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Product Operations on Pythagorean Co-Neutrosophic Graphs and its Application

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Abstract. Pythagorean Co-Neutrosophic Graphs (PyCNGs) have newly emerged after the foundation of Pythagorean fuzzy graph and Co-neutrosophic graphs, where the membership of indeterminacy(uncertain) is independent and membership of truth(existence) & false(non-existence) are dependent. This paper has a wide discussion of different product operations carried out in PyCNGs. Some properties like strong, complete with consideration of a vertex's degree in the graph are explored with examples. Also, an application related to the product on PyCNGs is demonstrated with brain network analysis and its function.

Keywords: Pythagorean co-neutrosophic graph; Modular product; Homomorphic product; Symmetric difference.

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1. Introduction

Graph theory attains global importance in the kind of real-life related discussion based on the theoretical concepts that already exist. Though it gives a detailed graphical structure for many problems of the real world, but uncertainty and vagueness prevail in the final output. Zadeh [39] refined crisp set theory and laid a new foundation in name of fuzzy set theory. Then, Kaufmann [15] excavated ideas on fuzzy graphical system which shows high accuracy when compared to crisp graphs. Further innovations regarding fuzzy graph models and their properties were discussed by Rosenfeld [29]. Bhattacharya [9], Bhuttani [10], and Nagoor Gani [18–22] laid an immense base by various discussions on structural representations of fuzzy graph and labelings. Other fuzzy field researchers followed up with more successful fuzzy graph-based works. With prior knowledge from Mordeson and Peng's [17] work, Nagoor Gani and Radha [20] applied conjunction over fuzzy graphs. Following that, various types of products on fuzzy graphs were developed and discussed briefly by Radha et al. [27, 28], and Shovan Dogra [32].

Krassimir Atannosov's [8] effort to reduce the inaccuracy occurs in fuzzy kind had succeeded by the introduction of intuitionistic fuzzy set(InFS), where explicit membership of non-existence becomes essential for coping with unforeseeable circumstances. R. Parvathi et al. [25, 26] extended the fuzzy graph concept to an intuitionistic fuzzy graph(InFG) with the help of InFS. M. Akram [5] and Nagoor Gani [23] elaborated on various properties of InFG which help to compare actual world components with InFG structure. Different types of products on InFG were demonstrated by Sankar Sahoo and Madhumangal Pal [30]. Yager [36–38] founded the Pythagorean fuzzy set(PyFS) to increase the flexibility of InFS, by adjusting the membership grades. M. Akram et al. [6] undertook research on Pythagorean fuzzy graphs (PyFG) and how they pertain to decision-making using PyFS.

F. Smarandache [33,34] successively enhanced the InFS by introducing a neutrosophic set(with explicit indeterminacy membership), which reduces the uncertain results of InFS in reality. The development of neutrosophic graphs, which include the expansion of neutrosophic sets, has been rendered viable thanks to the contributions of Smarandache. Wang et al. [35] devised a model for single-valued neutrosophic set (SVNS) via the neutrosophic set as its basis. Broumi and Smarandache [11] invented the notion of a single-valued neutrosophic graph (SVNG) and extensively investigated numerous features associated with its vertices and edges. Dhavaseelan et al. [12] analyzed the attributes of a single valued co-neutrosophic graph (SVCNG) by including anti-behavior aspects. They dealt with the degree and regularity of the reconstituted SVNG. With the Pythagorean neutrosophic set definition, Ajay et al. [1–4] innovated Pythagorean neutrosophic fuzzy graph (PyNFG). They exclusively dealt with labeling and regular property on PyNFG. Kartick Mohanta et al. [14] offered an in-depth investigation of a multitude of products derived from neutrosophic graphs. They particularly addressed the ideas of degree and total degree with these products. This work was extended to SVNG by Zeng [40] with some applications.

Pythagorean co-neutrosophic graphs(PyCNGs) are extensible when compared to neutrosophic graphs. Therefore, the results acquired through PyCNGs will be more accurate than neutrosophic graphs. In this paper, PyCNGs derived by using Pythagorean neutrosophic set(PyNS) have been taken and various types of products on PyCNGs were discussed and verified briefly with some additional properties. Finally, the product of the PyCNGs concept has been related to brain network analysis and its applications have been found.

This paper comprises the subsequent sections: Section 2 includes a comprehensive overview of the fundamental terminology of the PyCNGs. In section 3, different kinds of products concerned to PyCNGs has been structured and some properties like strong, complete are investigated with theorems. Section 4 demonstrates the utilization of product on PyCNGs, by comparing the brain functions of two individuals.

2. Preliminaries

Definition 2.1. *([1])*
A neutrosophic set C is defined on Z (universal set) is given by $C = \{\langle a, \alpha_N(a), \beta_N(a), \gamma_N(a)\rangle / a \in Z\}$, *where* $\alpha_C : Z \to [0,1]$, $\beta_C : Z \to [0,1]$ *and* $\gamma_C : Z \to [0,1]$ *represent the membership degree of existence, uncertain and non-existence function of vertex a on C, respectively with the requirement* $0 \leq \alpha_C(a) + \beta_C(a) + \gamma_C(a) \leq 3$, $\forall\ a \in Z$.

Definition 2.2. *([1])*
A neutrosophic graph on Z is a graph$(Gr) = (C, D)$ *with neutrosophic set C on Z and a neutrosophic relation D on Z, where* $C = \{a_1, a_2, \cdots, a_n\}$ *such that*
(i) α_C, β_C *and* γ_C *defined from Z to* $[0,1]$ *represent the membership degree of existence, uncertain and non-existence function of the vertex* $a \in Z$ *respectively with* $0 \leq \alpha_C(a) + \beta_C(a) + \gamma_C(a) \leq 3$, *for all* $a \in Z$.
(ii) $D \subseteq Z \times Z$ *with* α_D, β_D *and* γ_D *defined from* $Z \times Z$ *to* $[0,1]$ *represent the membership degree of existence, uncertain and non-existence function of the edges* $ab \in Z \times Z$ *respectively such that*
$\alpha_D(ab) \leq min[\alpha_C(a), \alpha_C(b)]$,
$\beta_D(ab) \leq min[\beta_C(a), \beta_C(b)]$,
$\gamma_D(ab) \leq max[\gamma_C(a), \gamma_C(b)]$ *and*
$0 \leq \alpha_D(ab) + \beta_D(ab) + \gamma_D(ab) \leq 3$, *for every ab.*

Definition 2.3. *([1])*
A Pythagorean neutrosophic set (PyNS) on an universal set Z is an element of type $C = \{\langle a, \alpha_C(a), \beta_C(a), \gamma_C(a)\rangle / a \in Z\}$, *where* $\alpha_C : Z \to [0,1]$, $\beta_C : Z \to [0,1]$ *and* $\gamma_C : Z \to [0,1]$ *represent the membership degree of existence, uncertain and non-existence function of C respectively, with* α_C, β_C *and* γ_C *satisfy the requirement* $0 \leq \alpha_C^2(a) + \beta_C^2(a) + \gamma_C^2(a) \leq 2$, $\forall$ $a \in Z$.

Definition 2.4. *([1])*
A PyNS D on $Z \times Z$ *is said to be a relation of Pythagorean neutrosophic type on Z, denoted*

by $D = \{\langle ab, \alpha_D(ab), \beta_D(ab), \gamma_D(ab)\rangle / ab \in Z \times Z\}$, *where* $\alpha_D : Z \times Z \to [0,1]$, $\beta_D : Z \times Z \to [0,1]$ *and* $\gamma_D : Z \times Z \to [0,1]$ *represent the membership degree of existence, uncertain and non-existence function of D, with* α_D, β_D *and* γ_D *satisfy the requirement* $0 \leq \alpha_D^2(ab) + \beta_D^2(ab) + \gamma_D^2(ab) \leq 2$, $\forall\ ab \in Z \times Z$.

Definition 2.5. *([1])*
A Pythagorean neutrosophic graph (PyNG) on a non-empty set Z is a pair $Gr = (C, D)$ *with C, a PyNS on Z and D, a relation of Pythagorean neutrosophic type on Z such that*
$\alpha_D(ab) \leq \alpha_C(a) \wedge \alpha_C(b)$,
$\beta_D(ab) \leq \beta_C(a) \wedge \beta_C(b)$,
$\gamma_D(ab) \leq \gamma_C(a) \vee \gamma_C(b)$,
and $0 \leq \alpha_D^2(ab) + \beta_D^2(ab) + \gamma_D^2(ab) \leq 2$, $\forall\ a, b \in Z$, *where* $\alpha_D : Z \times Z \to [0,1]$, $\beta_D : Z \times Z \to [0,1]$ *and* $\gamma_D : Z \times Z \to [0,1]$ *represent the membership degree of existence, uncertain and non-existence function of D respectively.*

Definition 2.6. *([1])*
A Pythagorean neutrosophic preference relation on $Z = \{a_1, a_2, \cdots, a_n\}$ *is demonstrated by a matrix* $M_p = (m_{kl})$ *of order* $n \times n$, *where* $m_{kl} = (a_k a_l, \alpha(a_k a_l), \beta(a_k a_l), \gamma(a_k a_l))$, $\forall\ k, l = 1, 2, \cdots, n$. *Let* $m_{kl} = (\alpha_{kl}, \beta_{kl}, \gamma_{kl})$, *where* α_{kl} *indicates the degree to which the element* a_k *is preferred to the element* a_l, β_{kl} *indicates the uncertain degree to which the element* a_k *is preferred to the element* a_l, γ_{kl} *denotes the degree to which the element* a_k *is not preferred to the element* a_l, *and* $\pi_{kl} = 2 - \alpha_{kl}^2 - \beta_{kl}^2 - \gamma_{kl}^2$ *is implemented as a hesitancy degree, with the requirements:*
$\alpha_{kl}, \beta_{kl}, \gamma_{kl} \in [0,1]$, $\alpha_{kl}^2 + \beta_{kl}^2 + \gamma_{kl}^2 \leq 2$, $\alpha_{kl} = \beta_{lk}$, $\beta_{kl} = \gamma_{lk}$, $\gamma_{kl} = \alpha_{lk}$,
$\alpha_{kk} = \beta_{kk} = \gamma_{kk} = 1$, $\forall\ k, l = 1, 2, \cdots, n$.

