Solution of The KdV Equation With Asymptotic Degeneracy

Tapas Kumar Sinha, Joseph Mathew

Abstract—**Recently T. C. Au-Yeung, C.Au, and P. C. W. Fung [2] have given the solution of the KdV equation [1] to the boundary condition $u \to b$ as $x \to \pm\infty$, where b is a constant. We have further extended the method of [2] to find the solution of the KdV equation with asymptotic degeneracy. Via simulations we find both bright and dark Solitons (i.e. Solitons with opposite phases).**

Keywords—KdV equation, Asymptotic Degeneracy, Solitons, Inverse Scattering

I. INTRODUCTION

C.S GARDNER, J.M Greene, M. D. Kruskal and R. M. Muira [1] have obtained the solution of the KdV equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1}$$

with the boundary conditions $u \to 0 \, as \, x \to \pm\infty$. Using Lax operator formalism [6] and inverse scattering [19], [20], T. C. Au-Yeung, C.Au, and P. C. W. Fung [2] have extended the solution of the KdV equation to the boundary condition $u \to b$ as $x \to \pm\infty$, where b is a constant. The question naturally arises whether a solution to the KdV equation exists for asymptotically degenerate states $u \to \pm f(x - vt)$ as $x \to \pm\infty$, where $f(x - vt)$ is a soliton. state. Note that the $\pm$ sign implies asymptotic degeneracy. W. P. Su, J. R. Schrieffer and A Heeger (SSH) in their classic paper [3] showed that the asymptotic states $\pm \tanh(x - vt)$ of the phi4 equation correspond to the two degenerate of Polyacetylene. Further Polyacetylene polymer has been approximated by a random field ising model (RFIM) [16-18] which in the continuum limit gives the KdV equation [19]. Thus motivated by the work of [3] we look for the solutions to the KdV equation with asymptotic degeneracy.

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II. LAX OPERATORS

Define the operators $i\frac{\partial L(t)}{\partial t} = [B(t), L(t)]$

$$L(t) = -\frac{\partial^2}{\partial x^2} + V(x,t) = -\frac{\partial^2}{\partial x^2} + \sec h^2(kx) \tag{2}$$

$$B(t) = -4i\frac{\partial^3}{\partial x^3} - 6i\left[u(x,t)\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u(x,t)\right] \tag{3}$$

Here $L(t)$ and $B(t)$ constitute the Lax pair [6] for the KdV equation. Note that for the potential in (2) we have used $\sec h^2(kx)$ which is a solution of the KdV equation [4]. Further one defines the evolution equations

$$i\frac{\partial U(t)}{\partial t} = B(t)U(t), U(0) = I \tag{4}$$

which leads to

$$i\frac{\partial L(t)}{\partial t} = [B(t), L(t)] \tag{5}$$

We now look for asymptotic solutions of (2).

III. ASYMPTOTIC SOLUTIONS

Non Linear equations such as KdV, Sine-Gordon, Phi4, and Navier Stokes can be put in the form of a conservation equation

$$T_t + U_x = 0 \tag{6}$$

For KdV equation the spatial component of the conservation law (i.e. U) is the Schrodinger equation. For the other equations some variable transformations are required to obtain the conservation law. The solution of this Schrodinger equation determines the time evolution of both the continuum and bound states in the asymptotic limit.

To fix our ideas we consider a stationary soliton located at $x = x_0$. For $x \gg x_0$ we have

$$\phi(k,t) \approx \begin{cases} a_+(k,t)e^{ikx} + \\ \quad + a_-(k,t)e^{-ikx} \text{ as } x \to +\infty \\ b_+(k,t)e^{ikx} + \\ \quad + b_-(k,t)e^{-ikx} \text{ as } x \to -\infty \end{cases} \tag{7}$$

The time evolution of $\phi(k,t)$ is given by

$$i\frac{\partial \phi(k,t)}{\partial t} = B(t)\phi(k,t) \tag{8}$$

The asymptotic form of $B(t)$, defined in (3), is

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$$B(t) = -4i\frac{\partial^3}{\partial x^3} - 6i\sec h^2(kx)\frac{\partial}{\partial x} \qquad (9)$$

Using (7) and (9) in (8) we obtain

$$
\begin{aligned}
a_+(k,t) &= a_+(k,0)e^{i(4k^3-6k\sec h^2(kx))t} \\
a_-(k,t) &= a_-(k,0)e^{-i(4k^3-6k\sec h^2(kx))t} \\
b_+(k,t) &= b_+(k,0)e^{i(4k^3-6k\sec h^2(kx))t} \\
b_-(k,t) &= b_-(k,0)e^{-i(4k^3-6k\sec h^2(kx))t}
\end{aligned}
\qquad (10)
$$

