

# $n$ – Strongly Gorenstein projective, injective and flat modules

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**Abstract**—Let  $R$  be a ring and  $n$  a fixed positive integer, we investigate the properties of  $n$ -strongly Gorenstein projective, injective and flat modules. Using the homological theory, we prove that the tensor product of an  $n$ -strongly Gorenstein projective (flat) right  $R$ -module and projective (flat) left  $R$ -module is also  $n$ -strongly Gorenstein projective (flat). Let  $R$  be a coherent ring, we prove that the character module of an  $n$ -strongly Gorenstein flat left  $R$ -module is an  $n$ -strongly Gorenstein injective right  $R$ -module. At last, let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ , we establish the relation between  $n$ -strongly Gorenstein projective (injective, flat)  $R$ -modules and  $n$ -strongly Gorenstein projective (injective, flat)  $S^{-1}R$ -modules. All conclusions in this paper is helpful for the research of Gorenstein dimensions in future.

**Keywords**—Commutative Ring,  $n$ -Strongly Gorenstein Projective,  $n$ -Strongly Gorenstein Injective,  $n$ -Strongly Gorenstein Flat,  $S$ -Ring.

## I. INTRODUCTION

UNLESS stated otherwise, throughout this paper all rings are associative with identity and all modules are unitary modules. Let  $R$  be a ring. We denote by  $R\text{-Mod}(\text{Mod-}R)$  the category of left(right)  $R$ -modules respectively. For any  $R$ -module  $M$ ,  $\text{pd}_R(M)$ ,  $\text{id}_R(M)$  and  $\text{fd}_R(M)$  denote the projective, injective and flat dimension of  $M$ , respectively. The character module  $\text{Hom}_Z(M, Q/Z)$  is denoted by  $M^+$ .

When  $R$  is two-sided noetherian, Auslander and Bridger [1] introduced the G-dimension for every finitely generated  $R$ -module. Several decades later, Enochs and Jenda[2,3] extended the ideas of Auslander and Bridger and introduced three homological dimensions, called the Gorenstein projective, injective and flat dimensions. These have been studied extensively by their founders and by Avramov, Christensen, Foxby, Frankild, Holm, Martsinkovsky, and Xu among others[4-7] over arbitrary associative rings. They proved that these dimensions are similar to the classical homological dimensions; i.e., projective, injective and flat dimensions respectively. D.Bennis and N.Mahdou[8] studied a particular case of Gorenstein projective, injective and flat modules, which they call respectively, strongly Gorenstein projective, injective and flat modules. They proved that every Gorenstein projective (resp. Gorenstein injective, Gorenstein flat) module is a direct summand of a strongly Gorenstein projective (resp. strongly Gorenstein injective, strongly Gorenstein flat) module. For any  $n \geq 1$ ,

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Bennis and Mahdou[9] introduced the notions of  $n$ -strongly Gorenstein projective injective and flat modules, in which 1-strongly Gorenstein projective (resp. injective) modules are just strongly Gorenstein projective (resp. injective) modules. Then they proved that an  $n$ -strongly Gorenstein projective module is projective if and only if it has finite flat dimension. They also gave some equivalent characterizations of  $n$ -strongly Gorenstein projective modules in terms of the vanishing of some homological groups.

In this paper, based on the results mentioned above, we continue the study of  $n$ -strongly Gorenstein projective, injective and flat modules, and investigate the properties and relation among them. At last, we study these properties under change of rings such that localizations and Morita equivalences. This paper is organized as follows.

In Section 2, we give some definitions of  $n$ -strongly Gorenstein projective, injective and flat modules, and some known results about them.

In Section 3, we study some equivalent characterizations, and then give some properties such that tensor product, polynomial ring and the relation among  $n$ -strongly Gorenstein projective, injective and flat modules

In Section 4, we give the properties of  $n$ -strongly Gorenstein projective, injective and flat modules under the change of rings such that localizations and Morita equivalences.

## II. DEFINITION AND GENERAL RESULTS

At first we introduce and study the  $n$ -strongly Gorenstein projective and injective modules which are defined as follows:

**Definitions 2.1**([9]) Let  $R$  be a positive integer. A left  $R$ -module  $M$  is called  $n$ -strongly Gorenstein projective ( $n$ -SG-projective for short) if there exists an exact sequence:

$$0 \longrightarrow M \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

in  $R\text{-Mod}$  with  $P_i$  projective for any  $1 \leq i \leq n$ , such that  $\text{Hom}_R(-, P)$  leaves the sequence exact whenever  $P \in R\text{-Mod}$  is projective.