Definition 2.7. *([30])*
The direct product of Pythagorean co-neutrosophic graphs $Gr_1 = (V^*, E^*, \kappa^*, \lambda^*)$ *and* $Gr_2 = (V^{**}, E^{**}, \kappa^{**}, \lambda^{**})$ *such that* $V^* \cap V^{**} = \phi$, *is defined to be the PyNG* $Gr_1 \sqcap Gr_2 = (V, E, \kappa^* \sqcap \kappa^{**}, \lambda^* \sqcap \lambda^{**})$ *where* $V = V^* \times V^{**}$, $E = \{((a_1, b_1), (a_2, b_2))/(a_1, a_2) \in E^*, (b_1, b_2) \in E^{**}\}$. *The existence, uncertain and non-existence values of the vertex* (a, b) *in* $Gr_1 \sqcap Gr_2$ *are given by*
(i) $(\kappa_1^* \sqcap \kappa_1^{**})(a, b) = \kappa_1^*(a) \vee \kappa_1^{**}(b)$,
(ii) $(\kappa_2^* \sqcap \kappa_2^{**})(a, b) = \kappa_2^*(a) \vee \kappa_2^{**}(b)$ *and*
(iii) $(\kappa_3^* \sqcap \kappa_3^{**})(a, b) = \kappa_3^*(a) \wedge \kappa_3^{**}(b)$.
The existence, uncertain and non-existence values of the edge $((a_1, b_1), (a_2, b_2))$ *in* $Gr_1 \sqcap Gr_2$ *are given by*
$(\lambda_1^* \sqcap \lambda_1^{**})((a_1, b_1), (a_2, b_2)) = \lambda_1^*(a_1, a_2) \vee \lambda_1^{**}(b_1, b_2)$

$(\lambda_2^* \sqcap \lambda_2^{**})((a_1, b_1), (a_2, b_2)) = \lambda_2^*(a_1, a_2) \vee \lambda_2^{**}(b_1, b_2)$
$(\lambda_3^* \sqcap \lambda_3^{**})((a_1, b_1), (a_2, b_2)) = \lambda_3^*(a_1, a_2) \wedge \lambda_3^{**}(b_1, b_2)$.

Definition 2.8. *([13])*
The random walk graph kernel $k(Gr_1, Gr_2)$ of Gr_1 and Gr_2 is referred to be $k(Gr_1, Gr_2) := q^T(I - cW)^{-1}p$, where, W denotes the weight matrix, c refers to decay factor and p & q denote respectively the starting and stopping probabilities.

3. Products on Pythagorean Co-Neutrosophic Graphs

Definition 3.1.
A Pythagorean co-neutrosophic graph (PyCNG) on a non-empty set Z is a graph$(Gr) = (C, D)$ with C, a PyNS on Z and D, a relation of Pythagorean neutrosophic type on Z such that
$\alpha_D(ab) \geq \alpha_C(a) \vee \alpha_C(b)$,
$\beta_D(ab) \geq \beta_C(a) \vee \beta_C(b)$,
$\gamma_D(ab) \geq \gamma_C(a) \wedge \gamma_C(b)$,
and $0 \leq \alpha_D^2(ab) + \beta_D^2(ab) + \gamma_D^2(ab) \leq 2$, $\forall\ a, b \in Z$, where $\alpha_D : Z \times Z \to [0,1]$, $\beta_D : Z \times Z \to [0,1]$ and $\gamma_D : Z \times Z \to [0,1]$ represent the membership degree of existence, uncertain and non-existence function of D respectively.

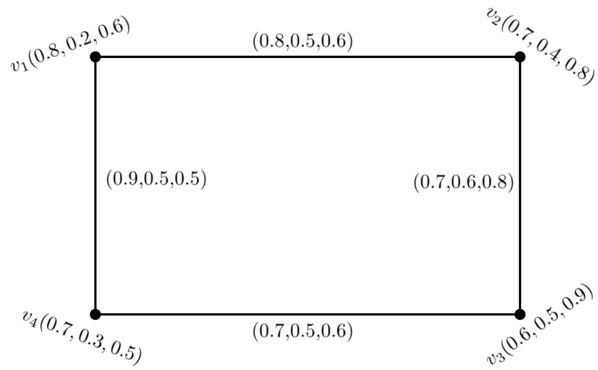

Figure 1. Pythagorean Co-Neutrosophic Graph

Definition 3.2. *(Modular product on Pythagorean co-neutrosophic graphs(PyCNGs))*
Consider $Gr_1 = (C_1, D_1)$ and $Gr_2 = (C_2, D_2)$ be PyCNGs of $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, respectively. The modular product $Gr_1 \odot Gr_2$ of Gr_1 and Gr_2 is denoted by $(C_1 \odot C_2, D_1 \odot D_2)$ with elemental vertex set $\mathscr{V}_1 \odot \mathscr{V}_2 = \{(a_1, b_1)/a_1 \in \mathscr{V}_1, b_1 \in \mathscr{V}_2\}$ and elemental edge set $\mathscr{E}_1 \odot \mathscr{E}_2 = \{(a_1, b_1)(a_2, b_2)/a_1a_2 \in \mathscr{E}_1, b_1b_2 \in \mathscr{E}_2$ or $a_1a_2 \notin \mathscr{E}_1, b_1b_2 \notin \mathscr{E}_2\}$ is defined as:
(i) $(\alpha_{C_1} \odot \alpha_{C_2})(a_1, b_1) = max(\alpha_{C_1}(a_1), \alpha_{C_2}(b_1))$
$(\beta_{C_1} \odot \beta_{C_2})(a_1, b_1) = max(\beta_{C_1}(a_1), \beta_{C_2}(b_1))$
$(\gamma_{C_1} \odot \gamma_{C_2})(a_1, b_1) = min(\gamma_{C_1}(a_1), \gamma_{C_2}(b_1))$,

where $a_1 \in \mathscr{V}_1$ *and* $b_1 \in \mathscr{V}_2$

(ii) $(\alpha_{D_1} \odot \alpha_{D_2})((a_1,b_1)(a_2,b_2)) = \begin{cases} max(\alpha_{D_1}(a_1a_2), \alpha_{D_2}(b_1b_2)) & if\ a_1a_2 \in \mathscr{E}_1, b_1b_2 \in \mathscr{E}_2 \\ max(\alpha_{C_1}(a_1), \alpha_{C_1}(a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)) & if\ a_1a_2 \notin \mathscr{E}_1, b_1b_2 \notin \mathscr{E}_2 \end{cases}$

$(\beta_{D_1} \odot \beta_{D_2})((a_1,b_1)(a_2,b_2)) = \begin{cases} max(\beta_{D_1}(a_1a_2), \beta_{D_2}(b_1b_2)) & if\ a_1a_2 \in \mathscr{E}_1, b_1b_2 \in \mathscr{E}_2 \\ max(\beta_{C_1}(a_1), \beta_{C_1}(a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)) & if\ a_1a_2 \notin \mathscr{E}_1, b_1b_2 \notin \mathscr{E}_2 \end{cases}$

$(\gamma_{D_1} \odot \gamma_{D_2})((a_1,b_1)(a_2,b_2)) = \begin{cases} min(\gamma_{D_1}(a_1a_2), \gamma_{D_2}(b_1b_2)) & if\ a_1a_2 \in \mathscr{E}_1, b_1b_2 \in \mathscr{E}_2 \\ min(\gamma_{C_1}(a_1), \gamma_{C_1}(a_2), \gamma_{C_2}(b_1), \gamma_{C_2}(b_2)) & if\ a_1a_2 \notin \mathscr{E}_1, b_1b_2 \notin \mathscr{E}_2 \end{cases}$

Theorem 3.3. *Consider PyCNGs* $Gr_1 = (C_1, D_1)$ *and* $Gr_2 = (C_2, D_2)$*, then modular product* $Gr_1 \odot Gr_2$ *is also a PyCNG.*

Proof.

Let $Gr_1 = (C_1, D_1)$ and $Gr_2 = (C_2, D_2)$ be PyCNGs of the graph $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, respectively. Here, $C^{Gr_1 \odot Gr_2}$ denotes the Pythagorean co-neutrosophic set of $\mathscr{V}_1 \odot \mathscr{V}_2$ and $D^{Gr_1 \odot Gr_2}$ denotes the Pythagorean co-neutrosophic set of $\mathscr{E}_1 \odot \mathscr{E}_2$.
For all $(a_1, b_1)(a_2, b_2) \in \mathscr{E}_1 \odot \mathscr{E}_2$,

$$\begin{aligned} (\alpha_{D_1} \odot \alpha_{D_2})((a_1,b_1)(a_2,b_2)) &= max(\alpha_{D_1}(a_1a_2), \alpha_{D_2}(b_1b_2)) \ \ if\, a_1a_2 \in \mathscr{E}_1, b_1b_2 \in \mathscr{E}_2 \\ &\geq max(\alpha_{C_1}(a_1), \alpha_{C_1}(a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)) \\ &\quad [since\ Gr_1\ \&\ Gr_2\ are\ PyCNGs] \end{aligned}$$

$$\begin{aligned} (\beta_{D_1} \odot \beta_{D_2})((a_1,b_1)(a_2,b_2)) &= max(\beta_{D_1}(a_1a_2), \beta_{D_2}(b_1b_2)) \ \ if\, a_1a_2 \in \mathscr{E}_1, b_1b_2 \in \mathscr{E}_2 \\ &\geq max(\beta_{C_1}(a_1), \beta_{C_1}(a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)) \\ &\quad [since\ Gr_1\ \&\ Gr_2\ are\ PyCNGs] \end{aligned}$$

$$\begin{aligned} (\gamma_{D_1} \odot \gamma_{D_2})((a_1,b_1)(a_2,b_2)) &= min(\gamma_{D_1}(a_1a_2), \gamma_{D_2}(b_1b_2)) \ \ if\, a_1a_2 \in \mathscr{E}_1, b_1b_2 \in \mathscr{E}_2 \\ &\geq min(\gamma_{C_1}(a_1), \gamma_{C_1}(a_2), \gamma_{C_2}(b_1), \gamma_{C_2}(b_2)) \\ &\quad [since\ Gr_1\ \&\ Gr_2\ are\ PyCNGs] \end{aligned}$$

$$\begin{aligned} (\alpha_{D_1} \odot \alpha_{D_2})((a_1,b_1)(a_2,b_2)) &= max(\alpha_{C_1}(a_1), \alpha_{C_1}(a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)), \ \ if\, a_1a_2 \notin \mathscr{E}_1, b_1b_2 \notin \mathscr{E}_2 \\ &= max((\alpha_{C_1} \odot \alpha_{C_2})(a_1,b_1), (\alpha_{C_1} \odot \alpha_{C_2})(a_2,b_2)) \end{aligned}$$

$$\begin{aligned} (\beta_{D_1} \odot \beta_{D_2})((a_1,b_1)(a_2,b_2)) &= max(\beta_{C_1}(a_1), \beta_{C_1}(a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)), \ \ if\, a_1a_2 \notin \mathscr{E}_1, b_1b_2 \notin \mathscr{E}_2 \\ &= max((\beta_{C_1} \odot \beta_{C_2})(a_1,b_1), (\beta_{C_1} \odot \beta_{C_2})(a_2,b_2)) \end{aligned}$$

$$\begin{aligned}(\gamma_{D_1} \odot \gamma_{D_2})((a_1, b_1)(a_2, b_2)) &= min(\gamma_{C_1}(a_1), \gamma_{C_1}(a_2), \gamma_{C_2}(b_1), \gamma_{C_2}(b_2)), \quad if\, a_1 a_2 \notin \mathscr{E}_1, b_1 b_2 \notin \mathscr{E}_2 \\ &= min((\gamma_{C_1} \odot \gamma_{C_2})(a_1, b_1), (\gamma_{C_1} \odot \gamma_{C_2})(a_2, b_2))\end{aligned}$$

Therefore,

$$\begin{aligned}(\alpha_{D_1} \odot \alpha_{D_2})((a_1, b_1)(a_2, b_2)) &\geq max((\alpha_{C_1} \odot \alpha_{C_2})(a_1, b_1), (\alpha_{C_1} \odot \alpha_{C_2})(a_2, b_2)) \\ (\beta_{D_1} \odot \beta_{D_2})((a_1, b_1)(a_2, b_2)) &\geq max((\beta_{C_1} \odot \beta_{C_2})(a_1, b_1), (\beta_{C_1} \odot \beta_{C_2})(a_2, b_2)) \\ (\gamma_{D_1} \odot \gamma_{D_2})((a_1, b_1)(a_2, b_2)) &\geq min((\gamma_{C_1} \odot \gamma_{C_2})(a_1, b_1), (\gamma_{C_1} \odot \gamma_{C_2})(a_2, b_2))\end{aligned}$$

This proves that $Gr_1 \odot Gr_2$ is also a PyCNG.

