The inverse problem of nonsymmetric matrices with a submatrix constraint and its approximation

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Abstract—In this paper, we first give the representation of the general solution of the following least-squares problem (LSP): Given matrices $X \in \mathbf{R}^{n \times p}, B \in \mathbf{R}^{p \times p}$ and $A_0 \in \mathbf{R}^{r \times r}$, find a matrix $A \in \mathbf{R}^{n \times n}$ such that $\|X^T A X - B\| = \min$, s. t. $A([1, r]) = A_0$, where $A([1, r])$ is the $r \times r$ leading principal submatrix of the matrix A. We then consider a best approximation problem: given an $n \times n$ matrix $\tilde{A}$ with $\tilde{A}([1, r]) = A_0$, find $\hat{A} \in \mathbf{S_E}$ such that $\|\tilde{A} - \hat{A}\| = \min_{A \in \mathbf{S_E}} \|\tilde{A} - A\|$, where $\mathbf{S_E}$ is the solution set of LSP. We show that the best approximation solution $\hat{A}$ is unique and derive an explicit formula for it.

Keywords—Inverse problem, Least-squares solution, model updating, Singular value decomposition(SVD), Optimal approximation.

I. INTRODUCTION

THROUGHOUT this paper, we denote the real $m \times n$ matrix space by $\mathbf{R}^{m \times n}$, the set of all orthogonal matrices in $\mathbf{R}^{n \times n}$ by $\mathbf{OR}^{n \times n}$, the transpose and the Moore-Penrose generalized inverse of a real matrix A by A^T and A^+, respectively. I_n represents the identity matrix of order n. For $A = [a_{ij}], B = [b_{ij}] \in \mathbf{R}^{m \times n}$, $A * B$ represents the Hadamard product of the matrices A and B, i.e., $A * B = [a_{ij} b_{ij}] \in \mathbf{R}^{m \times n}$. For $A, B \in \mathbf{R}^{m \times n}$, an inner product in $\mathbf{R}^{m \times n}$ is defined by $(A, B) = \mathrm{trace}(B^T A)$, then $\mathbf{R}^{m \times n}$ is a Hilbert space. The matrix norm $\| \cdot \|$ induced by the inner product is the Frobenius norm.

Using the finite element technique, the dynamic analysis of a structure is modelled by the generalized eigenvalue problem [3]

$$K_a x = \lambda M_a x, \qquad (1)$$

where $K_a, M_a \in \mathbf{R}^{n \times n}$ represent the analytical stiffness and mass matrices, respectively. High accuracy and large size structural applications require highly correlated finite element models to predict the system's dynamic behavior. Very often natural frequencies and mode shapes (eigenvalues and eigenvectors) of a finite element model described by (1) do not match very well with experimentally measured frequencies and mode shapes obtained from a real-life vibrating structure. Thus, a vibration engineer needs to update the theoretical finite

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element model of the structure such that the updated model predicts the observed dynamic behavior. The improved model may then be considered to be a better dynamic representation of the structure than the initial analytical model. This model can be used with greater confidence for the analysis of the structure under different boundary conditions or with physical structural changes.

Let $X \in \mathbf{R}^{n \times p}$ be the measured modal matrix, $\Lambda \in \mathbf{R}^{p \times p}$ the measured natural frequencies, where $p < n$, and Λ is diagonal. If the measured modal data X and Λ are correct, then the most common approach in finite element model updating is first to update the analytical mass or stiffness matrix to satisfy the basic orthogonality conditions

$$X^T M X = I_p, \quad X^T K X = \Lambda,$$

where $M, K \in \mathbf{R}^{n \times n}$ are to be updated mass and stiffness matrices, respectively. Some methods have been developed to improve the quality of the analytical finite element models using measured modal data [5, 6, 9, 11, 15, 16, 20]. However, the updated mass and stiffness matrices are adjusted globally. A spatial representation of the structural-element property changes that resulted from the model errors is generally preferred for engineering applications. Model errors may be localized by using sensitivity analysis [17], least-squares approach [13] and assigned partial eigenstructure [8]. Based on the localization of model errors, it is practice to adjust partial elements of the analytical mass and stiffness matrices M_a and K_a using measured modal data. On the other hand, it is well known that mode shapes of a given structure that are determined experimentally by vibration tests are usually nonorthogonal due to equipment calibration, excessive noise, misinterpretation of data, etc. Thus, the problem of updating the mass or stiffness matrix which is known as an inverse problem in structural dynamics can be mathematically formulated as follows.

