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The Convergence Theorems for Mixing Random Variable Sequences

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Abstract**—**In this paper, some limit properties for mixing random variables sequences were studied and some results on weak law of large number for mixing random variables sequences were presented. Some complete convergence theorems were also obtained. The results extended and improved the corresponding theorems in i.i.d random variables sequences.

Keywords**—**Complete convergence, mixing random variables, weak law of large numbers.

I. INTRODUCTION

SUPPOSE that (Ω, F, P) is a probability space and $\{X_n, n \geq 1\}$ be a sequence of random variable defined on (Ω, F).

Definition 1. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for any disjoint subsets A and B of $\{1, 2\cdots, n\}$ and any real coordinatewise nondecreasing functions f on R^A and g on R^B . $\mathrm{cov}(f(X_i, i \in A), g(X_j, j \in B)) \leq 0$, whenever the covariance exists. An infinite family of random variables $\{X_i, 1 \leq i < \infty\}$ is said to be NA if every finite subfamily is NA.

Definition 2. A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be ρ^--mixing if for any finite subsets $S, T \subset N$, $\rho^-(s) = \sup\{\rho^-(S,T); dist(S,T) \geq s\} \to 0, s \to \infty$. Here

$$\rho^-(S,T) = 0 \vee \sup\{\frac{\mathrm{cov}(f(X_i, i \in S), g(Y_j, j \in T))}{(Varf(X_i, i \in S))^{1/2}, (Varg(Y_j, j \in T))^{1/2}}, f, g \in \wp\},$$

$\wp$ be a class of functions, which are coordinatewise increasing.

It is easy to see that $\{X_i, 1 \leq i \leq n\}$ is NA if and only if $\rho^-(s) = 0$,for $s \geq 1$. So ρ^- -mixing is weaker than ρ^* -mixing and can be regarded as the asymptotically negative association or negative side ρ^*-mixing. Zhang [1] gives an example of a ρ^--mixing sequence which is neither NA nor ρ^*-mixing. Since introducing the concept, [2] pointed out and proved in their paper that a number of well-known multivariate distributions possess the NA property. Now people know that NA random variables have wide application in reliability theory and multivariate analysis.

Recently, [3] showed that NA structure plays an important role in risk management. Because of including NA and ρ^* -mixing random variables, the notions of ρ^--mixing random variables have received more and more attention in recent years. A great number of papers for ρ^--mixing random variables have appeared in the literature. See, for example, [4] for moment inequalities and application, [5], [6] for Central limit theorems, [7] for Inequalities of maximum of partial sums and weak convergence, [8], Strong consistency of M-estimator in nonlinear models etc. When these are compared with the corresponding results for sequences of independence random variables, there still remains much to be desired.

We assume in the whole paper that I_A be the indicator function of the set A . C denotes a positive constant which may be different in various places.

The main object of the paper is to study the limit properties on partial sums of ρ^- -mixing random variables sequences and try to obtain some new results. We establish the weak law of large numbers and complete convergence theorems. Our results in this paper extend and improve the corresponding results of [9] and [10]. The results depend on the following lemmas.

Lemma 1. (see [1]) Let $\{X_n, n \geq 1\}$ be a sequence of ρ^- -mixing random variables. $EX_i = 0$, $E|X_i|^q < \infty$ for some $q \geq 2$ and for every $i \geq 1$ .Then there exists a positive constant C depending only on q , such that

$$E(\max_{1\leq i\leq n}\left|\sum_{j=1}^{i} X_j\right|^q) \leq C(\sum_{i=1}^{n} E|X_i|^q + (\sum_{i=1}^{n} EX_i^2)^{q/2})$$

Lemma 2 Let $\{X_n, n \geq 1\}$ be a sequence of ρ^- -mixing random variables. Then for any $x \geq 0$,there exists a positive constant C such that for any $n \geq 1$

$$(1 - P(\max_{1\leq k\leq n}|X_k| > x))^2 \sum_{k=1}^{n} P(|X_k| > x) \leq CP(\max_{1\leq k\leq n}|X_k| > x)$$

Proof: Let $A_k = (|X_k| > x)$ and

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$$\alpha_n = 1 - P(\overset{n}{\underset{k=1}{\cup}} A_k) = 1 - P(\max_{1\le k\le n} |X_k| > x).$$

