Periodic orbits in a delayed Nicholson's blowflies model

Changjin Xu, Peiluan Li

Abstract—In this paper, a delayed Nicholson's blowflies model with a linear harvesting term is investigated. Regarding the delay as a bifurcation parameter, we show that Hopf bifurcation will occur when the delay crosses a critical value. Numerical simulations supporting the theoretical findings are carried out.

Keywords—Nicholson's blowflies model; Stability; Hopf bifurcation; Delay.

I. Introduction

It is well known that the population dynamics has been a subject of interest in mathematical biology. Recently, the theory of the population dynamics has made a remarkable progress and a great deal of results have been reported[1-3].

In 1980, to describe the population of the Australian sheep-blowfly and to agree with the experimental data obtained in [4], Gurney et al. [5] proposed the following nonlinear autonomous delay equation

$$\frac{dx(t)}{dt} = -\delta x(t) + Px(t-\tau)e^{-ax(t-\tau)} \tag{1}$$

where $x(t)$ is the size of the population at time t, P denotes the maximum per capita daily egg production, $\frac{1}{a}$ is the size at which the population reproduces at its maximum rate, δ is the per capita adult death rate, and τ is the generation time. The main dynamical behaviors of system (1) such as existence of positive solutions, persistence, permanence, oscillation and stability have been discussed by [6-11]. Assuming that a harvesting function is a function of the delayed estimate of the true population, Berezansky et al. [12] proposed Nicholson's blowflies model

$$\frac{dx(t)}{dt} = -\delta x(t) + Px(t-\tau)e^{-ax(t-\tau)} - Hx(t-\sigma), \tag{2}$$

where $\delta, p, \tau, a, H, \sigma \in (0, +\infty)$.

Based on former work [4-12], we further devote to explore the dynamical behaviors of system (2), i.e., we will investigate the natures of Hopf bifurcation of system (2). For simplification, we assume that $\sigma = \tau$, then system (2) takes the form

$$\frac{dx(t)}{dt} = -\delta x(t) + Px(t-\tau)e^{-ax(t-\tau)} - Hx(t-\tau). \tag{3}$$

The purpose of this paper is to investigate the existence of local Hopf bifurcation of model (3). This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are studied. In Section 3, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 4.

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II. Stability of the equilibrium and local Hopf bifurcations

It is easy to see that system (3) has a zero equilibrium point. If the condition $P > \delta + H$ holds, then system (3) has still another equilibrium

$$x^* = \frac{1}{a} \ln \frac{P}{\delta + H}.$$

In this paper, we only investigate the dynamical behavior of the zero equilibrium point. Clearly, the linearization of system (3) at zero equilibrium point takes the form

$$\frac{dx(t)}{dt} = -\delta x(t) + (P - H)x(t-\tau). \tag{4}$$

The associated characteristic equation of (4) is given by

$$\lambda + \delta - (P - H)e^{-\lambda\tau} = 0. \tag{5}$$

Let $\lambda = i\omega_0$ and substituting this into (5). Separating the real and imaginary parts, we have

$$(P - H)\cos\omega_0\tau = \delta, (P - H)\sin\omega_0\tau = -\omega_0. \tag{6}$$

Then we can obtain

$$(P - H)^2 = \delta^2 + \omega_0^2. \tag{7}$$

Obviously, if the condition

$$\text{(H1)}\ |P - H| > |\delta|$$

holds, then Eq.(7) has a pair of imaginary roots $\pm i\omega_0$ at a sequence of critical values τ_k, where

$$\omega_0 = \sqrt{(P - H)^2 - \delta^2}, \tag{8}$$

$$\tau_k = \frac{1}{\omega_0}\left[\arccos\frac{\delta}{P - H} + 2k\pi\right]. \tag{9}$$

When $\tau = 0$, (5) becomes

$$\lambda = (P - H) - \delta. \tag{10}$$

We assume that the condition

$$\text{(H2)}\ P - H < \delta$$

holds, then $\lambda = (P - H) - \delta < 0$. In view of above analysis, we have

Lemma 2.1. *If conditions (H1) and (H2) hold, then system (3) admits a pair of purely imaginary roots $\pm i\omega_0$ when $\tau = \tau_k, k = 0, 1, 2, \cdots$.*