Problem I. Given matrices $X \in \mathbf{R}^{n \times p}, B \in \mathbf{R}^{p \times p}$ and $A_0 \in \mathbf{R}^{r \times r}$, find a matrix $A \in \mathbf{R}^{n \times n}$ such that

$$\|X^T A X - B\| = \min, \ A([1, r]) = A_0,$$

where $A([1, r])$ is the $r \times r$ leading principal submatrix of the matrix A.

Problem II. Let $\mathbf{S_E}$ be the solution set of Problem I. Given a matrix $\tilde{A} \in \mathbf{R}^{n \times n}$ with $\tilde{A}([1, r]) = A_0$, find $\hat{A} \in \mathbf{S_E}$ such

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that

$$\|\tilde{A} - \hat{A}\| = \min_{A \in \mathbf{S_E}} \|\tilde{A} - A\|. \qquad (2)$$

There are many publications [7, 10, 18] concerning inverse problems for matrices. The inverse problems for symmetric, bisymmetric and centrosymmetric matrices with a submatrix constraint have been considered in [19], [14] and [2], respectively. In the present paper, we will discuss Problem I and II.

The paper is organized as follows. In Section 2, we give an expression of the general solution of Problem I using the generalized inverses and the singular value decompositions (SVDs) of matrices. As a by-product of our results on Problem I, we obtain a necessary and sufficient condition on X, B, A_0 for existence of $A \in \mathbf{R}^{n \times n}$ such that $X^T A X = B$, $A([1, r]) = A_0$, and a general form for all such A. In Section 3, we show that there exists a unique solution to Problem II and present the expression of the solution $\hat{A}$ of Problem II. Finally, in Section 4, a numerical algorithm to acquire the best approximation solution under the Frobenius norm sense is described and a numerical example is provided. Clearly, the results obtained are shown to include those given in [12] as particular cases.

II. THE SOLUTION OF PROBLEM I

To begin with, we introduce a lemma [4].

Lemma 1: If $E \in \mathbf{R}^{m \times l}$, $F \in \mathbf{R}^{q \times l}$ then the general solution of $\|ZE - F\| = \min$ is $Z = FE^+ + L(I_m - EE^+)$, where $L \in \mathbf{R}^{q \times m}$ is an arbitrary matrix.

Let the partition of the matrix X be

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \; X_1 \in \mathbf{R}^{r \times p}, \; X_2 \in \mathbf{R}^{(n-r) \times p}. \qquad (3)$$

Write

$$A = \begin{bmatrix} A_0 & F \\ G & H \end{bmatrix}, \qquad (4)$$

where $F \in \mathbf{R}^{r \times (n-r)}$, $G \in \mathbf{R}^{(n-r) \times r}$ and $H \in \mathbf{R}^{(n-r) \times (n-r)}$ are yet to be determined. From (3) and (4), we have

$$\|X^T A X - B\| = \\ \|X_2^T H X_2 + X_2^T G X_1 + X_1^T F X_2 - (B - X_1^T A_0 X_1)\|. \qquad (5)$$

Let the singular value decomposition of the matrix X_2 be

$$X_2 = P \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} Q^T, \qquad (6)$$

where $P = [P_1, P_2] \in \mathbf{OR}^{(n-r) \times (n-r)}, V = [Q_1, Q_2] \in \mathbf{OR}^{p \times p}, \Omega = \mathrm{diag}(\omega_1, \cdots, \omega_s), \; \omega_i > 0 \; (i = 1, \cdots, s), s = \mathrm{rank}(X_2), P_1 \in \mathbf{R}^{(n-r) \times s}, Q_1 \in \mathbf{R}^{p \times s}$, and let

$$P^T H P = \begin{bmatrix} H_{11} & H_{12} \\ H_{13} & H_{14} \end{bmatrix} \begin{array}{c} s \\ n-r-s \end{array} \; .$$
$$\begin{array}{cc} s & n-r-s \end{array}$$