Without loss of generality, assume $\alpha_n > 0$. By the Cauchy-Schwarz inequality and lem1, we have

$$\sum_{k=1}^{n} P(A_k) = \sum_{k=1}^{n} P(A_k, \overset{n}{\underset{j=1}{\cup}} A_j) = \sum_{k=1}^{n} E(I_{A_k} I_{\overset{n}{\underset{j=1}{\cup}} A_j})$$

$$= E(\sum_{k=1}^{n} (I_{A_k} - EI_{A_k})) I_{\overset{n}{\underset{j=1}{\cup}} A_j} + \sum_{k=1}^{n} P(A_k) P(\overset{n}{\underset{j=1}{\cup}} A_j)$$

$$\le (C \frac{(1-\alpha_n)}{\alpha_n} \alpha_n \sum_{k=1}^{n} P(A_k))^{1/2} + (1-\alpha_n) \sum_{k=1}^{n} P(A_k)$$

$$\le \frac{1}{2}(C \frac{(1-\alpha_n)}{\alpha_n} + \alpha_n \sum_{k=1}^{n} P(A_k)) + (1-\alpha_n) \sum_{k=1}^{n} P(A_k)$$

Thus, we have

$$\alpha_n^{\,2} \sum_{k=1}^{n} P(A_k) \le C(1-\alpha_n)$$

i.e.

$$(1 - P(\max_{1\le k\le n} |X_k| > x))^2 \sum_{k=1}^{n} P(|X_k| > x) \le CP(\max_{1\le k\le n} |X_k| > x)$$

Lemma 3. (see [10]) Let $\{X_n, n \ge 1\}$ be a random variable sequences, X is a random variable such that

$$P(|X_n| \ge t) \le cP(|X| \ge t) \text{ for any } t > 0, n \ge 1$$

Then for any $\beta > 0$, $t > 0$, we have

$$E|X_n|^\beta I_{(|X_n|>t)} \le C E|X|^\beta I_{(|X|>t)}$$

$$E|X_n|^\beta I_{(|X_n|\le t)} \le C(E|X|^\beta I_{(|X|\le t)} + t^\beta P(|X| > t))$$

II. Weak Convergence and Proof

Theorem 1. Let $\{X_n, n \ge 1\}$ be a sequence of ρ^--mixing random variables, $\{X_n\} < X$, satisfying

$$\lim_{n\to\infty} nP(|X| > n^p) = 0 \text{ for } p > 1/2 \tag{1}$$

Then

$$S_n / n^p - \sum_{i=1}^{n} EX_i I_{(|X_i|\le n^p)} / n^p \xrightarrow{P} 0 \tag{2}$$

Remark 1. Let $\{X_n, n \ge 1\}$ be a sequence of ρ^--mixing identically distributed random variables (i.d.r.v.), then

$$S_n / n^p - n^{1-p} EX_1 I_{(|X_1|\le n^p)} \xrightarrow{P} 0 \tag{3}$$

Remark 2. When $p = 1$ and $\{X_n, n \ge 1\}$ i.i.d, then Theorem 1 is the weak law of large numbers due to Feller. So theorem 1 extends Feller's weak law of large numbers to a ρ^--mixing setting.

Proof: Let $Y_i = X_i I_{(|X_i|\le n^p)}$,for $1 \le i \le n$ and $S_n^{'} = \sum_{i=1}^{n} Y_i$. Then, for each $n \ge 2$, $\{Y_i, i \ge 1\}$ are ρ^--mixing r.v.s. and for every $\varepsilon > 0$

$$P\left(\left|\frac{S_n}{n^p} - \frac{S_n^{'}}{n^p}\right| > \varepsilon\right) = P(\frac{S_n}{n^p} \ne \frac{S_n^{'}}{n^p}) = P(\bigcup_{i=1}^{n} (X_i \ne Y_i))$$

$$\le \sum_{i=1}^{n} P(|X_i| > n^p) \le CnP(|X| > n^p) \to 0$$

due to (1). So that (1) entails $\frac{S_n}{n^p} - \frac{S_n^{'}}{n^p} \xrightarrow{P} 0$. Thus to prove (2), it suffices to verify that

$$S_n^{'} / n^p - \sum_{i=1}^{n} EX_i I_{(|X_i|\le n^p)} / n^p \xrightarrow{P} 0, n \to \infty \tag{4}$$