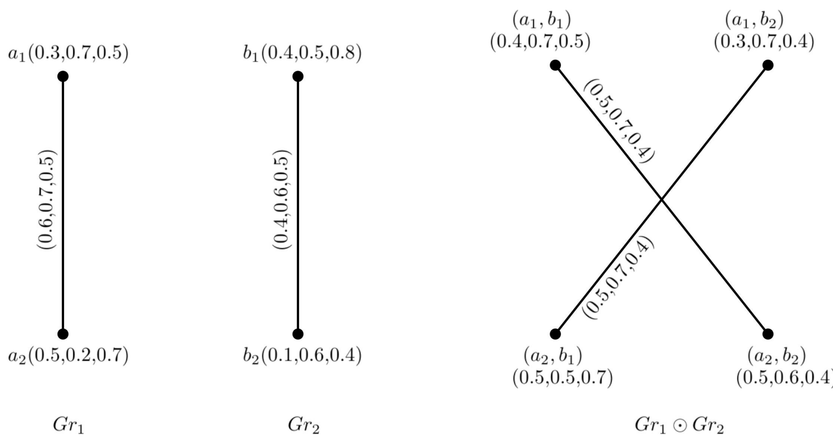

FIGURE 2. Modular product on PyCNGs

Theorem 3.4. *Consider strong PyCNGs $Gr_1 = (C_1, D_1)$ and $Gr_2 = (C_2, D_2)$. Then the modular product $Gr_1 \odot Gr_2$ is again a strong PyCNG.*

Proof.

Let us take $Gr_1 = (C_1, D_1)$ and $Gr_2 = (C_2, D_2)$ be strong PyCNGs of graphs $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, respectively. Here, $C^{Gr_1 \odot Gr_2}$ denotes the Pythagorean co-neutrosophic set of $\mathscr{V}_1 \odot \mathscr{V}_2$ and $D^{Gr_1 \odot Gr_2}$ denotes the Pythagorean co-neutrosophic set of $\mathscr{E}_1 \odot \mathscr{E}_2$.
$\forall (a_1, b_1)(a_2, b_2) \in \mathscr{E}_1 \odot \mathscr{E}_2$,

$$\begin{aligned}(\alpha_{D_1} \odot \alpha_{D_2})((a_1, b_1)(a_2, b_2)) &= max(\alpha_{D_1}(a_1 a_2), \alpha_{D_2}(b_1 b_2)) \;\; if\, a_1 a_2 \in \mathscr{E}_1, b_1 b_2 \in \mathscr{E}_2 \\ &= max(\alpha_{C_1}(a_1), \alpha_{C_1}(a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)) \\ &\quad [since\ Gr_1\ \&\ Gr_2\ are\ strong\ PyCNGs]\end{aligned}$$

$$\begin{aligned}(\beta_{D_1} \odot \beta_{D_2})((a_1, b_1)(a_2, b_2)) &= max(\beta_{D_1}(a_1 a_2), \beta_{D_2}(b_1 b_2)) \ \ if\, a_1 a_2 \in \mathscr{E}_1, b_1 b_2 \in \mathscr{E}_2 \\ &= max(\beta_{C_1}(a_1), \beta_{C_1}(a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)) \\ & \quad [since\ Gr_1\ \&\ Gr_2\ are\ strong\ PyCNGs]\end{aligned}$$

$$\begin{aligned}(\gamma_{D_1} \odot \gamma_{D_2})((a_1, b_1)(a_2, b_2)) &= min(\gamma_{D_1}(a_1 a_2), \gamma_{D_2}(b_1 b_2)) \ \ if\, a_1 a_2 \in \mathscr{E}_1, b_1 b_2 \in \mathscr{E}_2 \\ &= min(\gamma_{C_1}(a_1), \gamma_{C_1}(a_2), \gamma_{C_2}(b_1), \gamma_{C_2}(b_2)) \\ & \quad [since\ Gr_1\ \&\ Gr_2\ are\ strong\ PyCNGs]\end{aligned}$$

$$\begin{aligned}(\alpha_{D_1} \odot \alpha_{D_2})((a_1, b_1)(a_2, b_2)) &= max(\alpha_{C_1}(a_1), \alpha_{C_1}(a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)), \ \ if\, a_1 a_2 \notin \mathscr{E}_1, b_1 b_2 \notin \mathscr{E}_2 \\ &= max((\alpha_{C_1} \odot \alpha_{C_2})(a_1, b_1), (\alpha_{C_1} \odot \alpha_{C_2})(a_2, b_2))\end{aligned}$$

$$\begin{aligned}(\beta_{D_1} \odot \beta_{D_2})((a_1, b_1)(a_2, b_2)) &= max(\beta_{C_1}(a_1), \beta_{C_1}(a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)), \ \ if\, a_1 a_2 \notin \mathscr{E}_1, b_1 b_2 \notin \mathscr{E}_2 \\ &= max((\beta_{C_1} \odot \beta_{C_2})(a_1, b_1), (\beta_{C_1} \odot \beta_{C_2})(a_2, b_2))\end{aligned}$$

$$\begin{aligned}(\gamma_{D_1} \odot \gamma_{D_2})((a_1, b_1)(a_2, b_2)) &= min(\gamma_{C_1}(a_1), \gamma_{C_1}(a_2), \gamma_{C_2}(b_1), \gamma_{C_2}(b_2)), \ \ if\, a_1 a_2 \notin \mathscr{E}_1, b_1 b_2 \notin \mathscr{E}_2 \\ &= min((\gamma_{C_1} \odot \gamma_{C_2})(a_1, b_1), (\gamma_{C_1} \odot \gamma_{C_2})(a_2, b_2))\end{aligned}$$

Therefore,

$$\begin{aligned}(\alpha_{D_1} \odot \alpha_{D_2})((a_1, b_1)(a_2, b_2)) &= max((\alpha_{C_1} \odot \alpha_{C_2})(a_1, b_1), (\alpha_{C_1} \odot \alpha_{C_2})(a_2, b_2)) \\ (\beta_{D_1} \odot \beta_{D_2})((a_1, b_1)(a_2, b_2)) &= max((\beta_{C_1} \odot \beta_{C_2})(a_1, b_1), (\beta_{C_1} \odot \beta_{C_2})(a_2, b_2)) \\ (\gamma_{D_1} \odot \gamma_{D_2})((a_1, b_1)(a_2, b_2)) &= min((\gamma_{C_1} \odot \gamma_{C_2})(a_1, b_1), (\gamma_{C_1} \odot \gamma_{C_2})(a_2, b_2))\end{aligned}$$

This proves that $Gr_1 \odot Gr_2$ is again a strong PyCNG.

Definition 3.5. *(Degree of modular product on PyCNGs)*
Let $Gr_1 = (C_1, D_1)$ *and* $Gr_2 = (C_2, D_2)$ *be PyCNGs of the graph* $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ *and* $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$*, respectively. At any vertex* $(a_1, b_1) \in \mathscr{V}_1 \times \mathscr{V}_2$*:*

$$\begin{aligned}(d_\alpha)_{Gr_1 \odot Gr_2}(a_1, b_1) &= \sum_{((a_1,b_1),(a_2,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \alpha_{D_1} \odot \alpha_{D_2}((a_1, b_1), (a_2, b_2)) \\ &= \sum_{\substack{(a_1,a_2) \in \mathscr{E}_1, \\ (b_1,b_2) \in \mathscr{E}_2}} max(\alpha_{D_1}(a_1, a_2), \alpha_{D_2}(b_1, b_2)) \\ &\quad + \sum_{\substack{(a_1,a_2) \notin \mathscr{E}_1, \\ (b_1,b_2) \notin \mathscr{E}_2}} max(\alpha_{C_1}(a_1), \alpha_{C_1}(a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)),\end{aligned}$$

$$
\begin{aligned}
(d_\beta)_{Gr_1 \odot Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \beta_{D_1} \odot \beta_{D_2}((a_1,b_1),(a_2,b_2)) \\
&= \sum_{\substack{(a_1,a_2)\in \mathscr{E}_1,\\ (b_1,b_2)\in \mathscr{E}_2}} max(\beta_{D_1}(a_1,a_2), \beta_{D_2}(b_1,b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\notin \mathscr{E}_1,\\ (b_1,b_2)\notin \mathscr{E}_2}} max(\beta_{C_1}(a_1), \beta_{C_1}(a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)),
\end{aligned}
$$

$$
\begin{aligned}
(d_\gamma)_{Gr_1 \odot Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \gamma_{D_1} \odot \gamma_{D_2}((a_1,b_1),(a_2,b_2)) \\
&= \sum_{\substack{(a_1,a_2)\in \mathscr{E}_1,\\ (b_1,b_2)\in \mathscr{E}_2}} min(\gamma_{D_1}(a_1,a_2), \gamma_{D_2}(b_1,b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\notin \mathscr{E}_1,\\ (b_1,b_2)\notin \mathscr{E}_2}} min(\gamma_{C_1}(a_1), \gamma_{C_1}(a_2), \gamma_{C_2}(b_1), \gamma_{C_2}(b_2))
\end{aligned}
$$

Definition 3.6. *(Total degree of modular product on PyCNGs)*
Consider $Gr_1 = (C_1, D_1)$ and $Gr_2 = (C_2, D_2)$ to be PyCNGs of the graph $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, respectively. At any vertex $(a_1, b_1) \in \mathscr{V}_1 \times \mathscr{V}_2$:

$$
\begin{aligned}
(td_\alpha)_{Gr_1 \odot Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \alpha_{D_1} \odot \alpha_{D_2}((a_1,b_1),(a_2,b_2)) + \alpha_{C_1} \odot \alpha_{C_2}(a_1,b_1) \\
&= \sum_{\substack{(a_1,a_2)\in \mathscr{E}_1,\\ (b_1,b_2)\in \mathscr{E}_2}} max(\alpha_{D_1}(a_1,a_2), \alpha_{D_2}(b_1,b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\notin \mathscr{E}_1,\\ (b_1,b_2)\notin \mathscr{E}_2}} max(\alpha_{C_1}(a_1), \alpha_{C_1}(a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)) \\
&\quad + max(\alpha_{C_1}(a_1), \alpha_{C_2}(b_1)),
\end{aligned}
$$

$$
\begin{aligned}
(td_\beta)_{Gr_1 \odot Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \beta_{D_1} \odot \beta_{D_2}((a_1,b_1),(a_2,b_2)) + \beta_{C_1} \odot \beta_{C_2}(a_1,b_1) \\
&= \sum_{\substack{(a_1,a_2)\in \mathscr{E}_1,\\ (b_1,b_2)\in \mathscr{E}_2}} max(\beta_{D_1}(a_1,a_2), \beta_{D_2}(b_1,b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\notin \mathscr{E}_1,\\ (b_1,b_2)\notin \mathscr{E}_2}} max(\beta_{C_1}(a_1), \beta_{C_1}(a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)) \\
&\quad + max(\beta_{C_1}(a_1), \beta_{C_2}(b_1)),
\end{aligned}
$$

$$\begin{aligned}(td_\gamma)_{Gr_1\odot Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in\mathscr{E}_1\times\mathscr{E}_2}\gamma_{D_1}\odot\gamma_{D_2}((a_1,b_1),(a_2,b_2))+\gamma_{C_1}\odot\gamma_{C_2}(a_1,b_1)\\ &= \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\(b_1,b_2)\in\mathscr{E}_2}} min(\gamma_{D_1}(a_1,a_2),\gamma_{D_2}(b_1,b_2))\\ &\quad+\sum_{\substack{(a_1,a_2)\notin\mathscr{E}_1,\\(b_1,b_2)\notin\mathscr{E}_2}} min(\gamma_{C_1}(a_1),\gamma_{C_1}(a_2),\gamma_{C_2}(b_1),\gamma_{C_2}(b_2))\\ &\quad+min(\gamma_{C_1}(a_1),\gamma_{C_2}(b_1)),\end{aligned}$$

Example 3.7.