Then the relation of (5) is equivalent to

$$\|X^T A X - B\|^2 \\ = \|\Omega H_{11} \Omega + \Omega P_1^T G X_1 Q_1 \\ + Q_1^T X_1^T F P_1 \Omega - Q_1^T (B - X_1^T A_0 X_1) Q_1\|^2 \\ + \|\Omega P_1^T G X_1 Q_2 - Q_1^T (B - X_1^T A_0 X_1) Q_2\|^2 \\ + \|Q_2^T X_1^T F P_1 \Omega - Q_2^T (B - X_1^T A_0 X_1) Q_1\|^2 \\ + \|Q_2^T (B - X_1^T A_0 X_1) Q_2\|^2. \qquad (7)$$

It follows from (7) that $\|X^T A X - B\| = \min$ if and only if

$$H_{11} = \Omega^{-1} [Q_1^T (B - X_1^T A_0 X_1) Q_1 - \\ \Omega P_1^T G X_1 Q_1 - Q_1^T X_1^T F P_1 \Omega] \Omega^{-1}, \qquad (8)$$

$$\|\Omega P_1^T G X_1 Q_2 - Q_1^T (B - X_1^T A_0 X_1) Q_2\| = \min \qquad (9)$$

and

$$\|Q_2^T X_1^T F P_1 \Omega - Q_2^T (B - X_1^T A_0 X_1) Q_1\| = \min. \qquad (10)$$

Assume that the singular value decomposition of the matrix $X_1 Q_2$ is

$$X_1 Q_2 = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T, \qquad (11)$$

where $U = [U_1, U_2] \in \mathbf{OR}^{r \times r}$, $V = [V_1, V_2] \in \mathbf{OR}^{(p-s) \times (p-s)}$, $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_t)$, $\sigma_i > 0 \; (i = 1, \cdots, t)$, $t = \mathrm{rank}(X_1 Q_2)$, $U_1 \in \mathbf{R}^{r \times t}$, $V_1 \in \mathbf{R}^{(p-s) \times t}$. Solving the minimization problems (9) and (10) by means of Lemma 1, we obtain

$$G = G_0 + P_2 L_1 + J_1 U_2^T, \qquad (12)$$

$$F = F_0 + U_2 L_2 + J_2 P_2^T, \qquad (13)$$

where

$$G_0 = P_1 \Omega^{-1} Q_1^T (B - X_1^T A_0 X_1) Q_2 (X_1 Q_2)^+, \\ F_0 = (Q_2^T X_1^T)^+ Q_2^T (B - X_1^T A_0 X_1) Q_1 \Omega^{-1} P_1^T, \qquad (14)$$

and $J_1 \in \mathbf{R}^{(n-r) \times (r-t)}$, $L_1 \in \mathbf{R}^{(n-r-s) \times r}$, $L_2 \in \mathbf{R}^{(r-t) \times (n-r)}$, $J_2 \in \mathbf{R}^{r \times (n-r-s)}$ are arbitrary matrices. Substituting (12) and (13) into (8) yields

$$H_{11} = H_{110} - P_1^T J_1 U_2^T X_1 Q_1 \Omega^{-1} - \Omega^{-1} Q_1^T X_1^T U_2 L_2 P_1, \qquad (15)$$

where

$$H_{110} = \Omega^{-1} Q_1^T (B - X_1^T A_0 X_1) Q_1 \Omega^{-1} \\ - P_1^T G_0 X_1 Q_1 \Omega^{-1} - \Omega^{-1} Q_1^T X_1^T F_0 P_1. \qquad (16)$$

By now, we have proved the following result.

