(\eta))} \\ &= \sup_{\eta\in[0,1]} \frac{1-\sqrt{\frac{2}{\pi}}\eta e^{-\eta^2/2}}{\frac{2}{\pi}e^{-\eta^2} - \sqrt{\frac{2}{\pi}}\eta e^{-\eta^2/2}(1-2A(\eta))} \\ &= \sup_{\eta\in[0,1]} Y(\eta) \ . \end{aligned}$$

Elementary calculus shows that the maximizing η solves the equation

$$1 - 2A(\eta) = \frac{2}{\pi}e^{-\eta^2}$$

and then $Y(\eta) = \frac{\pi}{2}e^{\eta^2}$. Numerical computations show that the maximizing η is approximately .25573021316621 and the corresponding value of Y is approximately 1.676956674215576. This corresponds to $\lambda = .25573$, $h = .18009$, and $a = .77222$.

We may summarize all these calculations as:

Theorem 3 *Let $\eta \in [0,1]$ solve the equation*

$$1 - 2\sqrt{\frac{2}{\pi}} \int_0^\eta e^{-z^2/2} dz = \frac{2}{\pi} e^{-\eta^2} .$$

Then $K_G \geq \frac{\pi}{2} e^{\eta^2}$.

Related Example

Here is another example, seemingly more concrete than the last. For fixed n, let S_n be the unit sphere in $\mathbf{R}^n$. (Surface of sphere, not solid ball.) Let μ be rotationally invariant measure on S_n, normalized so $\mu(S_n) = 1$. Let $L^\infty = L^\infty(S_n, \mu)$ be the bounded functions on S_n and $L^1 = L^1(S_n, \mu)$ be the integrable functions on S_n. Let $k : S_n \times S_n \to \mathbf{R}$ be the inner product function $k(x, y) = (x, y)$. Finally, define $T_n : L^\infty \to L^1$ by

$$(T_n f)(x) = \int_{S_n} k(x, u) f(u) du,$$

and for $\lambda \in \mathbf{R}$ define $T_{n,\lambda} : L^\infty \to L^1$ by

$$T_{n,\lambda} f = T_n f - \lambda f .$$

Arguing as in the previous sections, one obtains for each n a similar lower bound for K_G. When n becomes large, the numerical values seem to converge to the value obtained above. It is easy to believe that T and T_n are "essentially the same".

Now, decompose $L^2(S_n, \mu) = \oplus \mathcal{H}_j$ into the spaces of spherical harmonics [S], $\mathcal{H}_j$ being the degree j harmonic polynomial functions on $\mathbf{R}^n$, restricted to S_n. Let P_j denote the projection onto $\mathcal{H}_j$. Then the operators T_n and $T_{n\lambda}$ may be written in the form

$$\begin{aligned} T_n &= \alpha P_1 \\ T_{n\lambda} &= \alpha P_1 - \lambda \sum_{j=0}^{\infty} P_j \ . \end{aligned}$$

It would be interesting to attempt the direct computation of the norm of some more general operator $\sum \alpha_j P_j$, where the α_j are not all of the same sign, but the methods of this paper probably do not extend beyond the case where only one of the α_j is positive. (We know from [R] that if all $\alpha_j \geq 0$ then the Grothendieck ratio $r(\sum \alpha_j P_j)$ cannot exceed $\pi/2$, so improved lower bounds on K_G require some alternation in sign.)

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References

[F] Fishburn, P. C. and J. A. Reeds, "Bell Inequalities, Grothendieck's Constant, and Root Two." Unpublished MS, 1990.

[G] Grothendieck, A. "Résumé de la théorie métrique des produits tensoriels topologiques," *Bol. Soc. Math. São Paulo* **8**, 1–79 (1956).

[Ha] Halmos, P. R. "The range of a vector measure," *Bull. A.M.S.* **54**, 416–421 (1948).

[Hu] Humphreys, J. *Introduction to Lie Algebras and Representation Theory*, Springer, New York, 1972.

[K] Krivine, J. L. "Constantes de Grothendieck et fonctions de type positif sur les sphères," *Adv. Math.* **31**, 16–30 (1979).

[L] Lindenstrauss, J. and Tzafriri, L. *Classical Banach Spaces I*, Springer, New York, 1977.

[P] Pisier, G. *Factorization of linear operators and geometry of Banach spaces*, Providence, RI: American Mathematical Society, 1986.

[R] Rietz, R. E., "A proof of the Grothendieck inequality," *Israel J. Math.* **19**, 271–276 (1974).

[S] Stein, E. M. and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, Princeton, NJ, 1971.