Dually, a  $R$ -module  $N$  is called  $n$ -strongly Gorenstein injective ( $n$ -SG-injective, for short) if there exists an exact sequence:

$$0 \longrightarrow N \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^n \longrightarrow N \longrightarrow 0$$

in  $R\text{-Mod}$  with  $I^i$  injective for any  $1 \leq i \leq n$ , such that  $\text{Hom}_R(I, -)$  leaves the sequence exact whenever  $I \in R\text{-Mod}$  is injective.

An left  $R$ -module  $L$  is said to be  $n$ -strongly Gorenstein flat ( $n$ -SG-flat for short), if there exists an exact sequence of  $R$ -modules

$$0 \rightarrow L \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow L \rightarrow 0$$

in  $R$ -Mod with  $F_i$  flat for any  $1 \leq i \leq n$  and which remains exact after applying  $E \otimes_R -$  for any injective right  $R$ -module  $E$ .

Note that 1-SG- projective (resp. injective,flat) modules are just SG-projective (resp. injective, flat) modules. It is trivial that a 1-SG-projective (resp.injective. flat) module is  $n$ -SG-projective(resp.injective, flat) for any  $n \geq 1$ . The class of all  $n$ - strongly Gorenstein projective  $R$ - modules is denoted by  $n$ -SGP( $R$ ); the class of all the  $n$ -strongly Gorenstein injective modules is denoted by  $n$ -SGI( $R$ ); We denote the class of  $n$  strongly Gorenstein flat modules by  $n$ -SDF( $R$ ).

We can easily obtain the following result by definitions.

**Proposition 2.2** For any  $n \geq 1$ , we have

- (1)  $n$ -SGP( $R$ ) is closed under direct sums.
- (2)  $n$ -SGI( $R$ ) is closed under direct products.

### III. THE RELATION OF N-STRONGLY GORENSTEIN PROJECTIVE ,INJECTIVE AND FLAT MODULE

In this section we study some equivalent characterizations , properties and the relation among  $n$ -strongly Gorenstein projective , injective and flat modules .

**Proposition 3.1** For any positive integer  $n$ ,a module  $M \in R$ -Mod is  $n$ -SG-projective if and only if there exist an exact sequence  $0 \rightarrow M \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow M \rightarrow 0$  where each  $P_i$  is projective for any  $1 \leq i \leq n$  and  $\text{Ext}_R^i(M, Q) = 0$  for any projective module  $Q$  and any  $i \geq 1$ .

**Proof** Let  $M$  be  $n$ -SG-projective , there exist an exact sequence  $0 \rightarrow M \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow M \rightarrow 0$  where each  $P_i$  is projective , such that  $\text{Hom}_R(-, Q)$ leaves the sequence exact whenever  $Q \in \text{Mod } R$  is projective. We get  $\text{Ext}_R^i(M, Q) = \text{Ext}_R^{1+i}(M, Q)$  for all  $i \geq 1$ .By induce, we have that  $\text{Ext}_R^i(M, Q) = 0$  for all  $i \geq 1$ .

Dually, we can gives some characterizations of  $n$ -SG-injective modules and  $n$ -SG-flat module.

**Proposition 3.2** For any positive integer  $n$ ,a module  $N \in R$ -Mod is  $n$ -SG-injective if and only if there exist an exact sequence  $0 \rightarrow N \rightarrow I^1 \rightarrow \dots \rightarrow I^n \rightarrow N \rightarrow 0$  where each  $I^i$  is injective and  $\text{Ext}_R^i(E, N) = 0$  for any injective left  $R$ -module  $E$  and any  $i \geq 1$ .

**Proposition 3.3** For any positive integer  $n$ ,a module  $M \in R$ -Mod is  $n$ -SG-flat if and only if there exist an exact sequence  $0 \rightarrow L \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow L \rightarrow 0$  where each  $F_i$  is flat and  $\text{Tor}_i^R(I, M) = 0$  for any injective right  $R$ -module  $I$  and any  $i \geq 1$ .