By the Toeplity Lemma and (1), we have

$$\frac{\sum_{k=1}^{n} k^{2p-2} kP(|X| > k^p)}{\sum_{j=1}^{n} j^{2p-2}} \to 0, n \to \infty$$

With this and $\sum_{j=1}^{n} j^{2p-2} = O(n^{2p-1})$ for $p > 1/2$, we have

$$n^{-2p+1} \sum_{k=1}^{n} k^{2p-1} P(|X| > k^p) \to 0, n \to \infty$$

which, in conjunction with Lemma 1 and Lemma 3 for every $\varepsilon > 0$,

$$P(|S_n^{'} - ES_n^{'}| > \varepsilon n^p) \le Cn^{-2p} E(S_n^{'} - ES_n^{'})^2 = Cn^{-2p} E(\sum_{j=1}^{n} (Y_j - EY_j))^2$$

$$\le Cn^{-2p} E \sum_{j=1}^{n} (Y_j - EY_j)^2 \le Cn^{-2p} \sum_{j=1}^{n} EX_j^{\,2} I_{(|X_j|\le n^p)}$$

$$\le Cn^{-2p} \sum_{j=1}^{n} (E|X|^2 I_{(|X|\le n^p)} + n^{2p} P(|X| > n^p))$$

$$\le Cn^{-2p+1} \sum_{k=1}^{n} E|X|^2 I_{((k-1)^p < |X| \le k^p)} + CnP(|X| > n^p)$$

$$\le Cn^{-2p+1} [\sum_{k=1}^{n} k^{2p-1} P(|X| > k^p) + 1] + CnP(|X| > n^p) \to 0$$

Thus

$$S_n^{'} / n^p - ES_n^{'} / n^p = S_n^{'} / n^p - \sum_{i=1}^{n} EX_i I_{\left(|X_i| \le n^p\right)} / n^p \xrightarrow{P} 0$$

i.e. (4) holds.

III. Complete Convergence

Definition 3. A function $l(x) > 0 \ (x > 0)$ is said to be a slowly varying function if for any $c > 0$, $\lim_{x\to\infty} l(cx)/l(x) = 1$

Lemma 4. (see [11]) Let $l(x)$ be a slowly varying function, then

(i). $\lim_{k\to\infty} \sup_{2^k \le x < 2^{k+1}} l(x)/l(2^k) = 1$

(ii). For any $r > 0$, $\eta > 0$ and any natural number k, there exist constants $c_1, c_2 > 0$ such that

$$c_1 2^{kr} l(2^k \eta) \le \sum_{j=1}^{k} 2^{jr} l(2^j \eta) \le c_2 2^{kr} l(2^k \eta)$$

(iii).For any $r < 0$, $\eta > 0$ and any natural number k,there exist constants $d_1, d_2 > 0$ such that

$$d_1 2^{kr} l(2^k \eta) \le \sum_{j=k}^{\infty} 2^{jr} l(2^j \eta) \le d_2 2^{kr} l(2^k \eta)$$

Theorem 2. Let $\{X_n, n \ge 1\}$ be a sequence of ρ^--mixing i.d.r.v.s. and $l(x)$ be a slowly varying function. Then for $0 < p < 2$, $\alpha p \ge 1$ and $EX_1 = 0$, the following statements are equivalent

$$E(\left|X_1\right|^p l(\left|X_1\right|^{1/\alpha})) < \infty \tag{5}$$

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} l(n) P(\max_{1 \le j \le n} \left|S_j\right| > \varepsilon n^{\alpha}) < \infty, \forall \varepsilon > 0 \tag{6}$$

Remark 3. When $l(x) = 1$ and $\{X_n, n \ge 1\}$ i.i.d., then Theorem 2 becomes the Baum and Katz complete convergence theorem. So theorem 4 extends and improves the Baum and Katz complete convergence theorem for i.i.d.r.v.s to a ρ^--mixing i.d.r.v.s.