Examine the modular product on PyCNGs, as shown in Figure 2. Then the degree of the vertex (a_1,b_1) *is* $d_{Gr_1\odot Gr_2}(a_1,b_1)=((d_\alpha)_{Gr_1\odot Gr_2}(a_1,b_1),(d_\beta)_{Gr_1\odot Gr_2}(a_1,b_1),(d_\gamma)_{Gr_1\odot Gr_2}(a_1,b_1))$ = *(0.5,0.7,0.4) and the total degree of the vertex* (a_1,b_1) *is* $td_{Gr_1\odot Gr_2}(a_1,b_1)=((td_\alpha)_{Gr_1\odot Gr_2}(a_1,b_1),(td_\beta)_{Gr_1\odot Gr_2}(a_1,b_1),(td_\gamma)_{Gr_1\odot Gr_2}(a_1,b_1))=$ *(0.9,1.4,0.9). In the same way, the remaining vertices degree and total degree are obtained by* $d_{Gr_1\odot Gr_2}(a_1,b_2)=$ *(0.5,0.7,0.4),* $td_{Gr_1\odot Gr_2}(a_1,b_2)=$ *(0.8,1.4,0.8),* $d_{Gr_1\odot Gr_2}(a_2,b_1)=$ *(0.5,0.7,0.4),* $td_{Gr_1\odot Gr_2}(a_2,b_1)=$ *(1.0,1.2,1.1),* $d_{Gr_1\odot Gr_2}(a_2,b_2)=$ *(0.5,0.7,0.4) and* $td_{Gr_1\odot Gr_2}(a_2,b_2)=$ *(1.0,1.3,0.8).*

Theorem 3.8. *Consider complete PyCNGs* $Gr_1=(C_1,D_1)$ *and* $Gr_2=(C_2,D_2)$*.*
(i) Let $\alpha_{D_1}\geq\alpha_{D_2}$, $\beta_{D_1}\geq\beta_{D_2}$, $\gamma_{D_1}\leq\gamma_{D_2}$ *then* $d_{Gr_1\odot Gr_2}(a_1,a_2)=d_{Gr_1}(a_1)$, *where* $d_{Gr_1\odot Gr_2}(a_1,a_2)=((d_\alpha)_{Gr_1}(a_1),(d_\beta)_{Gr_1}(a_1),(d_\gamma)_{Gr_1}(a_1))$
(ii) If $\alpha_{D_1}\leq\alpha_{D_2}$, $\beta_{D_1}\leq\beta_{D_2}$, $\gamma_{D_1}\geq\gamma_{D_2}$ *then* $d_{Gr_1\odot Gr_2}(a_1,a_2)=d_{Gr_2}(a_1)$, *where* $d_{Gr_1\odot Gr_2}(a_1,a_2)=((d_\alpha)_{Gr_1}(a_1),(d_\beta)_{Gr_1}(a_1),(d_\gamma)_{Gr_1}(a_1))$

Proof.

(i) We have, degree of each membership of a vertex in modular product as

$$\begin{aligned}(d_\alpha)_{Gr_1\odot Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in\mathscr{E}_1\times\mathscr{E}_2}\alpha_{D_1}\odot\alpha_{D_2}((a_1,b_1),(a_2,b_2))\\ &= \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\(b_1,b_2)\in\mathscr{E}_2}} max(\alpha_{D_1}(a_1,a_2),\alpha_{D_2}(b_1,b_2))\\ &\quad+\sum_{\substack{(a_1,a_2)\notin\mathscr{E}_1,\\(b_1,b_2)\notin\mathscr{E}_2}} max(\alpha_{C_1}(a_1),\alpha_{C_1}(a_2),\alpha_{C_2}(b_1),\alpha_{C_2}(b_2))\\ &= \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\(b_1,b_2)\in\mathscr{E}_2}} max(\alpha_{D_1}(a_1,a_2),\alpha_{D_2}(b_1,b_2))\ \ [since\ both\ PyCNGs\ are\ complete]\\ &= \sum_{(a_1,a_2)\in\mathscr{E}_1}\alpha_{D_1}(a_1,a_2)\ \ [since\ \alpha_{D_1}\geq\alpha_{D_2}]\\ &= (d_\alpha)_{Gr_1}(a_1),\end{aligned}$$

$$\begin{aligned}(d_\beta)_{Gr_1\odot Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in\mathscr{E}_1\times\mathscr{E}_2} \beta_{D_1}\odot\beta_{D_2}((a_1,b_1),(a_2,b_2))\\
&= \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\(b_1,b_2)\in\mathscr{E}_2}} max(\beta_{D_1}(a_1,a_2),\beta_{D_2}(b_1,b_2))\\
&\quad+\sum_{\substack{(a_1,a_2)\notin\mathscr{E}_1,\\(b_1,b_2)\notin\mathscr{E}_2}} max(\beta_{C_1}(a_1),\beta_{C_1}(a_2),\beta_{C_2}(b_1),\beta_{C_2}(b_2))\\
&= \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\(b_1,b_2)\in\mathscr{E}_2}} max(\beta_{D_1}(a_1,a_2),\beta_{D_2}(b_1,b_2))\ \ [since\ both\ PyCNGs\ are\ complete]\\
&= \sum_{(a_1,a_2)\in\mathscr{E}_1} \beta_{D_1}(a_1,a_2)\ \ [since\ \beta_{D_1}\geq\beta_{D_2}]\\
&= (d_\beta)_{Gr_1}(a_1),\end{aligned}$$

$$\begin{aligned}(d_\gamma)_{Gr_1\odot Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in\mathscr{E}_1\times\mathscr{E}_2} \gamma_{D_1}\odot\gamma_{D_2}((a_1,b_1),(a_2,b_2))\\
&= \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\(b_1,b_2)\in\mathscr{E}_2}} min(\gamma_{D_1}(a_1,a_2),\gamma_{D_2}(b_1,b_2))\\
&\quad+\sum_{\substack{(a_1,a_2)\notin\mathscr{E}_1,\\(b_1,b_2)\notin\mathscr{E}_2}} min(\gamma_{C_1}(a_1),\gamma_{C_1}(a_2),\gamma_{C_2}(b_1),\gamma_{C_2}(b_2))\\
&= \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\(b_1,b_2)\in\mathscr{E}_2}} min(\gamma_{D_1}(a_1,a_2),\gamma_{D_2}(b_1,b_2))\ \ [since\ both\ PyCNGs\ are\ complete]\\
&= \sum_{(a_1,a_2)\in\mathscr{E}_1} \gamma_{D_1}(a_1,a_2)\ \ [since\ \gamma_{D_1}\leq\gamma_{D_2}]\\
&= (d_\gamma)_{Gr_1}(a_1)\end{aligned}$$

Therefore, $d_{Gr_1\odot Gr_2}(a_1,a_2)= d_{Gr_1}(a_1)$.

(ii) Similar proof by reversing minimum and maximum conditions of (i).

Definition 3.9. *(Homomorphic product on Pythagorean co-neutrosophic graphs(PyCNGs)) Consider PyCNGs $Gr_1=(C_1,D_1)$ and $Gr_2=(C_2,D_2)$ of $\mathscr{G}_1=(\mathscr{V}_1,\mathscr{E}_1)$ and $\mathscr{G}_2=(\mathscr{V}_2,\mathscr{E}_2)$, respectively. Then homomorphic product $Gr_1\diamond Gr_2$ of Gr_1 and Gr_2 is denoted by $(C_1\diamond C_2, D_1\diamond D_2)$ with elemental vertex set $\mathscr{V}_1\diamond\mathscr{V}_2=\{(a_1,b_1)/a_1\in\mathscr{V}_1, b_1\in\mathscr{V}_2\}$ and elemental edge set $\mathscr{E}_1\diamond\mathscr{E}_2=\{(a_1,b_1)(a_2,b_2)/a_1=a_2, b_1b_2\in\mathscr{E}_2\ or\ a_1a_2\in\mathscr{E}_1, b_1b_2\notin\mathscr{E}_2\}$ is defined as:*

(i) $(\alpha_{C_1}\diamond\alpha_{C_2})(a_1,b_1)=max(\alpha_{C_1}(a_1),\alpha_{C_2}(b_1))$

$(\beta_{C_1}\diamond\beta_{C_2})(a_1,b_1)=max(\beta_{C_1}(a_1),\beta_{C_2}(b_1))$

$(\gamma_{C_1}\diamond\gamma_{C_2})(a_1,b_1)=min(\gamma_{C_1}(a_1),\gamma_{C_2}(b_1))$,

$\forall(a_1, b_1) \in \mathscr{V}$ *and*

(ii)

$$(\alpha_{D_1} \diamond \alpha_{D_2})((a_1, b_1)(a_2, b_2)) = \begin{cases} max(\alpha_{C_1}(a_1), \alpha_{D_2}(b_1 b_2)) & if\ a_1 = a_2, b_1 b_2 \in \mathscr{E}_2 \\ max(\alpha_{D_1}(a_1, a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)) & if\ a_1 a_2 \in \mathscr{E}_1, b_1 b_2 \notin \mathscr{E}_2 \end{cases}$$

$$(\beta_{D_1} \diamond \beta_{D_2})((a_1, b_1)(a_2, b_2)) = \begin{cases} max(\beta_{C_1}(a_1), \beta_{D_2}(b_1 b_2)) & if\ a_1 = a_2, b_1 b_2 \in \mathscr{E}_2 \\ max(\beta_{D_1}(a_1, a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)) & if\ a_1 a_2 \in \mathscr{E}_1, b_1 b_2 \notin \mathscr{E}_2 \end{cases}$$

$$(\gamma_{D_1} \diamond \gamma_{D_2})((a_1, b_1)(a_2, b_2)) = \begin{cases} min(\gamma_{C_1}(a_1), \gamma_{D_2}(b_1 b_2)) & if\ a_1 = a_2, b_1 b_2 \in \mathscr{E}_2 \\ min(\gamma_{D_1}(a_1, a_2), \gamma_{C_2}(b_1), \gamma_{C_2}(b_2)) & if\ a_1 a_2 \in \mathscr{E}_1, b_1 b_2 \notin \mathscr{E}_2 \end{cases}$$