**Proposition 3.4** Let  $R$  be a commutative ring and  $Q$  a projective left  $R$ -module. If  $M$  is an  $n$ -SG-projective,then  $M \otimes_R Q$  is an  $n$ -SG-projective  $R$ -module.

**Proof** There is an exact sequence  $0 \rightarrow M \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow M \rightarrow 0$  where each  $P_i$  is projective .Then  $0 \rightarrow M \otimes_R Q \rightarrow P_n \otimes_R Q \rightarrow \dots \rightarrow P_1 \otimes_R Q \rightarrow M \otimes_R Q \rightarrow 0$  is exact and  $P_i \otimes_R Q$  is projective  $R$ -modules

for all  $1 \leq i \leq n$ . Let  $Q'$  be any projective  $R$ -module, Then  $\text{Ext}_R^i(M \otimes_R Q, Q') \simeq \text{Hom}_R(Q, \text{Ext}_R^i(M, Q')) = 0$  by [10, P258, 9.20] for all  $i \geq 1$ . Hence  $M \otimes_R Q$  is an  $n$ - SG-projective  $R$ -module by Proposition 3.1.

**Proposition 3.5** Let  $R$  be a commutative ring. If  $M$  is an  $n$ -SG-projective  $R$ -module, then  $M[x]$  is an  $n$ -SG-projective  $R$ -module.

**Proof** There is an exact sequence  $0 \rightarrow M \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow M \rightarrow 0$  where each  $P_i$  is a projective  $R$ -module for any  $1 \leq i \leq n$ . So  $0 \rightarrow M[x] \rightarrow P_n[x] \rightarrow \dots \rightarrow P_1[x] \rightarrow M[x] \rightarrow 0$  where each  $P_i[x]$  is projective  $R[x]$ -module for any  $1 \leq i \leq n$ . Let  $Q$  be any projective  $R$ -module, then  $Q[x] \simeq R[x] \otimes_R Q \simeq R^{(N)} \otimes_R Q \simeq Q^{(N)}$ . Hence  $Q[x]$  is a projective  $R[x]$ -module, and so  $Q$  is a projective  $R$ -module by [11, Proposition 5.11]. Thus  $\text{Ext}_{R[x]}^i(M[x], Q) \simeq \text{Ext}_R^i(M, Q) = 0$  by [10, p. 258, 9.21] for all  $i \geq 1$ , and  $M[x]$  is an  $n$ -SG-projective  $R$ -module.

**Proposition 3.6** Let  $R$  be a commutative ring and  $F$  a flat  $R$ -module. If  $M$  is an  $n$ -SG-flat,then  $M \otimes_R F$  is an  $n$ -SG-flat  $R$ -module.

**Proof** There is an exact sequence  $0 \rightarrow M \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow M \rightarrow 0$  where each  $F_i$  is flat .Then  $0 \rightarrow M \otimes_R Q \rightarrow F_n \otimes_R Q \rightarrow \dots \rightarrow F_1 \otimes_R Q \rightarrow M \otimes_R Q \rightarrow 0$  is exact and  $F_i \otimes_R Q$  is flat  $R$ -module for all  $1 \leq i \leq n$ . Let  $I$  be any injective  $R$ -module and  $\mathcal{F}$  be a flat resolution of  $I$ , Then  $\text{Tor}_i^R(M \otimes_R Q, I) = H_i(M \otimes_R Q \otimes_R \mathcal{F}) \simeq \text{Tor}_i^R(M, Q \otimes_R I) = 0$  for all  $i \geq 1$ , Since  $F \otimes_R I$  is an injective  $R$ -module. Hence  $M \otimes_R Q$  is an  $n$ - SG-flat  $R$ -module.

**Theorem 3.7** Let  $R$  be right coherent and  $M$  is an  $n$ -SG-flat left  $R$ -module , then character module  $M^+ = \text{Hom}_Z(M, Q/Z)$  is an  $n$ -SG-injective right  $R$ -module.

**Proof** There exists an exact sequence  $0 \rightarrow M \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow M \rightarrow 0$  in  $R$ -Mod with  $F_i$  flat for any  $1 \leq i \leq n$  Then  $0 \rightarrow M^+ \rightarrow F_1^+ \rightarrow \dots \rightarrow F_n^+ \rightarrow M^+ \rightarrow 0$  is exact in Mod- $R$ . and  $F_i^+$  is injective for any  $1 \leq i \leq n$ . Let  $I$  be an injective right  $R$ -module. Then  $\text{Ext}_R^i(I, M^+) = \text{Tor}_i^R(I, M)^+ = 0$  for all  $i \geq 1$ , and hence  $M^+$  is an  $n$ -SG-injective right  $R$ -module.