Theorem 1: Suppose that $X \in \mathbf{R}^{n \times p}, B \in \mathbf{R}^{p \times p}$ and $A_0 \in \mathbf{R}^{r \times r}$. Let the partition of the matrix X be (3), and the SVDs of the matrices X_2 and $X_1 Q_2$ be given by (6) and (11), respectively. Then the solution set $\mathbf{S_E}$ of Problem I can be expressed as

$$\mathbf{S_E} = \left\{ A \in \mathbf{R}^{n \times n} \middle| A = \begin{bmatrix} A_0 & F \\ G & H \end{bmatrix} \right\},$$

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where

$$H = P \begin{bmatrix} H_{11} & H_{12} \\ H_{13} & H_{14} \end{bmatrix} P^T,$$

G, F, G_0, F_0, H_{11} are given by (12), (13), (14) and (15), respectively, and $L_1, L_2, J_1, J_2, H_{12}, H_{13}, H_{14}$ are arbitrary matrices.

From (7) and Theorem 1, we can easily obtain the following result [12].

Corollary 2: Under the same assumptions as in Theorem 1. Then the matrix equation

$$X^T A X = B, \ A([1, r]) = A_0 \tag{17}$$

have a solution $A \in \mathbf{R}^{n \times n}$ if and only if

$$Q_2^T (B - X_1^T A_0 X_1) Q_2 = 0, \tag{18}$$
$$Q_1^T (B - X_1^T A_0 X_1) Q_2 V_2 = 0, \tag{19}$$
$$V_2^T Q_2^T (B - X_1^T A_0 X_1) Q_1 = 0, \tag{20}$$

in which case, the general solution of the equation (17) is

$$A = \begin{bmatrix} A_0 & F_0 + U_2 L_2 + J_2 P_2^T \\ G_0 + P_2 L_1 + J_1 U_2^T & P \begin{bmatrix} H_{11} & H_{12} \\ H_{13} & H_{14} \end{bmatrix} P^T \end{bmatrix},$$

where G_0, F_0, H_{11} are given by (14) and (15), respectively, and $L_1, L_2, J_1, J_2, H_{12}, H_{13}, H_{14}$ are arbitrary matrices.

III. THE SOLUTION OF PROBLEM II

It is easy to verify that $\mathbf{S_E}$ is a closed convex subset of $\mathbf{R}^{n \times n}$. From the best approximation theorem (see [1]), we know there exists a unique solution $\hat{A}$ in $\mathbf{S_E}$ such that (2) holds.

We now focus our attention on seeking the unique solution $\hat{A}$ in $\mathbf{S_E}$. For the given matrix $\tilde{A} \in \mathbf{R}^{n \times n}$ with $\tilde{A}([1, r]) = A_0$, write

$$\tilde{A} = \begin{bmatrix} A_0 & \tilde{F} \\ \tilde{G} & \tilde{H} \end{bmatrix} \begin{matrix} r \\ n-r \end{matrix} \tag{21}$$
$$\begin{matrix} r & n-r \end{matrix}$$

and

$$P^T \tilde{H} P = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{13} & \tilde{H}_{14} \end{bmatrix}, \tag{22}$$

where $\tilde{H}_{11} \in \mathbf{R}^{s \times s}$, $\tilde{H}_{14} \in \mathbf{R}^{(n-r-s) \times (n-r-s)}$.

For any matrix $A \in \mathbf{S_E}$, by using (12), (13), (15), (21) and (22), we have

$$\begin{aligned} & \| A - \tilde{A} \|^2 \\ = \ & \| P_1^T J_1 U_2^T X_1 Q_1 \Omega^{-1} + \Omega^{-1} Q_1^T X_1^T U_2 L_2 P_1 \\ & - (H_{110} - \tilde{H}_{11}) \|^2 \\ & + \| U_2 L_2 + J_2 P_2^T - (\tilde{F} - F_0) \|^2 \\ & + \| P_2 L_1 + J_1 U_2^T - (\tilde{G} - G_0) \|^2 \\ & + \| H_{12} - \tilde{H}_{12} \|^2 + \| H_{13} - \tilde{H}_{13} \|^2 + \| H_{14} - \tilde{H}_{14} \|^2. \end{aligned} \tag{23}$$

Notice that

$$\begin{aligned} & \| U_2 L_2 + J_2 P_2^T - (\tilde{F} - F_0) \|^2 \\ = \ & \left\| U_2 L_2 + [0, J_2] \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} - (\tilde{F} - F_0) \right\|^2 \\ = \ & \| U_2 L_2 P_1 - (\tilde{F} - F_0) P_1 \|^2 \\ & + \| U_2 L_2 P_2 + J_2 - (\tilde{F} - F_0) P_2 \|^2. \end{aligned} \tag{24}$$