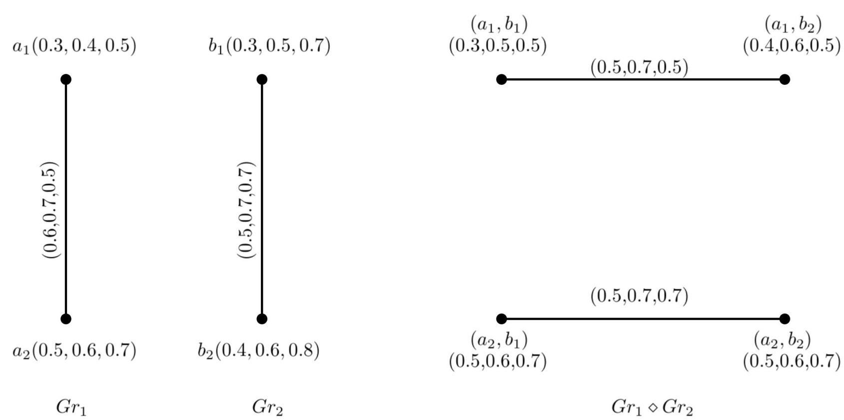

FIGURE 3. Homomorphic product on PyCNGs

Definition 3.10. *(Degree of homomorphic product on PyCNGs)*
Consider PyCNGs $Gr_1 = (C_1, D_1)$ *and* $Gr_2 = (C_2, D_2)$ *of the graph* $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ *and* $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$*, respectively. At any vertex* $(a_1, a_2) \in \mathscr{V}_1 \times \mathscr{V}_2$*:*

$$\begin{aligned} (d_\alpha)_{Gr_1 \diamond Gr_2}(a_1, b_1) &= \sum_{((a_1,b_1),(a_2,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \alpha_{D_1} \diamond \alpha_{D_2}((a_1, b_1), (a_2, b_2)) \\ &= \sum_{\substack{a_1 = a_2, \\ (b_1, b_2) \in \mathscr{E}_2}} max(\alpha_{C_1}(a_1), \alpha_{D_2}(b_1, b_2)) \\ &\quad + \sum_{\substack{(a_1, a_2) \in \mathscr{E}_1, \\ (b_1, b_2) \notin \mathscr{E}_2}} max(\alpha_{D_1}(a_1, a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)), \end{aligned}$$

$$\begin{aligned} (d_\beta)_{Gr_1 \diamond Gr_2}(a_1, b_1) &= \sum_{((a_1,b_1),(a_2,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \beta_{D_1} \diamond \beta_{D_2}((a_1, b_1), (a_2, b_2)) \\ &= \sum_{\substack{a_1 = a_2, \\ (b_1, b_2) \in \mathscr{E}_2}} max(\beta_{C_1}(a_1), \beta_{D_2}(b_1, b_2)) \\ &\quad + \sum_{\substack{(a_1, a_2) \in \mathscr{E}_1, \\ (b_1, b_2) \notin \mathscr{E}_2}} max(\beta_{D_1}(a_1, a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)), \end{aligned}$$

$$
\begin{aligned}
(d_{\gamma})_{Gr_1 \diamond Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in \mathscr{E}_1\times\mathscr{E}_2} \gamma_{D_1} \diamond \gamma_{D_2}((a_1,b_1),(a_2,b_2)) \\
&= \sum_{\substack{a_1=a_2,\\ (b_1,b_2)\in\mathscr{E}_2}} min(\gamma_{C_1}(a_1), \gamma_{D_2}(b_1,b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\ (b_1,b_2)\notin\mathscr{E}_2}} min(\gamma_{D_1}(a_1,a_2), \gamma_{C_2}(b_1), \gamma_{C_2}(b_2))
\end{aligned}
$$

Definition 3.11. *(Total degree of homomorphic product on PyCNGs)*
Consider PyCNGs $Gr_1 = (C_1, D_1)$ and $Gr_2 = (C_2, D_2)$ of the graph $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, respectively. At any vertex $(a_1, a_2) \in \mathscr{V}_1 \times \mathscr{V}_2$:

$$
\begin{aligned}
(td_{\alpha})_{Gr_1 \diamond Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in \mathscr{E}_1\times\mathscr{E}_2} \alpha_{D_1} \diamond \alpha_{D_2}((a_1,b_1),(a_2,b_2)) + \alpha_{C_1} \diamond \alpha_{C_2}(a_1,b_1) \\
&= \sum_{\substack{a_1=a_2,\\ (b_1,b_2)\in\mathscr{E}_2}} max(\alpha_{C_1}(a_1), \alpha_{D_2}(b_1,b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\ (b_1,b_2)\notin\mathscr{E}_2}} max(\alpha_{D_1}(a_1,a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)) \\
&\quad + max(\alpha_{C_1}(a_1), \alpha_{C_2}(b_1)),
\end{aligned}
$$

$$
\begin{aligned}
(td_{\beta})_{Gr_1 \diamond Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in \mathscr{E}_1\times\mathscr{E}_2} \beta_{D_1} \diamond \beta_{D_2}((a_1,b_1),(a_2,b_2)) + \beta_{C_1} \diamond \beta_{C_2}(a_1,b_1) \\
&= \sum_{\substack{a_1=a_2,\\ (b_1,b_2)\in\mathscr{E}_2}} max(\beta_{C_1}(a_1), \beta_{D_2}(b_1,b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\ (b_1,b_2)\notin\mathscr{E}_2}} max(\beta_{D_1}(a_1,a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)) \\
&\quad + min(\beta_{C_1}(a_1), \beta_{C_2}(b_1)),
\end{aligned}
$$

$$
\begin{aligned}
(td_{\gamma})_{Gr_1 \diamond Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in \mathscr{E}_1\times\mathscr{E}_2} \gamma_{D_1} \diamond \gamma_{D_2}((a_1,b_1),(a_2,b_2)) + \gamma_{C_1} \diamond \gamma_{C_2}(a_1,b_1) \\
&= \sum_{\substack{a_1=a_2,\\ (b_1,b_2)\in\mathscr{E}_2}} min(\gamma_{C_1}(a_1), \gamma_{D_2}(b_1,b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\ (b_1,b_2)\notin\mathscr{E}_2}} min(\gamma_{D_1}(a_1,a_2), \gamma_{C_2}(b_1), \gamma_{C_2}(b_2)) \\
&\quad + max(\gamma_{C_1}(a_1), \gamma_{C_2}(b_1))
\end{aligned}
$$

Example 3.12.
Examine
the homomorphic product on PyCNGs, as shown in the Figure 3. Then the degree of the vertex (a_1, b_1) is $d_{Gr_1 \diamond Gr_2}(a_1,b_1) = ((d_{\alpha})_{Gr_1 \diamond Gr_2}(a_1,b_1), (d_{\beta})_{Gr_1 \diamond Gr_2}(a_1,b_1), (d_{\gamma})_{Gr_1 \diamond Gr_2}(a_1,b_1))$

= *(0.5,0.7,0.5)* *and the total degree of the vertex* (a_1, b_1) *is* $td_{Gr_1 \diamond Gr_2}(a_1, b_1)$= $((td_\alpha)_{Gr_1\diamond Gr_2}(a_1, b_1), (td_\beta)_{Gr_1\diamond Gr_2}(a_1, b_1), (td_\gamma)_{Gr_1\diamond Gr_2}(a_1, b_1))$= *(0.8,1.2,1.0). Likewise, the degree and total degree of remaining vertices are obtained by* $d_{Gr_1\diamond Gr_2}(a_1, b_2)$= *(0.5,0.7,0.5),* $td_{Gr_1\diamond Gr_2}(a_1, b_2)$= *(0.9,1.3,1.0),* $d_{Gr_1\diamond Gr_2}(a_2, b_1)$= *(0.5,0.7,0.7),* $td_{Gr_1\diamond Gr_2}(a_2, b_1)$= *(1.0,1.3,1.4),* $d_{Gr_1\diamond Gr_2}(a_2, b_2)$= *(0.5,0.7,0.7) and* $td_{Gr_1\diamond Gr_2}(a_2, b_2)$= *(1.0,,1.3,1.4).*

Theorem 3.13. *Consider PyCNGs* $Gr_1 = (C_1, D_1)$ *and* $Gr_2 = (C_2, D_2)$. *If* Gr_2 *is a Pythagorean co-neutrosophic complete graph and* $\alpha_{C_1} \geq \alpha_{D_2}$, $\beta_{C_1} \geq \beta_{D_2}$, $\gamma_{C_1} \leq \gamma_{D_2}$, *then* $d_{Gr_1\diamond Gr_2}(a_1, b_1) = ((|\mathscr{V}_2| - 1)\alpha_{C_1}(a_1), |\mathscr{V}_2| - 1)\beta_{C_1}(a_1), |\mathscr{V}_2| - 1)\gamma_{C_1}(a_1))$.

Proof. We have, vertex degree of each membership in a homomorphic product as

$$\begin{aligned}
(d_\alpha)_{Gr_1\diamond Gr_2}(a_1, b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in \mathscr{E}_1\times\mathscr{E}_2} \alpha_{D_1} \diamond \alpha_{D_2}((a_1, b_1), (a_2, b_2)) \\
&= \sum_{\substack{a_1=a_2,\\ (b_1,b_2)\in\mathscr{E}_2}} max(\alpha_{C_1}(a_1), \alpha_{D_2}(b_1, b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\ (b_1,b_2)\notin\mathscr{E}_2}} max(\alpha_{D_1}(a_1, a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)) \\
&= \sum_{\substack{a_1=a_2,\\ (b_1,b_2)\in\mathscr{E}_2}} max(\alpha_{C_1}(a_1), \alpha_{D_2}(b_1, b_2)) \\
&\quad [since\ Gr_2\ is\ a\ pythagorean\ co-neutrosophic\ complete\ graph] \\
&= \sum_{a_1=a_2} \alpha_{C_1}(a_1) \quad [since\ \alpha_{C_1} \geq \alpha_{D_2}] \\
&= (|\mathscr{V}_2| - 1)\alpha_{C_1}(a_1),
\end{aligned}$$

$$\begin{aligned}
(d_\beta)_{Gr_1\diamond Gr_2}(a_1, b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in \mathscr{E}_1\times\mathscr{E}_2} \beta_{D_1} \diamond \beta_{D_2}((a_1, b_1), (a_2, b_2)) \\
&= \sum_{\substack{a_1=a_2,\\ (b_1,b_2)\in\mathscr{E}_2}} max(\beta_{C_1}(a_1), \beta_{D_2}(b_1, b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\ (b_1,b_2)\notin\mathscr{E}_2}} max(\beta_{D_1}(a_1, a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)) \\
&= \sum_{\substack{a_1=a_2,\\ (b_1,b_2)\in\mathscr{E}_2}} max(\beta_{C_1}(a_1), \beta_{D_2}(b_1, b_2)) \\
&\quad [since\ Gr_2\ is\ a\ pythagorean\ co-neutrosophic\ complete\ graph] \\
&= \sum_{a_1=a_2} \beta_{C_1}(a_1) \quad [since\ \beta_{C_1} \geq \beta_{D_2}] \\
&= (|\mathscr{V}_2| - 1)\beta_{C_1}(a_1),
\end{aligned}$$