In studying perfect rings, Bass[12] proved that a ring  $R$  is perfect if, and only if, every flat  $R$ -module  $M$  is projective. Motivated by this result, Sakhajev asked when, more generally, every finitely generated flat module is projective [13],who called the ring which satisfies the question a  $S$ -ring , to honor Sakhajev. D.Bennis gives a characterization of  $S$ -rings as follows.

**Lemma 3.8**([8])  $R$  is an  $S$ -ring if, and only if, every finitely generated strongly Gorenstein flat  $R$ -module is strongly Gorenstein projective.

Now, we give the relationship between  $n$ -SG-flat module and  $n$ -SG-projective module on  $S$ -ring.

**Theorem 3.9** Let  $R$  be an  $S$ -ring, a left  $R$ -module  $M$  is finitely generated  $n$ -SG- projective if, and only if, it is finitely generated  $n$ -SG-flat.

**Proof** ( $\implies$ ) Assume that  $M$  is a finitely generated  $n$ -SG-projective  $R$ -module, there is an exact sequence  $0 \rightarrow M \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow M \rightarrow 0$  where each  $P_i$  is finitely generated projective for any  $1 \leq i \leq n$  and  $\text{Ext}_R^i(M, R) = 0$

for all  $i \geq 1$ . Let  $I$  be an any injective  $R$ - module, we have an isomorphism [14]

$$\text{Tor}_i^R(I, M) \simeq \text{Hom}(\text{Ext}_R^i(M, R), I) = 0$$

for all  $i \geq 1$ . Thus  $\text{Tor}_i^R(I, M) = 0$  for all  $i \geq 1$ , and therefore  $M$  is finitely generated  $n$ -SG-flat.

( $\Leftarrow$ ) Assume  $M$  to be a finitely generated  $n$ -SG-flat module, From Proposition 3.3, we deduce that there exists an exact sequence  $0 \rightarrow M \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow M \rightarrow 0$  in  $R$ -Mod with  $F_i$  finitely generated flat for any  $1 \leq i \leq n$  and  $\text{Tor}_i^R(I, M) = 0$  for all  $i \geq 1$  and any injective module  $I$ . By hypothesis,  $F_i$  is finitely generated projective for all  $1 \leq i \leq n$ . we have an isomorphism

$$\text{Tor}_i^R(I, M) \simeq \text{Hom}(\text{Ext}_R^i(M, R), I) = 0.$$

If we assume  $I$  to be faithfully injective, the isomorphism above implies that  $\text{Ext}_R^i(M, R) = 0$ . This means, that  $M$  is finitely generated  $n$ -SG-projective.

#### IV. CHANGE OF RINGS

Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ . Then  $S^{-1}R = (R \times S)/\sim = [a/s | a \in R, s \in S]$  is a ring and  $S^{-1}M = (M \times S)/\sim = [x/s | x \in M, s \in S]$  is a  $S^{-1}R$ -module. If  $P$  is prime ideal of  $R$  and  $S = R - P$ , then we will denote  $S^{-1}M, S^{-1}R$  by  $M_P, R_P$ , respectively.

**Theorem 4.1** Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ . If  $\bar{B}$  is an finitely generated  $n$ -SG-projective  $S^{-1}R$ -module, then  $\bar{B}$  is an  $n$ -SG-flat  $R$ -module

**Proof** There exists an exact sequence  $0 \rightarrow \bar{B} \rightarrow \bar{P}_n \rightarrow \dots \rightarrow \bar{P}_1 \rightarrow \bar{B} \rightarrow 0$  in  $S^{-1}R$ -Mod with  $\bar{P}_i$  finitely generated projective for all  $1 \leq i \leq n$ . Then  $\bar{P}_i$  is flat  $R$ -module by [11.Theorem 5.18]. Let  $I$  be any injective  $R$ -module. Then we get  $\text{Ext}_{S^{-1}R}^i(\bar{B}, S^{-1}R) = 0$  for any  $i \geq 1$ , Since  $\bar{B}$  is an finitely generated  $n$ -SG-projective  $S^{-1}R$ -module. Therefore