Likewise,

$$\begin{aligned} & \| P_2 L_1 + J_1 U_2^T - (\tilde{G} - G_0) \|^2 \\ = \ & \| P_1^T J_1 U_2^T - P_1^T (\tilde{G} - G_0) \|^2 \\ & + \| L_1 + P_2^T J_1 U_2^T - P_2^T (\tilde{G} - G_0) \|^2. \end{aligned} \tag{25}$$

It follows from (23), (24) and (25) that $\| \tilde{A} - A \| = \min$ if and only if

$$H_{12} = \tilde{H}_{12}, \ H_{13} = \tilde{H}_{13}, \ H_{14} = \tilde{H}_{14}, \tag{26}$$

$$J_2 = (\tilde{F} - F_0) P_2 - U_2 L_2 P_2, \ L_1 = P_2^T (\tilde{G} - G_0) - P_2^T J_1 U_2^T, \tag{27}$$

and

$$\begin{aligned} f(J_1, L_2) = \ & \| U_2 L_2 P_1 - \tilde{C} \|^2 + \| P_1^T J_1 U_2^T - \tilde{D} \|^2 \\ & + \| P_1^T J_1 S + S^T L_2 P_1 - \tilde{E} \|^2 = \min, \end{aligned} \tag{28}$$

where

$$\begin{aligned} \tilde{C} = (\tilde{F} - F_0) P_1, \ & \tilde{D} = P_1^T (\tilde{G} - G_0), \\ \tilde{E} = H_{110} - \tilde{H}_{11}, \ & S = U_2^T X_1 Q_1 \Omega^{-1}. \end{aligned} \tag{29}$$

From (28) we have

$$\begin{aligned} f(J_1, L_2) = \ & \text{trace}(P_1^T L_2^T L_2 P_1) - 2\text{trace}(P_1^T L_2^T U_2^T \tilde{C}) \\ & + \text{trace}(\tilde{C}^T \tilde{C}) + \text{trace}(J_1^T P_1 P_1^T J_1) \\ & - 2\text{trace}(U_2 J_1^T P_1 \tilde{D}) + \text{trace}(\tilde{D}^T \tilde{D}) \\ & + \text{trace}(S^T J_1^T P_1 P_1^T J_1 S) + \text{trace}(P_1^T L_2^T S S^T L_2 P_1) \\ & + \text{trace}(\tilde{E}^T \tilde{E}) + 2\text{trace}(S^T J_1^T P_1 S^T L_2 P_1) \\ & - 2\text{trace}(P_1^T L_2^T S \tilde{E}) - 2\text{trace}(S^T J_1^T P_1 \tilde{E}). \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{\partial f(J_1, L_2)}{\partial J_1} = \ & 2 P_1 P_1^T J_1 + 2 P_1 P_1^T J_1 S S^T \\ & + 2 P_1 S^T L_2 P_1 S^T - 2 P_1 \tilde{E} S^T - 2 P_1 \tilde{D} U_2, \end{aligned}$$

$$\begin{aligned} \frac{\partial f(J_1, L_2)}{\partial L_2} = \ & 2 L_2 P_1 P_1^T + 2 S P_1^T J_1 S P_1^T \\ & + 2 S S^T L_2 P_1 P_1^T - 2 U_2^T \tilde{C} P_1^T - 2 S \tilde{E} P_1^T. \end{aligned}$$

Setting $\frac{\partial f(J_1, L_2)}{\partial J_1} = 0$ and $\frac{\partial f(J_1, L_2)}{\partial L_2} = 0$, we obtain

$$P_1^T J_1 + P_1^T J_1 S S^T + S^T L_2 P_1 S^T = \tilde{E} S^T + \tilde{D} U_2, \tag{30}$$

$$L_2 P_1 + S P_1^T J_1 S + S S^T L_2 P_1 = U_2^T \tilde{C} + S \tilde{E}. \tag{31}$$