$$
\begin{aligned}
(d_\gamma)_{Gr_1 \diamond Gr_2}(a_1,b_1) &= \sum_{((a_1,b_1),(a_2,b_2))\in \mathscr{E}_1\times\mathscr{E}_2} \gamma_{D_1}\diamond\gamma_{D_2}((a_1,b_1),(a_2,b_2)) \\
&= \sum_{\substack{u_1=u_2,\\(v_1,v_2)\in\mathscr{E}_2}} min(\gamma_{C_1}(a_1),\gamma_{D_2}(b_1,b_2)) \\
&\quad + \sum_{\substack{(a_1,a_2)\in\mathscr{E}_1,\\(b_1,b_2)\notin\mathscr{E}_2}} min(\gamma_{D_1}(a_1,a_2),\gamma_{C_2}(b_1),\gamma_{C_2}(b_2)) \\
&= \sum_{\substack{a_1=a_2,\\(b_1,b_2)\in\mathscr{E}_2}} min(\gamma_{C_1}(a_1),\gamma_{D_2}(b_1,b_2)) \\
&\quad [since\ Gr_2\ is\ a\ pythagorean\ co-neutrosophic\ complete\ graph] \\
&= \sum_{a_1=a_2}\gamma_{C_1}(a_1)\ \ [since\ \gamma_{C_1}\le\gamma_{D_2}] \\
&= (|\mathscr{V}_2|-1)\gamma_{C_1}(a_1)
\end{aligned}
$$

Therefore, we conclude that $d_{Gr_1\diamond Gr_2}(a_1,b_1) = ((|\mathscr{V}_2|-1)\alpha_{C_1}(a_1), |\mathscr{V}_2|-1)\beta_{C_1}(a_1), |\mathscr{V}_2|-1)\gamma_{C_1}(a_1))$.

Definition 3.14. *(Symmetric difference on PyCNGs)*
Consider PyCNGs $Gr_1 = (C_1, D_1)$ and $Gr_2 = (C_2, D_2)$ of the graphs $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, respectively. Then, the symmetric difference of Gr_1 and Gr_2 is denoted by $Gr_1 \oplus Gr_2 = (C_1 \oplus C_2, D_1 \oplus D_2)$ and defined as:

(a) $\forall (f,g) \in \mathscr{V}_1 \times \mathscr{V}_2$,
(i) $\alpha_{C_1} \oplus \alpha_{C_2}(f,g) = max(\alpha_{C_1}(f), \alpha_{C_2}(g))$,
(ii) $\beta_{C_1} \oplus \beta_{C_2}(f,g) = max(\beta_{C_1}(f), \beta_{C_2}(g))$ *and*
(iii) $\gamma_{C_1} \oplus \gamma_{C_2}(f,g) = min(\gamma_{C_1}(f), \gamma_{C_2}(g))$;

(b) $\forall f \in \mathscr{V}_1$ *and* $(g,h) \in \mathscr{E}_2$,
(i) $(\alpha_{D_1} \oplus \alpha_{D_2}((f,g),(f,h)) = max(\alpha_{C_1}(f), \alpha_{D_2}(g,h))$;
(ii) $(\beta_{D_1} \oplus \beta_{D_2}((f,g),(f,h)) = max(\beta_{C_1}(f), \beta_{D_2}(g,h))$;
(iii) $(\gamma_{D_1} \oplus \gamma_{D_2}((f,g),(f,h)) = min(\gamma_{C_1}(f), \gamma_{D_2}(g,h))$;

(c) $\forall f \in \mathscr{V}_2$ *and* $(g,h) \in \mathscr{E}_1$,
(i) $(\alpha_{D_1} \oplus \alpha_{D_2}((g,f),(h,f)) = max(\alpha_{D_1}(g,h), \alpha_{C_2}(f))$;
(ii) $(\beta_{D_1} \oplus \beta_{D_2}((g,f),(h,f)) = max(\beta_{D_1}(g,h), \beta_{C_2}(f))$;
(iii) $(\gamma_{D_1} \oplus \gamma_{D_2}((g,f),(h,f)) = min(\gamma_{D_1}(g,h), \gamma_{C_2}(f))$;

(d) $\forall (f,g) \notin \mathscr{E}_1$ *and* $(h,e) \in \mathscr{E}_2$,
(i) $(\alpha_{D_1} \oplus \alpha_{D_2}((f,h),(g,e)) = max(\alpha_{C_1}(f), \alpha_{C_1}(g), \alpha_{D_2}(h,e))$;
(ii) $(\beta_{D_1} \oplus \beta_{D_2}((f,h),(g,e)) = max(\beta_{C_1}(f), \beta_{C_1}(g), \beta_{D_2}(h,e))$;

(iii) $(\gamma_{D_1} \oplus \gamma_{D_2}((f,h),(g,e)) = min(\gamma_{C_1}(f), \gamma_{C_1}(g), \gamma_{D_2}(h,e));$

(e) $\forall (f,g) \in \mathscr{E}_1$ *and* $(h,e) \notin \mathscr{E}_2$,

(i) $(\alpha_{D_1} \oplus \alpha_{D_2}((f,h),(g,e)) = max(\alpha_{D_1}(f,g), \alpha_{C_2}(h), \alpha_{C_2}(e));$

(ii) $(\beta_{D_1} \oplus \beta_{D_2}((f,h),(g,e)) = max(\beta_{D_1}(f,g), \beta_{C_2}(h), \beta_{C_2}(e));$

(iii) $(\gamma_{D_1} \oplus \gamma_{D_2}((f,h),(g,e)) = min(\gamma_{D_1}(f,g), \gamma_{C_2}(h), \gamma_{C_2}(e)).$

Remark 3.15. *The strong product on PyCNGs is quite similar to the symmetric difference, if the conditions* (*d*) & (*e*) *of symmetric difference definition 3.14 are combined. With the same conditions* (*a*), (*b*), (*c*) *of symmetric difference, the following condition is needed to define strong product on PyCNGs:*

(f) $\forall (f,g) \in \mathscr{E}_1$ *and* $(h,e) \in \mathscr{E}_2$,

(i) $(\alpha_{D_1} \oplus \alpha_{D_2}((f,h),(g,e)) = max(\alpha_{D_1}(f,g), \alpha_{D_2}(h,e));$

(ii) $(\beta_{D_1} \oplus \beta_{D_2}((f,h),(g,e)) = max(\beta_{D_1}(f,g), \beta_{D_2}(h,e));$

(iii) $(\gamma_{D_1} \oplus \gamma_{D_2}((f,h),(g,e)) = min(\gamma_{D_1}(f,g), \gamma_{D_2}(h,e)).$

Theorem 3.16. *Consider PyCNGs* Gr_1 *and* Gr_2 *of the graphs* $\mathscr{G}_1$ *and* $\mathscr{G}_2$*, respectively. Then symmetric difference* $Gr_1 \oplus Gr_2$ *of* Gr_1 *and* Gr_2 *is a PyCNG.*

Proof.

Let us take $Gr_1 = (C_1, D_1)$ and $Gr_2 = (C_2, D_2)$ to be PyCNGs of the graphs $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, respectively.

(i) Let $a_1 \in \mathscr{V}_1$ and $(b_1, b_2) \in \mathscr{E}_2$. Then,

$$\begin{aligned}
(\alpha_{D_1} \oplus \alpha_{D_2})((a_1,b_1),(a_1,b_2)) &= max(\alpha_{C_1}(a_1), \alpha_{D_2}(b_1,b_2)) \\
&= max(\alpha_{C_1}(a_1), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)) \\
&= max(max(\alpha_{C_1}(a_1), \alpha_{C_2}(b_1)), max(\alpha_{C_1}(a_1), \alpha_{C_2}(b_2))) \\
&= max((\alpha_{C_1} \oplus \alpha_{C_2})(a_1,b_1), (\alpha_{C_1} \oplus \alpha_{C_2})(a_1,b_2)),
\end{aligned}$$

$$\begin{aligned}
(\beta_{D_1} \oplus \beta_{D_2})((a_1,b_1),(a_1,b_2)) &= max(\beta_{C_1}(a_1), \beta_{D_2}(b_1,b_2)) \\
&= max(\beta_{C_1}(a_1), \beta_{C_2}(b_1), \beta_{C_2}(b_2)) \\
&= max(max(\beta_{C_1}(a_1), \beta_{C_2}(b_1)), max(\beta_{C_1}(a_1), \beta_{C_2}(b_2))) \\
&= max((\beta_{C_1} \oplus \beta_{C_2})(a_1,b_1), (\beta_{C_1} \oplus \beta_{C_2})(a_1,b_2)),
\end{aligned}$$

$$\begin{aligned}(\gamma_{D_1} \oplus \gamma_{D_2})((a_1, b_1), (a_1, b_2)) &= min(\gamma_{C_1}(a_1), \gamma_{D_2}(b_1, b_2)) \\ &= min(\gamma_{C_1}(a_1), \gamma_{C_2}(b_1), \gamma_{C_2}(b_2)) \\ &= min(min(\gamma_{C_1}(a_1), \gamma_{C_2}(b_1)), min(\gamma_{C_1}(a_1), \gamma_{C_2}(b_2))) \\ &= min((\gamma_{C_1} \oplus \gamma_{C_2})(a_1, b_1), (\gamma_{C_1} \oplus \gamma_{C_2})(a_1, b_2)).\end{aligned}$$

(ii) Let $(a_1, a_2) \in \mathscr{E}_1$ and $b_1 \in \mathscr{V}_2$. Then,

$$\begin{aligned}(\alpha_{D_1} \oplus \alpha_{D_2})((a_1, b_1), (a_2, b_1)) &= max(\alpha_{D_1}(a_1, a_2), \alpha_{C_2}(b_1)) \\ &= max(\alpha_{C_1}(a_1), \alpha_{C_1}(a_2), \alpha_{C_2}(b_1)) \\ &= max(max(\alpha_{C_1}(a_1), \alpha_{C_2}(b_1)), max(\alpha_{C_1}(a_2), \alpha_{C_2}(b_1))) \\ &= max((\alpha_{C_1} \oplus \alpha_{C_2})(a_1, b_1), (\alpha_{C_1} \oplus \alpha_{C_2})(a_2, b_1)),\end{aligned}$$

$$\begin{aligned}(\beta_{D_1} \oplus \beta_{D_2})((a_1, b_1), (a_2, b_1)) &= max(\beta_{D_1}(a_1, a_2), \beta_{C_2}(b_1)) \\ &= max(\beta_{C_1}(a_1), \beta_{C_1}(a_2), \beta_{C_2}(b_1)) \\ &= max(max(\beta_{C_1}(a_1), \beta_{C_2}(b_1), max(\beta_{C_1}(a_2), \beta_{C_2}(b_1))) \\ &= max((\beta_{C_1} \oplus \beta_{C_2})(a_1, b_1), (\beta_{C_1} \oplus \beta_{C_2})(a_2, b_1)),\end{aligned}$$

$$\begin{aligned}(\gamma_{D_1} \oplus \gamma_{D_2})((a_1, b_1), (a_2, b_1)) &= min(\gamma_{D_1}(a_1, a_2), \gamma_{C_2}(b_1)) \\ &= min(\gamma_{C_1}(a_1), \gamma_{C_1}(a_2), \gamma_{C_2}(b_1)) \\ &= min(min(\gamma_{C_1}(a_1), \gamma_{C_2}(b_1)), min(\gamma_{C_1}(a_2), \gamma_{C_2}(b_1))) \\ &= min((\gamma_{C_1} \oplus \gamma_{C_2})(a_1, b_1), (\gamma_{C_1} \oplus \gamma_{C_2})(a_2, b_1)).\end{aligned}$$