$$\begin{aligned} & \text{Hom}_{S^{-1}R}(\text{Ext}_{S^{-1}R}^i(\bar{B}, S^{-1}R), S^{-1}I) \\ & \simeq \text{Tor}_i^{S^{-1}R}(S^{-1}I, \bar{B}) \simeq \text{Tor}_i^R(I, \bar{B}) \otimes_R S^{-1}R \end{aligned}$$

and Hence  $\text{Tor}_i^R(I, \bar{B}) = 0$  by [15. condition  $O_r$ ] for all  $i \geq 1$ . Therefore  $\bar{B}$  is an  $n$ -SG-flat  $R$ -module.

**Lemma 4.2**[16] Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ . If  $S^{-1}R$  is a projective  $R$ -module, then  $\bar{A}$  is a projective  $R$ -module if and only if  $\bar{A}$  is a projective  $S^{-1}R$ -module for any  $\bar{A} \in S^{-1}R$ -Mod.

**Theorem 4.3** Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ . If  $S^{-1}R$  is a projective  $R$ -module, then

(1) If  $A$  is an  $n$ -SG-projective  $R$ -module, then  $S^{-1}A$  is an  $n$ -SG-projective  $S^{-1}R$ -module;

(2) If  $S^{-1}R$  is a finitely generated  $R$ -module, then  $\bar{B}$  is an  $n$ -SG-projective  $R$ -module if and only if  $\bar{B}$  is an  $n$ -SG-projective  $S^{-1}R$ -module for any  $\bar{B} \in S^{-1}R$ -Mod.

**Proof** (1) There exists an exact sequence  $0 \rightarrow M \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow M \rightarrow 0$  with  $P_i$  projective, Then  $0 \rightarrow S^{-1}M \rightarrow S^{-1}P_n \rightarrow \dots \rightarrow S^{-1}P_1 \rightarrow$

$S^{-1}M \rightarrow 0$  is exact in  $S^{-1}R$ -module, and  $S^{-1}P_i$  is projective  $S^{-1}R$ -module. Let  $Q$  be any projective  $S^{-1}R$ -module, Then  $Q$  is projective  $R$ -module by Lemma 4.2. Since  $A$  is an  $n$ -SG-projective  $R$ -module, then we have  $\text{Ext}_R^i(A, Q) = 0$  for all  $i \geq 1$ . So  $\text{Ext}_{S^{-1}R}^i(S^{-1}A, Q) \simeq \text{Ext}_{S^{-1}R}^i(S^{-1}R \otimes_R A, Q) \simeq \text{Ext}_R^i(A, Q) = 0$  by [10.P258.9.21] for all  $i \geq 1$ . Hence  $S^{-1}A$  is an  $n$ -SG-projective  $S^{-1}R$ -module;

(2) ( $\Rightarrow$ ) By (1), Since  $B \simeq S^{-1}B$ .

( $\Leftarrow$ ) There exists an exact sequence  $0 \rightarrow \bar{B} \rightarrow \bar{P}_n \rightarrow \dots \rightarrow \bar{P}_1 \rightarrow \bar{B} \rightarrow 0$  in  $S^{-1}R$ -Mod with  $\bar{P}_i$  projective for all  $1 \leq i \leq n$ . Then  $\bar{P}_i$  is a projective  $R$ -module by Lemma 4.1. Then  $\text{Hom}_R(S^{-1}R, Q)$  is a projective  $S^{-1}R$ -module since  $S^{-1}R$  is a finitely generated projective  $R$ -module by Lemma 4.2. So  $\text{Ext}_R^i(\bar{B}, Q) \simeq \text{Ext}_R^i(S^{-1}R \otimes_{S^{-1}R} \bar{B}, Q) \simeq \text{Ext}_{S^{-1}R}^i(\bar{B}, \text{Hom}_R(S^{-1}R, Q)) = 0$  by [10.Prop.5.17] and [10.P258.9.21] for all  $i \geq 1$ . Hence  $B$  is an  $n$ -SG-projective  $R$ -module.

**Theorem 4.4** Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ . If  $S^{-1}R$  is a projective  $R$ -module, then

(1) If  $A$  is an  $n$ -SG-injective  $R$ -module, then  $\text{Hom}_R(S^{-1}R, A)$  is an  $n$ -SG-injective  $S^{-1}R$ -module;

(2) For any  $B \in R$ -Mod,  $\text{Hom}_R(S^{-1}R, B)$  is an  $n$ -SG-injective  $R$ -module if and only if  $\text{Hom}_R(S^{-1}R, B)$  is an  $n$ -SG-injective  $S^{-1}R$ -module.