IV. BOUND STATES OF THE SCHRODINGER EQUATION

We wish to find bound states of the operator $L(t)$, defined in equation (2). Consider the equation

$$\left[-\frac{\partial^2}{\partial x^2} + \sec h^2(kx)\right]\psi(k,x) = -k^2\psi(k,x) \qquad (11)$$

As $x \to \pm\infty$, the potential $\sec h^2(kx)$. In the limit $x \to \pm\infty$, the solution must satisfy

$$\psi(k,x) \to e^{\pm ikx} \text{ as } x \to \pm\infty \qquad (12)$$

The wave vector K_n of the bound states satisfy [3]

$$k = -K_n \qquad (13)$$

Substituting (13) in (11) one obtains the equation for the bound state

$$\left[-\frac{\partial^2}{\partial x^2} + \sec h^2(kx)\right]\psi_n = -K_n^2\psi_n \qquad (14)$$

where ψ_n satisfies the following boundary conditions

$$\psi_n \approx \left\{ \begin{array}{l} R_n(0)e^{-K_nx} \text{ as } x \to \infty \\ T_n(0)e^{K_nx} \text{ as } x \to -\infty \end{array} \right\} \qquad (15)$$

Here both $R_n(0), T_n(0)$ are normalization constants. Since we know the time evolution (10) we can write

$$\psi_n(x,t) = \psi_n(x,0)e^{-(4K_n^3+6K_n\sec h^2(x))t} \qquad (16)$$

Note the replacement of k by iK_n in (10) to obtain (16). It now remains to normalize (16). One can show that the normalization constant is

$$M_n(t) = e^{(8K_n^3+12K_n\sec h^2(x))t}M_n(0) \qquad (17)$$

V. GELFAND-LEVITAN EQUATION

In the inverse scattering method the Gelfand-Levitan equation [5] is used to determine the scattering potential $V(x,t)$ in (2) for all x and t. The scattering potential $V(x,t)$ satisfies

$$V(x,t) = -2\frac{dg(x,x)}{dx} \qquad (18)$$

where $g(x,y)$ for $x < y$ is the solution of the Gelfand-Levitan equation [5]

$$g(x,y) + K(x+y) + \int_x^\infty K(y+y')g(x,y')dy' = 0 \qquad (19)$$

with

$$K(y) = \frac{1}{2\pi}\int_{-\infty}^\infty R(k,t)e^{iky}dk + \sum_{n=1}^N M_ne^{-K_ny} \qquad (20)$$

where M_n are the normalization constants for the bound states of the of the operator defined in (2). $-K_n^2, n = 1, 2$ are the bound state energies of the operator (2). $R(k,t)$ is the reflection coefficient. Note that the very definition of $g(x,y)$ includes causality into the equation (19). In (19) x represents the source coordinates and y' the "effect" coordinates. To solve (19) we consider a single bound state of energy $-K^2$ and $R(k,0) = 0$. The latter condition implies that the potential is reflection-less. Using (17) in (19) we obtain

$$
\begin{aligned}
&g(x,y,t) + e^{(8K_n^3+12K_n\sec h^2(x))t}M(0)e^{-K(x+y)} + \\
&\quad + \int_x^\infty e^{(8K_n^3+12K_n\sec h^2(x))t}M(0) \cdot \\
&\qquad \cdot e^{-K(x+y)}g(x,y',t)dy' = 0
\end{aligned}
\qquad (21)
$$

Since we know that the bound state function has an exponential decay we can write

$$g(x,y,t) = e^{-Ky}h(x,t) \qquad (22)$$

we obtain

$$
\begin{aligned}
g(x,y,t) = &-M(0) \cdot \\
&\cdot \frac{e^{(8K^3+12K\sec h^2(x))t}e^{-K(x+y)}}{1 + \frac{M(0)}{2K}e^{(8K^3+12K\sec h^2(x))t}e^{-2Kx}}
\end{aligned}
\qquad (23)
$$

Now in the inverse scattering framework the scattering potential $V(x,t)$ is the solution of the equation (i.e. KdV equation). Accordingly, taking the derivative of (23) we obtain the solution

$$
\begin{aligned}
u(x,t) = &K^2 \sec h^2 \cdot \\
&\cdot \left[K\left(x - (4K^2 + 6\sec h^2(x))t - \frac{\delta}{K}\right)\right]
\end{aligned}
\qquad (24)
$$