Let the singular value decomposition of the matrix S be

$$S = T \begin{bmatrix} \Gamma & 0 \\ 0 & 0 \end{bmatrix} W^T, \tag{32}$$

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where $T = [T_1, T_2] \in \mathbf{OR}^{(r-t)\times(r-t)}$, $W = [W_1, W_2] \in \mathbf{OR}^{s\times s}$, $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_f)$, $\gamma_i > 0$ $(i = 1, \cdots, f)$, $f = \mathrm{rank}(S)$, $T_1 \in \mathbf{R}^{(r-t)\times f}$, $W_1 \in \mathbf{R}^{s\times f}$. Put

$$W^T P_1^T J_1 T = \left[\begin{array}{cc} J_{11} & J_{12} \\ J_{13} & J_{14} \end{array} \right] \begin{array}{c} f \\ s-f \end{array}, \quad (33)$$
$$\begin{array}{cc} f & r-t-f \end{array}$$

$$T^T L_2 P_1 W = \left[\begin{array}{cc} L_{21} & L_{22} \\ L_{23} & L_{24} \end{array} \right] \begin{array}{c} f \\ r-t-f \end{array}. \quad (34)$$
$$\begin{array}{cc} f & s-f \end{array}$$

After some algebraic manipulations, the equation of (30) is equivalent to

$$J_{11} + J_{11}\Gamma^2 + \Gamma L_{21}\Gamma = W_1^T(\tilde{D}U_2 + \tilde{E}S^T)T_1, \quad (35)$$
$$J_{13} + J_{13}\Gamma^2 = W_2^T(\tilde{D}U_2 + \tilde{E}S^T)T_1, \quad (36)$$
$$J_{12} = W_1^T(\tilde{D}U_2 + \tilde{E}S^T)T_2, \quad (37)$$
$$J_{14} = W_2^T(\tilde{D}U_2 + \tilde{E}S^T)T_2, \quad (38)$$

and the equation of (31) is equivalent to

$$L_{21} + \Gamma^2 L_{21} + \Gamma J_{11}\Gamma = T_1^T(U_2^T\tilde{C} + S\tilde{E})W_1, \quad (39)$$
$$L_{22} + \Gamma^2 L_{22} = T_1^T(U_2^T\tilde{C} + S\tilde{E})W_2, \quad (40)$$
$$L_{23} = T_2^T(U_2^T\tilde{C} + S\tilde{E})W_1, \quad (41)$$
$$L_{24} = T_2^T(U_2^T\tilde{C} + S\tilde{E})W_2. \quad (42)$$

Let $J_{11} = [j_{ik}] \in \mathbf{R}^{f\times f}$, $L_{21} = [l_{ik}] \in \mathbf{R}^{f\times f}$, $W_1^T(\tilde{D}U_2 + \tilde{E}S^T)T_1 = [d_{ik}] \in \mathbf{R}^{f\times f}$ and $T_1^T(U_2^T\tilde{C} + S\tilde{E})W_1 = [c_{ik}] \in \mathbf{R}^{f\times f}$. From (35) and (39) we have

$$j_{ik} + j_{ik}\gamma_k^2 + \gamma_i l_{ik}\gamma_k = d_{ik}, \quad i, k = 1, \cdots, f,$$
$$l_{ik} + \gamma_i^2 l_{ik} + \gamma_i j_{ik}\gamma_k = c_{ik}, \quad i, k = 1, \cdots, f.$$

Solving these linear equations with respect to $l_{ik}, j_{ik}(i, k = 1, \cdots, f)$, we obtain

$$l_{ik} = \frac{1}{1 + \gamma_i^2 + \gamma_k^2}(c_{ik}(1+\gamma_k^2) - \gamma_i d_{ik}\gamma_k), \text{ for } i, k = 1, \cdots, f, \quad (43)$$

$$j_{ik} = \frac{1}{1 + \gamma_i^2 + \gamma_k^2}((1+\gamma_i^2)d_{ik} - \gamma_i c_{ik}\gamma_k), \text{ for } i, k = 1, \cdots, f, \quad (44)$$