**Proof** (1) There exists an exact sequence  $0 \rightarrow A \rightarrow I^1 \rightarrow \dots \rightarrow I^n \rightarrow A \rightarrow 0$  with  $I^i$  injective for all  $1 \leq i \leq n$ . Then  $0 \rightarrow \text{Hom}_R(S^{-1}R, A) \rightarrow \text{Hom}_R(S^{-1}R, I^1) \rightarrow \dots \rightarrow \text{Hom}_R(S^{-1}R, I^n) \rightarrow \text{Hom}_R(S^{-1}R, A) \rightarrow 0$  is exact in  $S^{-1}R$ -Mod and  $\text{Hom}_R(S^{-1}R, I^i)$  is an injective  $S^{-1}R$ -module by [17.Theorem 3.2.9]. Let  $\bar{I}$  be any injective  $S^{-1}R$ -module, then  $\bar{I}$  is an injective  $R$ -module by [12.Lemma 1.2]. So  $\text{Ext}_{S^{-1}R}^i(\bar{I}, \text{Hom}_R(S^{-1}R, A)) \simeq \text{Ext}_R^i(\bar{I}, A) = 0$  by [10.P.258.9.21] for all  $i \geq 1$ , and Hence  $\text{Hom}_R(S^{-1}R, A)$  is an  $n$ -SG-injective  $S^{-1}R$ -module.

(2)( $\Rightarrow$ ) is obvious.

( $\Leftarrow$ ) There is an exact sequence  $0 \rightarrow \text{Hom}_R(S^{-1}R, B) \rightarrow \bar{I}^1 \rightarrow \dots \rightarrow \bar{I}^n \rightarrow \text{Hom}_R(S^{-1}R, B) \rightarrow 0$  in  $S^{-1}R$ -Mod with  $\bar{I}^i$  injective for all  $1 \leq i \leq n$ . Then  $E_i$  is injective  $R$ -module. Let  $I$  be any injective  $R$ -module, then  $S^{-1}I$  is an injective  $S^{-1}R$ -module. So

$$\begin{aligned} & \text{Ext}_R^i(I, \text{Hom}_R(S^{-1}R, B)) \simeq \text{Ext}_{S^{-1}R}^i(I, \text{Hom}_R(S^{-1}R, B)) \\ & \simeq \text{Ext}_{S^{-1}R}^i(S^{-1}I, \text{Hom}_R(S^{-1}R, B)) = 0 \end{aligned}$$

by [10.P.258.9.21] for all  $i \geq 1$ . Hence  $\text{Hom}_R(S^{-1}R, B)$  is an  $n$ -SG-injective  $R$ -module.

**Theorem 4.5** Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ . Then

(1) If  $A$  is an  $n$ -SG-flat  $R$ -module, then  $S^{-1}A$  is an  $n$ -SG-flat  $R$ -module;

(2) If  $A$  is an  $n$ -SG-flat  $R$ -module, then  $S^{-1}A$  is an  $n$ -SG-flat  $S^{-1}R$ -module;

(3) For any  $\bar{B} \in S^{-1}R$ -Mod,  $\bar{B}$  is an  $n$ -SG-flat  $R$ -module if and only if  $\bar{B}$  is an  $n$ -SG-flat  $S^{-1}R$ -module;

**Proof** (1) There is an exact sequence  $0 \rightarrow A \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow A \rightarrow 0$  in  $R\text{-Mod}$  where each  $F_i$  is flat for  $1 \leq i \leq n$ . Then  $0 \rightarrow S^{-1}A \rightarrow S^{-1}F_n \rightarrow \dots \rightarrow S^{-1}F_1 \rightarrow S^{-1}A \rightarrow 0$  is exact and  $S^{-1}F_i$  is flat  $S^{-1}R$ -module for all  $1 \leq i \leq n$ . Hence  $S^{-1}F_i$  is flat  $S^{-1}R$ -module. Let  $I$  be any injective  $R$ -module. Then  $\text{Tor}_i^{S^{-1}R}(I, S^{-1}A) \simeq \text{Tor}_i^R(S^{-1}I, A) = 0$  by [15.Prop.5.17] Since  $S^{-1}I$  is injective  $R$ -module by [12.Lemma 1.2]. Hence  $S^{-1}A$  is an  $n$ -SG-flat  $R$ -module.