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of Eq.(5) near $\tau = \tau_k$ satisfying $\alpha(\tau_k) = 0,\ \omega(\tau_k) = \omega_0$. Due to functional differential equation theory, for every $\tau_k, k = 0, 1, 2, \cdots$, there exists a $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_k| < \varepsilon$. Substituting $\lambda(\tau)$ into the left hand side of (5) and taking the derivative of λ with respect to τ, we get

$$\begin{aligned}\left[\frac{d\lambda}{d\tau}\right]^{-1} &= -\frac{1}{(P-H)\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda} \\ &= -\frac{e^{\lambda\tau}}{(P-H)\lambda} - \frac{\tau}{\lambda}.\end{aligned}$$

It follows together with (6) that

$$\begin{aligned}\text{Re}\left[\frac{d\lambda}{d\tau}\right]\Big|_{\tau=\tau_k}^{-1} &= -\text{Re}\left\{\frac{e^{\lambda\tau}}{(P-H)\lambda}\right\}_{\tau=\tau_k} \\ &= -\text{Re}\left\{\frac{\cos\omega_0\tau_k + i\sin\omega_0\tau_k}{(P-H)\omega_0 i}\right\} \\ &= -\frac{(P-H)\omega_k\sin\omega_0\tau_k}{(P-H)^2\omega_0^2} \\ &= \frac{\omega_0^2}{(P-H)^2\omega_0^2}.\end{aligned}$$

Thus

$$\text{sign}\left\{\text{Re}\left[\frac{d\lambda}{d\tau}\right]\Big|_{\tau=\tau_k}\right\} = \text{sign}\left\{\text{Re}\left[\frac{d\lambda}{d\tau}\right]\Big|_{\tau=\tau_k}^{-1}\right\} > 0.$$

According to the results of Kuang [1] and Hale [2], we have

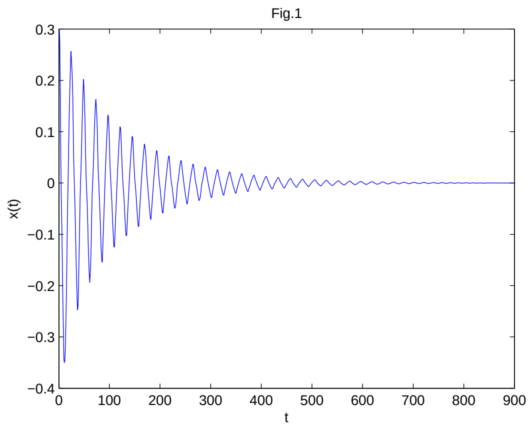
Theorem 2.1. *If conditions (H1) and (H2) hold, the zero equilibrium of system (3) is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau \geq \tau_0$. System (3) underdoes a Hopf bifurcation at the zero equilibrium when $\tau = \tau_k, k = 0, 1, 2, \cdots$.*

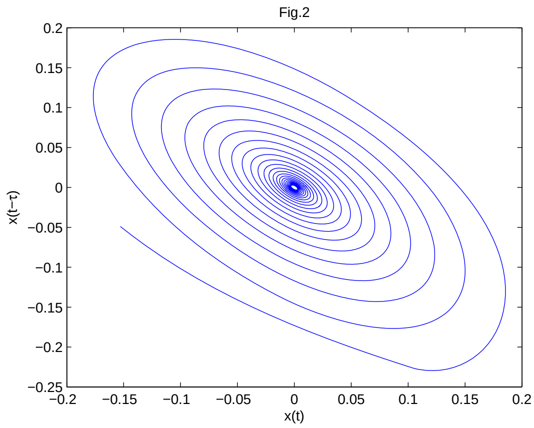
III. Numerical Examples

In this section, we use the formulae obtained in Section 2 to verify the existence of local Hopf bifurcation. We consider the following special case of system (3)

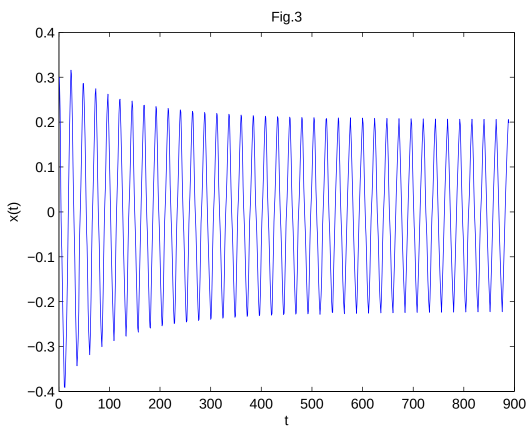
$$\frac{dx(t)}{dt} = -1.2x(t) + 0.2x(t-\tau)e^{-3x(t-\tau)} - 2x(t-\tau). \quad (11)$$