Let $\Phi = [\frac{1}{1+\gamma_i^2+\gamma_k^2}] \in \mathbf{R}^{f\times f}$, then (43) and (44) may be expressed as

$$L_{21} = \Phi * [T_1^T(U_2^T\tilde{C} + S\tilde{E})W_1(I_f + \Gamma^2) - \Gamma W_1^T(\tilde{D}U_2 + \tilde{E}S^T)T_1\Gamma], \quad (45)$$

$$J_{11} = \Phi * [((I_f + \Gamma^2)W_1^T(\tilde{D}U_2 + \tilde{E}S^T)T_1 - \Gamma T_1^T(U_2^T\tilde{C} + S\tilde{E})W_1\Gamma]. \quad (46)$$

From (36) and (40), we have

$$J_{13} = W_2^T(\tilde{D}U_2 + \tilde{E}S^T)T_1(I_f + \Gamma^2)^{-1}, \quad (47)$$

$$L_{22} = (I_f + \Gamma^2)^{-1}T_1^T(U_2^T\tilde{C} + S\tilde{E})W_2. \quad (48)$$

Thus, from (33) and (34) we have

$$P_1^T J_1 = W \left[\begin{array}{cc} J_{11} & J_{12} \\ J_{13} & J_{14} \end{array} \right] T^T, \quad (49)$$

$$L_2 P_1 = T \left[\begin{array}{cc} L_{21} & L_{22} \\ L_{23} & L_{24} \end{array} \right] W^T, \quad (50)$$

where J_{11}, J_{12}, J_{13} and J_{14} are given by (46), (37), (47) and (38), respectively, and L_{21}, L_{22}, L_{23} and L_{24} are given by (45), (48), (41) and (42), respectively.
Inserting J_2, L_1 in (27) into (12) and (13), we obtain

$$F = F_0 P_1 P_1^T + \tilde{F}P_2 P_2^T + U_2 L_2 P_1 P_1^T,$$
$$G = P_1 P_1^T G_0 + P_2 P_2^T \tilde{G} + P_1 P_1^T J_1 U_2^T.$$

Summing up above discussion, we have proved the following result.

Theorem 3: For the given matrix $\tilde{A} \in \mathbf{R}^{n\times n}$ with $\tilde{A}([1, r]) = A_0$, then the matrix best approximation problem (2) has a unique solution $\hat{A} \in \mathbf{S_E}$. Furthermore, let the partition of $\tilde{A}$ be (21) and $P^T\tilde{H}P$ be (22). Then the unique solution of Problem II can be expressed as

$$\hat{A} = \left[\begin{array}{cc} A_0 & \hat{F} \\ \hat{G} & \hat{H} \end{array} \right], \quad (51)$$

where

$$\hat{F} = F_0 P_1 P_1^T + \tilde{F}P_2 P_2^T + U_2 L_2 P_1 P_1^T,$$
$$\hat{G} = P_1 P_1^T G_0 + P_2 P_2^T \tilde{G} + P_1 P_1^T J_1 U_2^T,$$
$$\hat{H} = P \left[\begin{array}{cc} H_{11} & \tilde{H}_{12} \\ \tilde{H}_{13} & \tilde{H}_{14} \end{array} \right] P^T,$$

$P_1^T J_1$, $L_2 P_1$ and H_{11} are given by (49), (50) and (15), respectively.

IV. A NUMERICAL EXAMPLE

According to the previous discussion, we now give a numerical algorithm for solving Problem I and Problem II as follows.

Algorithm 1.
1) Input matrices X, B, A_0, and $\tilde{A}$;
2) Form the matrix X_1, X_2 according to (3);
3) Compute the SVD (6) of the matrix X_2 and then compute the SVD (11) of $X_1 Q_2$;
4) Compute G_0, F_0 and H_{110} by (14) and (16), respectively;
5) Partition matrix $\tilde{A}$ as in (21) to get $\tilde{F}$, $\tilde{G}$, $\tilde{H}$;
6) Compute $\tilde{H}_{1j}$, $j = 1, 2, 3, 4$ by (22);
7) Compute the matrices $\tilde{C}, \tilde{D}, \tilde{E}$ and S by (29);
8) Compute the SVD (32) of the matrix S;
9) Compute J_{11}, J_{12}, J_{13} and J_{14} are given by (46), (37), (47) and (38), respectively;

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10) Compute L_{21}, L_{22}, L_{23} and L_{24} are given by (45), (48), (41) and (42), respectively;

11) Compute $P_1^T J_1$, $L_2 P_1$ by (49) and (50), and then compute H_{11} by (15);

12) Compute the unique solution $\hat{A}$ of Problem II according to (51).