(iii) Let $(a_1, a_2) \notin \mathscr{E}_1$ and $(b_1, b_2) \in \mathscr{E}_2$. Then,

$$\begin{aligned}(\alpha_{D_1} \oplus \alpha_{D_2})((a_1, b_1), (a_2, b_2)) &= max(\alpha_{C_1}(a_1), \alpha_{C_1}(a_2), \alpha_{D_2}(b_1, b_2)) \\ &= max(\alpha_{C_1}(a_1), \alpha_{C_1}(a_2), \alpha_{C_2}(b_1), \alpha_{C_2}(b_2)) \\ &= max(\alpha_{C_1}(a_1), \alpha_{C_2}(b_1), \alpha_{C_1}(a_2), \alpha_{C_2}(b_2)) \\ &= max((\alpha_{C_1} \oplus \alpha_{C_2})(a_1, b_1), (\alpha_{C_1} \oplus \alpha_{C_2})(a_2, b_2)),\end{aligned}$$

$$\begin{aligned}(\beta_{D_1} \oplus \beta_{D_2})((a_1, b_1), (a_2, b_2)) &= max(\beta_{C_1}(a_1), \beta_{C_1}(a_2), \beta_{D_2}(b_1, b_2)) \\ &= max(\beta_{C_1}(a_1), \beta_{C_1}(a_2), \beta_{C_2}(b_1), \beta_{C_2}(b_2)) \\ &= max(\beta_{C_1}(a_1), \beta_{C_2}(b_1), \beta_{C_1}(a_2), \beta_{C_2}(b_2)) \\ &= max((\beta_{C_1} \oplus \beta_{C_2})(a_1, b_1), (\beta_{C_1} \oplus \beta_{C_2})(a_2, b_2)),\end{aligned}$$

$$\begin{aligned}(\gamma_{D_1}\oplus\gamma_{D_2})((a_1,b_1),(a_2,b_2)) &= min(\gamma_{C_1}(a_1),\gamma_{C_1}(a_2),\gamma_{D_2}(b_1,b_2))\\ &= min(\gamma_{C_1}(a_1),\gamma_{C_1}(a_2),\gamma_{C_2}(b_1),\gamma_{C_2}(b_2))\\ &= min(\gamma_{C_1}(a_1),\gamma_{C_2}(b_1),\gamma_{C_1}(a_2),\gamma_{C_2}(b_2))\\ &= min((\gamma_{C_1}\oplus\gamma_{C_2})(a_1,b_1),(\gamma_{C_1}\oplus\gamma_{C_2})(a_2,b_2)).\end{aligned}$$

(iv) Let $(a_1,a_2)\in\mathscr{E}_1$ and $(b_1,b_2)\notin\mathscr{E}_2$. Then,

$$\begin{aligned}(\alpha_{D_1}\oplus\alpha_{D_2})((a_1,b_1),(a_2,b_2)) &= max(\alpha_{D_1}(a_1,a_2),\alpha_{C_2}(b_1),\alpha_{C_2}(b_2))\\ &= max(\alpha_{C_1}(a_1),\alpha_{C_1}(a_2),\alpha_{C_2}(b_1),\alpha_{C_2}(b_2))\\ &= max(\alpha_{C_1}(a_1),\alpha_{C_2}(b_1),\alpha_{C_1}(a_2),\alpha_{C_2}(b_2))\\ &= max((\alpha_{C_1}\oplus\alpha_{C_2})(a_1,b_1),(\alpha_{C_1}\oplus\alpha_{C_2})(a_2,b_2)),\end{aligned}$$

$$\begin{aligned}(\beta_{D_1}\oplus\beta_{D_2})((a_1,b_1),(a_2,b_2)) &= max(\beta_{D_1}(a_1,a_2),\beta_{C_2}(b_1),\beta_{C_2}(b_2))\\ &= max(\beta_{C_1}(a_1),\beta_{C_1}(a_2),\beta_{C_2}(b_1),\beta_{C_2}(b_2))\\ &= max(\beta_{C_1}(a_1),\beta_{C_2}(b_1),\beta_{C_1}(a_2),\beta_{C_2}(b_2))\\ &= max((\beta_{C_1}\oplus\beta_{C_2})(a_1,b_1),(\beta_{C_1}\oplus\beta_{C_2})(a_2,b_2)),\end{aligned}$$

$$\begin{aligned}(\gamma_{D_1}\oplus\gamma_{D_2})((a_1,b_1),(a_2,b_2)) &= min(\gamma_{D_1}(a_1,a_2),\gamma_{C_2}(b_1),\gamma_{C_2}(b_2))\\ &= min(\gamma_{C_1}(a_1),\gamma_{C_1}(a_2),\gamma_{C_2}(b_1),\gamma_{C_2}(b_2))\\ &= min(\gamma_{C_1}(a_1),\gamma_{C_2}(b_1),\gamma_{C_1}(a_2),\gamma_{C_2}(b_2))\\ &= min((\gamma_{C_1}\oplus\gamma_{C_2})(a_1,b_1),(\gamma_{C_1}\oplus\gamma_{C_2})(a_2,b_2)).\end{aligned}$$

Hence the proof.

Definition 3.17. *(Degree of the symmetric difference on PyCNGs)*
Let us take $Gr_1=(C_1,D_1)$ and $Gr_2=(C_2,D_2)$ to be PyCNGs of the graphs $\mathscr{G}_1=(\mathscr{V}_1,\mathscr{E}_1)$ and $\mathscr{G}_2=(\mathscr{V}_2,\mathscr{E}_2)$, respectively. At any vertex $(a_1,a_2)\in\mathscr{V}_1\times\mathscr{V}_2$:

$$\begin{aligned}(d_\alpha)_{Gr_1\oplus Gr_2}(a_1,a_2) &= \sum_{((a_1,a_2),(b_1,b_2))\in\mathscr{E}_1\times\mathscr{E}_2}\alpha_{D_1}\oplus\alpha_{D_2}((a_1,a_2),(b_1,b_2))\\ &= \sum_{\substack{a_1=b_1\in\mathscr{V}_1,\\(a_2,b_2)\in\mathscr{E}_2}} max(\alpha_{C_1}(a_1),\alpha_{D_2}(a_2,b_2))+\sum_{\substack{(a_1,b_1)\in\mathscr{E}_1,\\a_2=b_2\in\mathscr{V}_2}} max(\alpha_{D_1}(a_1,b_1),\alpha_{C_2}(b_1))\\ &\quad+\sum_{\substack{(a_1,b_1)\notin\mathscr{E}_1,\\(a_2,b_2)\in\mathscr{E}_2}} max(\alpha_{C_1}(a_1),\alpha_{C_1}(b_1),\alpha_{D_2}(a_2,b_2))\\ &\quad+\sum_{\substack{(a_1,b_1)\in\mathscr{E}_1,\\(a_2,b_2)\notin\mathscr{E}_2}} max(\alpha_{C_1}(a_1,b_1),\alpha_{C_2}(a_2),\alpha_{C_2}(b_2)),\end{aligned}$$

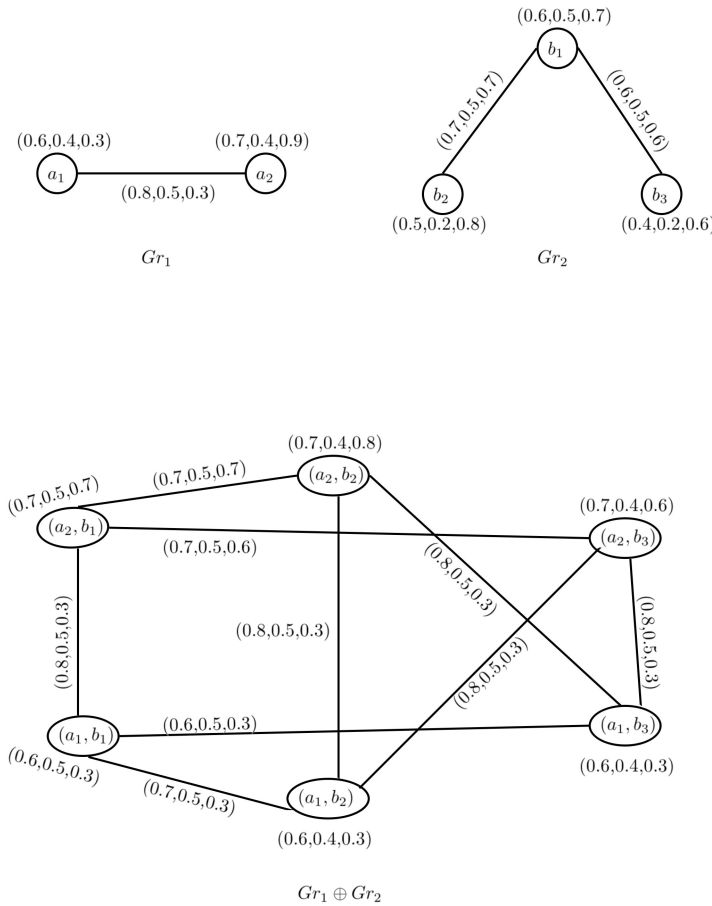

FIGURE 4. Gr_1 & Gr_2, Symmetric difference $Gr_1 \oplus Gr_2$

$$
\begin{aligned}
(d_\beta)_{Gr_1 \oplus Gr_2}(a_1, a_2) &= \sum_{((a_1,a_2),(b_1,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \beta_{D_1} \oplus \beta_{D_2}((a_1, a_2), (b_1, b_2)) \\
&= \sum_{\substack{a_1 = b_1 \in \mathscr{V}_1, \\ (a_2,b_2) \in \mathscr{E}_2}} max(\beta_{C_1}(a_1), \beta_{D_2}(a_2, b_2)) + \sum_{\substack{(a_1,b_1) \in \mathscr{E}_1, \\ a_2 = b_2 \in \mathscr{V}_2}} max(\beta_{D_1}(a_1, b_1), \beta_{C_2}(b_1)) \\
&\quad + \sum_{\substack{(a_1,b_1) \notin \mathscr{E}_1, \\ (a_2,b_2) \in \mathscr{E}_2}} max(\beta_{C_1}(a_1), \beta_{C_1}(b_1), \beta_{D_2}(a_2, b_2)) \\
&\quad + \sum_{\substack{(a_1,b_1) \in \mathscr{E}_1, \\ (a_2,b_2) \notin \mathscr{E}_2}} max(\beta_{C_1}(a_1, b_1), \beta_{C_2}(a_2), \beta_{C_2}(b_2)),
\end{aligned}
$$

$$
\begin{aligned}
(d_{\gamma})_{Gr_1 \oplus Gr_2}(a_1, a_2) &= \sum_{((a_1,a_2),(b_1,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \gamma_{D_1} \oplus \gamma_{D_2}((a_1, a_2), (b_1, b_2)) \\
&= \sum_{\substack{a_1=b_1 \in \mathscr{V}_1, \\ (a_2,b_2) \in \mathscr{E}_2}} min(\gamma_{C_1}(a_1), \gamma_{D_2}(a_2, b_2)) + \sum_{\substack{(a_1,b_1) \in \mathscr{E}_1, \\ a_2=b_2 \in \mathscr{V}_2}} min(\gamma_{D_1}(a_1, b_1), \gamma_{C_2}(b_1)) \\
&\quad + \sum_{\substack{(a_1,b_1) \notin \mathscr{E}_1, \\ (a_2,b_2) \in \mathscr{E}_2}} min(\gamma_{C_1}(a_1), \gamma_{C_1}(b_1), \gamma_{D_2}(a_2, b_2)) \\
&\quad + \sum_{\substack{(a_1,b_1) \in \mathscr{E}_1, \\ (a_2,b_2) \notin \mathscr{E}_2}} min(\gamma_{C_1}(a_1, b_1), \gamma_{C_2}(a_2), \gamma_{C_2}(b_2)).
\end{aligned}
$$