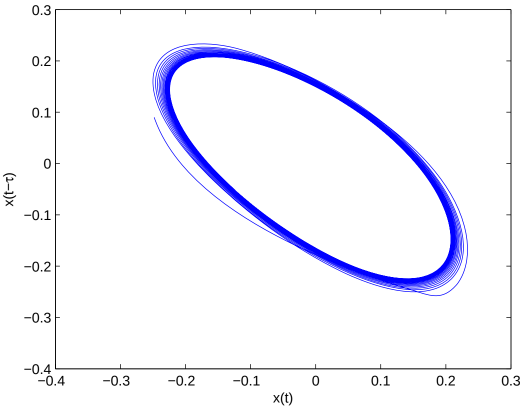
It is easy to see that the conditions (H1) and (H2) hold, then system (11) has a unique zero equilibrium $x^* = 0$. By direct computation by means of Matlab 7.0, we get $\omega_0 \approx 1.3416, \tau_0 \approx 1.6$. Thus the zero equilibrium $x^* = 0$ is stable when $\tau < \tau_0$ which is illustrated by the computer simulations (see Figs.1-2). When τ passes through the critical value $\tau_0 \approx 1.6$, the zero equilibrium $x^* = 0$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the zero equilibrium $x^* = 0$ which are depicted in Figs.3-4. When τ is large, chaotic phenomena will appear as shown in Figs.5-6.



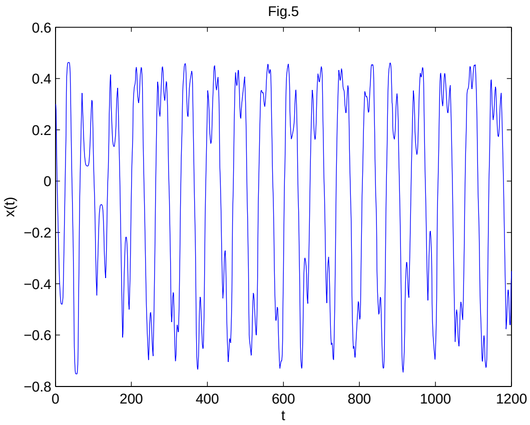


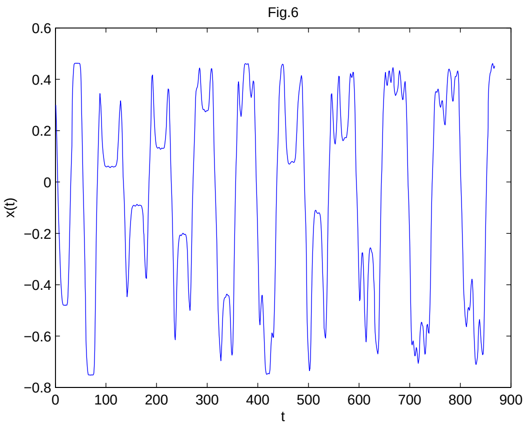
Figs.1-2 Dynamic behavior of system (11): times series of x. A Matlab simulation of the asymptotically stable zero equilibrium to system (11) with $\tau = 1.45 < \tau_0 \approx 1.6$. The initial value is 0.3.





Figs.3-4 Dynamic behavior of system (11): times series of x. A Matlab simulation of a Hopf bifurcation from the zero equilibrium to system (11) with $\tau = 1.8 > \tau_0 \approx 1.6$. The initial value is 0.3.





Figs.5-6 Dynamic behavior of system (11): times series of x. A Matlab simulation of chaotic phenomena appears to system (11) with $\tau = 12$ and $\tau = 24$. The initial value is 0.3.

IV. Conclusions

In this paper, we have dealt with the dynamics of a delayed Nicholson's blowflies model with a linear harvesting term. We show that under a certain condition, there exists a critical value τ_0 of the delay τ for the stability of the Nicholson's blowflies system. If $\tau \in [0, \tau_0)$, the zero equilibrium of theNicholson's blowflies system is asymptotically stable which means that the size of the population will keep in a steady state. When the delay τ passes through some critical values $\tau = \tau_k, k = 0, 1, 2, \cdots$, the zero equilibrium of the population system loses its stability and a Hopf bifurcation will occur. Moreover, it is shown that the chaotic phenomena appears when delay is large enough.

Acknowledgment

This work is supported by National Natural Science Foundation of China (No.10961008), Soft Science and Technology Program of Guizhou Province (No.2011 LKC2030), Natural Science and Technology Foundation of Guizhou Province(J[2012]2100), Governor Foundation of Guizhou Province(2012) and Doctoral Foundation of Guizhou University of Finance and Economics (2010).

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