Example 1. (An example for updating the mass matrix of a vibrating system described in (1)).

Let A_0, X, B and $\tilde{A}$ be given by

$$A_0 = \begin{bmatrix} 0.3333 & 0.1667 & 0 & 0 \\ 0.1667 & 0.6667 & 0.1667 & 0 \\ 0 & 0.1667 & 0.6667 & 0.1667 \\ 0 & 0 & 0.1667 & 0.6667 \end{bmatrix},$$

$$X = \begin{bmatrix} 39.230 & 347.67 & -45.500 \\ -22.312 & -289.62 & 54.211 \\ 68.151 & 640.05 & -90.167 \\ -61.888 & -640.58 & 100.18 \\ 206.84 & 491.56 & 141.90 \\ -427.7 & -1474.9 & -161.45 \\ 315.58 & 1375.1 & 36.539 \\ -184.24 & -993.56 & 33.587 \end{bmatrix},$$

$B = I_3$ and

$$\tilde{A} = \begin{bmatrix} 0.3333 & 0.1667 & 0 & 0 \\ 0.1667 & 0.6667 & 0.1667 & 0 \\ 0 & 0.1667 & 0.6667 & 0.1667 \\ 0 & 0 & 0.1667 & 0.6667 \\ 0.1000 & 0.4500 & 0.1800 & 0.4000 \\ 0.3500 & 0.7000 & 0.2100 & 0.3000 \\ 0.6000 & 0.8000 & 0.3900 & 0.2000 \\ 0.2000 & 0.2800 & 0.4700 & 0.1000 \end{bmatrix}$$

$$\begin{bmatrix} 0.1000 & 0.3500 & 0.6000 & 0.2000 \\ 0.4500 & 0.7000 & 0.8000 & 0.2800 \\ 0.1800 & 0.2100 & 0.3900 & 0.4700 \\ 0.4000 & 0.3000 & 0.2000 & 0.1000 \\ 0.1200 & 0.4800 & 0.4680 & 0.3600 \\ 0.4800 & 0.8400 & 0.6060 & 0.3480 \\ 0.4680 & 0.6060 & 0.4680 & 0.4020 \\ 0.3600 & 0.3480 & 0.4020 & 0.1200 \end{bmatrix},$$

According to algorithm 1 we obtain the unique solution of Problem II as follows.

$$\hat{A} = \begin{bmatrix} 0.3333 & 0.1667 & 0 & 0 \\ 0.1667 & 0.6667 & 0.1667 & 0 \\ 0 & 0.1667 & 0.6667 & 0.1667 \\ 0 & 0 & 0.1667 & 0.6667 \\ 0.1224 & 0.3938 & 0.2359 & 0.3212 \\ 0.3447 & 0.7322 & 0.1893 & 0.3376 \\ 0.5783 & 0.8287 & 0.3459 & 0.2507 \\ 0.2297 & 0.2283 & 0.5353 & 0.0182 \end{bmatrix}$$

$$\begin{bmatrix} 0.0498 & 0.4734 & 0.4967 & 0.2684 \\ 0.3744 & 0.7667 & 0.8069 & 0.2387 \\ 0.3182 & 0.0434 & 0.4384 & 0.4913 \\ 0.3728 & 0.2461 & 0.3087 & -0.0093 \\ 0.0684 & 0.5068 & 0.4982 & 0.3091 \\ 0.5068 & 0.8209 & 0.5974 & 0.3681 \\ 0.4982 & 0.5974 & 0.4406 & 0.4404 \\ 0.3091 & 0.3681 & 0.4404 & 0.0622 \end{bmatrix},$$

Although we do not need to verify the consistency conditions (18)-(20), we note that the conditions (18)-(20) do hold for this example. Furthermore, we can figure out

$$\|X^T \hat{A} X - B\| = 7.2881e - 010, \quad \|\tilde{A} - \hat{A}\| = 0.4216.$$

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