Definition 3.18. *(Total degree of the symmetric difference on PyCNGs)*
Consider PyCNGs $Gr_1 = (C_1, D_1)$ and $Gr_2 = (C_2, D_2)$ of the graphs $\mathscr{G}_1 = (\mathscr{V}_1, \mathscr{E}_1)$ and $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$, respectively. At any vertex $(a_1, a_2) \in \mathscr{V}_1 \times \mathscr{V}_2$:

$$
\begin{aligned}
(td_{\alpha})_{Gr_1 \oplus Gr_2}(a_1, a_2) &= \sum_{((a_1,a_2),(b_1,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \alpha_{D_1} \oplus \alpha_{D_2}((a_1, a_2), (b_1, b_2)) \\
&= \sum_{\substack{a_1=b_1 \in \mathscr{V}_1, \\ (a_2,b_2) \in \mathscr{E}_2}} max(\alpha_{C_1}(a_1), \alpha_{D_2}(a_2, b_2)) + \sum_{\substack{(a_1,b_1) \in \mathscr{E}_1, \\ a_2=b_2 \in \mathscr{V}_2}} max(\alpha_{D_1}(a_1, b_1), \alpha_{C_2}(b_1)) \\
&\quad + \sum_{\substack{(a_1,b_1) \notin \mathscr{E}_1, \\ (a_2,b_2) \in \mathscr{E}_2}} max(\alpha_{C_1}(a_1), \alpha_{C_1}(b_1), \alpha_{D_2}(a_2, b_2)) \\
&\quad + \sum_{\substack{(a_1,b_1) \in \mathscr{E}_1, \\ (a_2,b_2) \notin \mathscr{E}_2}} max(\alpha_{C_1}(a_1, b_1), \alpha_{C_2}(a_2), \alpha_{C_2}(b_2)) + max(\alpha_{C_1}(a_1), \alpha_{C_2}(a_2)),
\end{aligned}
$$

$$
\begin{aligned}
(td_{\beta})_{Gr_1 \oplus Gr_2}(a_1, a_2) &= \sum_{((a_1,a_2),(b_1,b_2)) \in \mathscr{E}_1 \times \mathscr{E}_2} \beta_{D_1} \oplus \beta_{D_2}((a_1, a_2), (b_1, b_2)) \\
&= \sum_{\substack{a_1=b_1 \in \mathscr{V}_1, \\ (a_2,b_2) \in \mathscr{E}_2}} max(\beta_{C_1}(a_1), \beta_{D_2}(a_2, b_2)) + \sum_{\substack{(a_1,b_1) \in \mathscr{E}_1, \\ a_2=b_2 \in \mathscr{V}_2}} max(\beta_{D_1}(a_1, b_1), \beta_{C_2}(b_1)) \\
&\quad + \sum_{\substack{(a_1,b_1) \notin \mathscr{E}_1, \\ (a_2,b_2) \in \mathscr{E}_2}} max(\beta_{C_1}(a_1), \beta_{C_1}(b_1), \beta_{D_2}(a_2, b_2)) \\
&\quad + \sum_{\substack{(a_1,b_1) \in \mathscr{E}_1, \\ (a_2,b_2) \notin \mathscr{E}_2}} max(\beta_{C_1}(a_1, b_1), \beta_{C_2}(a_2), \beta_{C_2}(b_2)) + max(\beta_{C_1}(a_1), \beta_{C_2}(a_2)),
\end{aligned}
$$

$$\begin{aligned}(td_\gamma)_{Gr_1\oplus Gr_2}(a_1,a_2) &= \sum_{((a_1,a_2),(b_1,b_2))\in\mathscr{E}_1\times\mathscr{E}_2} \gamma_{D_1}\oplus\gamma_{D_2}((a_1,a_2),(b_1,b_2))\\ &= \sum_{\substack{a_1=b_1\in\mathscr{V}_1,\\(a_2,b_2)\in\mathscr{E}_2}} min(\gamma_{C_1}(a_1),\gamma_{D_2}(a_2,b_2)) + \sum_{\substack{(a_1,b_1)\in\mathscr{E}_1,\\a_2=b_2\in\mathscr{V}_2}} min(\gamma_{D_1}(a_1,b_1),\gamma_{C_2}(b_1))\\ &\quad+\sum_{\substack{(a_1,b_1)\notin\mathscr{E}_1,\\(a_2,b_2)\in\mathscr{E}_2}} min(\gamma_{C_1}(a_1),\gamma_{C_1}(b_1),\gamma_{D_2}(a_2,b_2))\\ &\quad+\sum_{\substack{(a_1,b_1)\in\mathscr{E}_1,\\(a_2,b_2)\notin\mathscr{E}_2}} min(\gamma_{C_1}(a_1,b_1),\gamma_{C_2}(a_2),\gamma_{C_2}(b_2)) + min(\gamma_{C_1}(a_1),\gamma_{C_2}(a_2)).\end{aligned}$$

Example 3.19.

As per Figure 4, examine the symmetric difference on PyCNGs. Then the vertex degree of (a_1,b_1) *is* $d_{Gr_1\oplus Gr_2}(a_1,b_1)= ((d_\alpha)_{Gr_1\oplus Gr_2}(a_1,b_1),(d_\beta)_{Gr_1\oplus Gr_2}(a_1,b_1),(d_\gamma)_{Gr_1\oplus Gr_2}(a_1,b_1))=$ *(2.1,1.5,0.9) and the vertex total degree of* (a_1,b_1) *is* $td_{Gr_1\oplus Gr_2}(a_1,b_1)= ((td_\alpha)_{Gr_1\oplus Gr_2}(a_1,b_1),(td_\beta)_{Gr_1\oplus Gr_2}(a_1,b_1),(td_\gamma)_{Gr_1\oplus Gr_2}(a_1,b_1))=$ *(2.7,2.0,1.2). Likewise, the vertex degree and vertex total degree of the balance vertices are obtained by* $d_{Gr_1\oplus Gr_2}(a_2,b_1)=$ *(2.2,1.5,1.6),* $td_{Gr_1\oplus Gr_2}(a_2,b_1)=$ *(2.9,2.0,2.3),* $d_{Gr_1\oplus Gr_2}(a_1,b_2)=$ *(2.3,1.5,0.9),* $td_{Gr_1\oplus Gr_2}(a_1,b_2)=$ *(2.9,1.9,1.2),* $d_{Gr_1\oplus Gr_2}(a_2,b_2)=$ *(2.3,1.5,1.3),* $td_{Gr_1\oplus Gr_2}(a_2,b_2)=$ *(3.0,1.9,2.1),* $d_{Gr_1\oplus Gr_2}(a_1,b_3)=$ *(2.2,1.5,0.9),* $td_{Gr_1\oplus Gr_2}(a_1,b_3)=$ *(2.8,1.9,1.2),* $d_{Gr_1\oplus Gr_2}(a_2,b_3)=$ *(2.3,1.5,1.2),* $td_{Gr_1\oplus Gr_2}(a_2,b_3)=$ *(3.0,1.9,1.8).*

4. Application of Product on PyCNG in Brain Network Analysis

The human brain is a complex structure of the nervous system, which controls many activities in our body. It comprises different brain regions which are interconnected by neural pathways to perform and regulate various functions like thinking, decision-making, breathing, etc. Graph theory is much useful to structure and demonstrate the brain regions and their interconnecting networks. Here, we consider a Pythagorean co-neutrosophic graphical environment to deal with brain network analysis, since the function of a particular region and the connecting neural path can be studied deeply with three membership functions and their individual weights.

(i) Let us consider the brain graphs $Gr = (V,E)$ of two autism patients with brain's functional regions like the sensory cortex, motor cortex, frontal lobe, parietal lobe, and temporal lobe, where vertex set V denotes the brain regions and edge set E denotes the effective connectivity between different brain regions. Since we are going to deal with Pythagorean co-neutrosophic graph, an explicit declaration about each membership of vertex and edge is essential. The existence, uncertain, and non-existence memberships of vertices denotes brain region's function, inconsistency in functioning, and failure in function, respectively. For edges, the corresponding memberships may be considered as effective connectivity, inconsistency due to causal influences, and failure in connectivity. In general, kernels are used to define the similarity between a pair of elements. Here, we take the random walk graph kernel, which analyzes the communication network between two different brain network graphs and is used to measure the similarity in the number of random walks by comparing their corresponding random walk probability distributions. If a pair of Pythagorean co-neutrosophic graphs(PyCNGs) Gr_1 and Gr_2 are given, then the kernel executes the random walks on both graphs and compute the amount of matching walks, which will be equivalent to the performance of the random walk on their direct product graph. By using definitions 2.7 and 2.8, we can get the direct product of PyCNGs and the random walk graph kernel of graphs Gr_1 and Gr_2, where the weight matrix and decay factor must be found for each membership. Finally, the functional similarity and communications between these two autism patients will be analyzed by considering the output-direct product graph.

(ii) The shortest path graph kernel is another type of graph kernel, which measures the similarity between two PyCNGs by considering the shortest paths between their vertices. This kernel is generally suitable for graphs, where the distance between vertices is considered as road transport, social networks, molecular distance, etc. Here, the shortest path graph kernel will be used to measure the similarity by considering the shortest path between the neural elements. Also, the product of the wiener indices of two PyCNGs is equivalent to the shortest path kernel on graphs.

5. **Conclusion**

This paper deals with various products on Pythagorean co-neutrosophic graphs(PyCNGs), and some properties like degree, total degree are explained in the resultant PyCNG with examples, which give more flexible results than Pythagorean fuzzy graphs. Also, an application related to brain network analysis is given as an initiative to continue the progress in PyCNG.

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