(2) There is an exact sequence  $0 \rightarrow A \rightarrow F_n \rightarrow \dots \rightarrow F_1 \rightarrow A \rightarrow 0$  in  $R\text{-Mod}$  where each  $F_i$  is flat for  $1 \leq i \leq n$ . Then  $0 \rightarrow S^{-1}A \rightarrow S^{-1}F_n \rightarrow \dots \rightarrow S^{-1}F_1 \rightarrow S^{-1}A \rightarrow 0$  is exact and  $S^{-1}F_i$  is flat  $S^{-1}R$ -module for all  $1 \leq i \leq n$ . Let  $\bar{I}$  be any injective  $S^{-1}R$ -module. Then  $\bar{I}$  be any injective  $R$ -module by [11.Lemma 1.2]. So  $\text{Tor}_i^{S^{-1}R}(\bar{I}, S^{-1}A) \simeq \text{Tor}_i^R(\bar{I}, A) \otimes_R S^{-1}R = 0$  for all  $i \geq 1$ . Hence  $S^{-1}A$  is an  $n$ -SG-flat  $S^{-1}R$ -module.

(3)( $\implies$ ) by (2)

$\Leftarrow$  There is an exact sequence  $0 \rightarrow \bar{B} \rightarrow \bar{F}_n \rightarrow \dots \rightarrow \bar{F}_1 \rightarrow \bar{B} \rightarrow 0$  in  $S^{-1}R\text{-Mod}$  where each  $\bar{F}_i$  is flat for  $1 \leq i \leq n$ . Then  $\bar{F}_i$  is flat  $R$ -module. Let  $I$  be any injective  $R$ -module. Then  $\text{Tor}_i^R(I, \bar{B}) \simeq \text{Tor}_i^{S^{-1}R}(S^{-1}I, \bar{B}) = 0$ . So  $\bar{B}$  is an  $n$ -SG-flat  $R$ -module.

At last, we discuss the relation under rings equivalence as the following Proposition.

**Proposition 4.6** Let  $R$  and  $S$  be equivalent rings via equivalences  $\mathcal{F} : R\text{-Mod} \rightarrow S\text{-Mod}$  and  $\mathcal{G} : S\text{-Mod} \rightarrow R\text{-Mod}$ . Then

(1)  $M \in n\text{-SGP}(R)$  if and only if  $\mathcal{F}(M) \in n\text{-SGP}(S)$  for all  $M \in R\text{-Mod}$ ;

(2)  $M \in n\text{-SGI}(R)$  if and only if  $\mathcal{F}(M) \in n\text{-SGI}(S)$  for all  $M \in R\text{-Mod}$ ;

(3)  $M \in n\text{-SGF}(R)$  if and only if  $\mathcal{F}(M) \in n\text{-SGF}(S)$  for all  $M \in R\text{-Mod}$ .

**Proof** (1)( $\implies$ ) There exists an exact sequence  $0 \rightarrow M \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow M \rightarrow 0$  with  $P_i$  projective, Then  $0 \rightarrow \mathcal{F}(M) \rightarrow \mathcal{F}(P_n) \rightarrow \dots \rightarrow \mathcal{F}(P_1) \rightarrow \mathcal{F}(M) \rightarrow 0$  is exact in  $S\text{-Mod}$ , and  $\mathcal{F}(P_i)$  is projective  $S$ -module. Let  $Q$  be any projective  $S$ -module, then  $\mathcal{G}(Q)$  is a projective  $R$ -module. Since  $M$  is an  $n$ -SG-projective  $R$ -module, then we have  $\text{Ext}_R^i(M, \mathcal{G}(Q)) = 0$  for all  $i \geq 1$ . So  $\text{Ext}_S^i(\mathcal{F}(M), Q) \simeq \text{Ext}_R^i(M, \mathcal{G}(Q)) = 0$ . Therefore  $\mathcal{F}(M) \in n\text{-SGP}(S)$ .

( $\Leftarrow$ ) By  $\mathcal{G}\mathcal{F}(M) \simeq M$ .

(2) and (3) By analogy with the proof of(1).

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