Proof: $(5) \Rightarrow (6)$ Let $Y_i = X_i I_{(|X_i| \le n^{\alpha})}$. We have

$$n^{-\alpha} \max_{1 \le j \le n} \left|\sum_{i=1}^{j} EY_i\right| \to 0 \quad n \to \infty \tag{7}$$

So, $\forall \varepsilon > 0$,as n large enough, we have

$$\max_{1 \le j \le n} \left|\sum_{i=1}^{j} EY_i\right| < \frac{\varepsilon n^{\alpha}}{2} \quad \left|\sum_{i=1}^{j} EY_i\right| < \frac{\varepsilon n^{\alpha}}{2}$$

Thus

$$\begin{aligned}
&\{\max_{1\le j\le n} |S_j| \ \ge \ \varepsilon n^{\alpha}\} \\
&\cup \{\max_{1\le j\le n} |S_j| \ge \varepsilon n^{\alpha}, \forall i: 1 \le i \le n, |X_i| \le n^{\alpha}\} \\
&= \{\max_{1\le j\le n} |S_j| \ge \varepsilon n^{\alpha}, \exists i: 1 \le i \le n, |X_i| > n^{\alpha}\} \\
&\subseteq \bigcup_{i=1}^{n} \{|X_i| > n^{\alpha}\} \cup \{\max_{1\le j\le n} \sum_{i=1}^{j} Y_i | \ge \varepsilon n^{\alpha}\} \\
&\subseteq \bigcup_{i=1}^{n} \{|X_i| > n^{\alpha}\} \cup \{\max_{1\le j\le n} \sum_{i=1}^{j} (Y_i - EY_i) | \ge \varepsilon n^{\alpha} / 2\}
\end{aligned}$$

Without loss of generality, to prove (6), it suffices to prove that

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} l(\mathrm{n}) \mathrm{P}\ (\bigcup_{i=1}^{n} \{|X_i| > n^{\alpha}\}) < \infty \tag{8}$$

$$\sum_{n=1}^{\infty} n^{\alpha p - 2} l(\mathrm{n}) \mathrm{P}\ (\max_{1\le j\le n} \left|\sum_{i=1}^{j} (Y_i - EY_i)\right| \ge \varepsilon n^{\alpha}) < \infty \tag{9}$$

By Lemma 4 and (5), it is easy to see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p - 2} l(\mathrm{n}) \mathrm{P}(\bigcup_{i=1}^{n} \{|X_i| > n^{\alpha}\}) \le \sum_{n=1}^{\infty} n^{\alpha p - 1} l(\mathrm{n}) \mathrm{P}(|X_1| > n^{\alpha}) \\
&= \sum_{j=1}^{\infty} \sum_{2^j \le n < 2^{j+1}} n^{\alpha p - 1} l(\mathrm{n}) \mathrm{P}(|X_1| > n^{\alpha}) \\
&\le C \sum_{j=1}^{\infty} 2^{j(\alpha p - 1)} 2^j l(2^j) \mathrm{P}(|X_1| > 2^{j\alpha}) \\
&= C \sum_{j=1}^{\infty} 2^{j\alpha p} l(2^j) \sum_{k=j}^{\infty} \mathrm{P}(2^{\alpha k} < |X_1| \le 2^{\alpha(k+1)}) \\
&\le C \sum_{k=1}^{\infty} 2^{k\alpha p} l(2^k) \mathrm{P}(2^{\alpha k} < |X_1| \le 2^{\alpha(k+1)}) \\
&\le CE(\left|X_1\right|^p l(\left|X_1\right|^{1/\alpha})) < \infty
\end{aligned}$$

i.e. (8) holds.

By the Lemma 1, Lemma 4, Markov inequality and (5), we obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p - 2} l(\mathrm{n}) \mathrm{P}\ (\max_{1\le j\le n} \left|\sum_{i=1}^{j} (Y_i - EY_i)\right| \ge \varepsilon n^{\alpha}) \\
&\le C \sum_{n=1}^{\infty} n^{\alpha p - 2 - 2\alpha} l(\mathrm{n}) \sum_{i=1}^{n} E(Y_i - EY_i)^2
\end{aligned}$$