$$\text{where} \quad \delta = \frac{1}{2}\ln\frac{M(0)}{2K} \qquad (25)$$

VI. RESULTS

Our simulation results are summarized in figs 1-3. Figure 1 shows the general soliton profile. Figure 2 shows the Soliton profile for a fixed K (Bound state energy) and varying delta (phase). There appears to be a sudden increase in both amplitude and asymptotic value at certain value of delta. This is indicative of a phase transition and will be investigated later. For a fixed delta, change in K has a profound effect
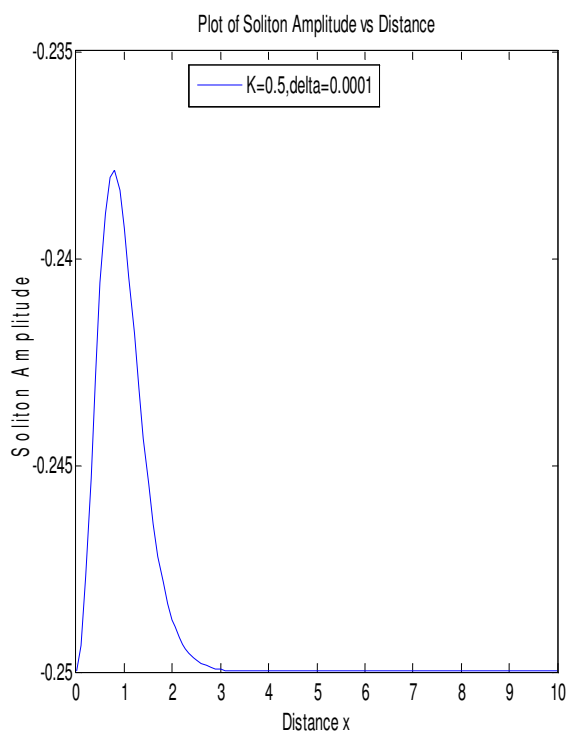
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Fig. 1.   Plot of Soliton Amplitude vs Distance
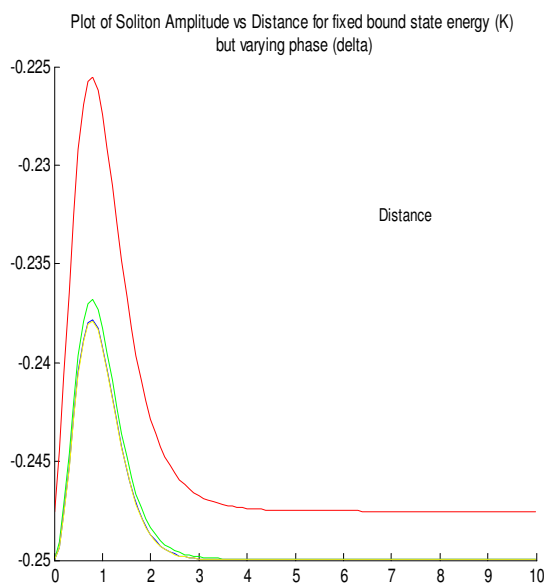
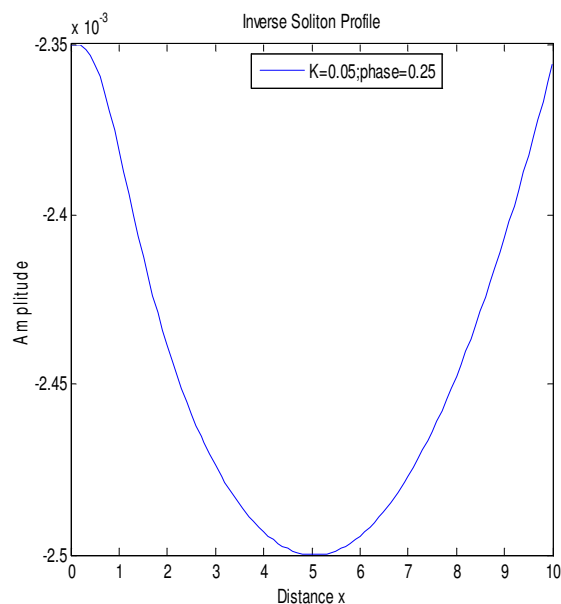Fig. 2.   Plot of Soliton Profile for Fixed K Varying Delta

Fig. 3.   Plot of Soliton Profile for Fixed Delta Varying K

VII. CONCLUSION

Systems with asymptotic degeneracy form an important class of problems which have diverse applications e.g. Polyacetylene polymers, Persistence problems [7]–[12] in the case of ising spins, effect of past DNA interactions on present and future DNA expression [13]–[15], astrophysics (influence of past Galactic events on present and future). We have used the inverse scattering framework to solve this problem. Via simulation we find that while the phase factor (defined in (25)) has very little effect on the Soliton shape, the bound state energy parameter K has a profound effect on the Soliton profile. For fixed K, change in phase, after a certain value, results in a sudden increase in amplitude and asymptotic value. On the other hand, for a fixed phase change in K, results in inversion of the Soliton amplitude.

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on Soliton profile (Fig. 3). Note the inverted amplitude of the Soliton as well as its increased width. Thus depending on the relative values of K and delta both bright and dark Solitons are possible. These will correspond to the different conformations of Polyacetylene (trans, cis).

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