$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} l(\mathrm{n}) EX_1^2 I_{(|X_1|\leq n^{\alpha})}$$
$$= C\sum_{j=1}^{\infty} \sum_{2^{j-1}\leq n<2^{j}} n^{\alpha p-1-2\alpha} l(\mathrm{n}) EX_1^2 I_{(|X_1|\leq n^{\alpha})}$$
$$\leq C\sum_{j=1}^{\infty} 2^{\alpha j(p-2)} l(2^j) EX_1^2 I_{(|X_1|\leq 2^{j\alpha})}$$
$$= C\sum_{j=1}^{\infty} 2^{\alpha j(p-2)} l(2^j) \sum_{k=1}^{j} EX_1^2 I_{(2^{\alpha(k-1)}<|X_1|\leq 2^{\alpha k})}$$
$$= C\sum_{k=1}^{\infty}\sum_{j=k}^{\infty} 2^{\alpha j(p-2)} l(2^j) EX_1^2 I_{(2^{\alpha(k-1)}<|X_1|\leq 2^{\alpha k})}$$
$$\leq CE(|X_1|^p\, l(|X_1|^{1/\alpha})) < \infty .$$

$(6) \Rightarrow (5)$

Obviously, (6) implies that

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\max_{1\leq k\leq n} |X_k| \geq \varepsilon n^{\alpha}) < \infty \tag{10}$$

Noting $\alpha p - 2 + 1 > 0$, we have

$$\sum_{m=1}^{\infty} P(\max_{1\leq j\leq 2^m} |X_j| \geq \varepsilon 2^{\alpha(m+1)}) \leq C\sum_{m=1}^{\infty} \sum_{2^m\leq n<2^{m+1}} \frac{1}{n} P(\max_{1\leq j\leq n} |X_j| \geq \varepsilon n^{\alpha})$$
$$= C\sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1\leq j\leq n} |X_j| \geq \varepsilon n^{\alpha})$$
$$\leq C\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) P(\max_{1\leq j\leq n} |X_j| \geq \varepsilon n^{\alpha}) < \infty$$

Thus

$$\max_{2^{(m-1)}\leq n<2^m} P(\max_{1\leq j\leq n} |X_j| \geq \varepsilon 2^{2\alpha} n^{\alpha}) \leq P(\max_{1\leq j<2^m} |X_j| \geq \varepsilon 2^{\alpha(m+1)}) \to 0$$

Therefore, for n sufficiently large, we have

$$P(\max_{1\leq j\leq n} |X_j| \geq 2\varepsilon n^{\alpha}) < \frac{1}{2}$$

By lemma 2, we have

$$\sum_{k=1}^{n} P(|X_k| \geq \varepsilon 2^{2\alpha} n^{\alpha}) \leq 4CP(\max_{1\leq j\leq n} |X_j| \geq \varepsilon 2^{2\alpha} n^{\alpha})$$

In conjunction with (10), we get

$$\sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P(|X_1| \geq \varepsilon 2^{2\alpha} n^{\alpha}) < \infty, \forall \varepsilon > 0$$

Thus by Lemma 4, we finally have

$$\infty > \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) P(|X_1| \geq \varepsilon 2^{2\alpha} n^{\alpha})$$
$$= \sum_{j=1}^{\infty} \sum_{2^j\leq n<2^{j+1}} n^{\alpha p-1} l(n) P(|X_1| \geq \varepsilon 2^{2\alpha} n^{\alpha})$$
$$\geq C\sum_{j=1}^{\infty} 2^{j\alpha p} l(2^j) P(|X_1| \geq \varepsilon 2^{\alpha} 2^{(j+1)\alpha} \square\ \varepsilon_0 2^{\alpha j})$$
$$= C\sum_{j=1}^{\infty} 2^{j\alpha p} l(2^j) \sum_{k=j}^{\infty} P(\varepsilon_0 2^{\alpha k} \leq |X_1| < \varepsilon_0 2^{\alpha(k+1)})$$
$$= C\sum_{k=1}^{\infty}\sum_{j=1}^{k} 2^{j\alpha p} l(2^j) P(\varepsilon_0 2^{\alpha k} \leq |X_1| < \varepsilon_0 2^{\alpha(k+1)})$$
$$\geq C\sum_{k=1}^{\infty} 2^{k\alpha p} l(2^k) P(\varepsilon_0 2^{\alpha k} \leq |X_1| < \varepsilon_0 2^{\alpha(k+1)})$$
$$\geq CE(|X_1|^p\, l(|X_1|^{1/\alpha}))$$

It completes the proof of Theorem 2.

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