Introduction to Magnetohydrodynamics

Nick Murphy

Harvard-Smithsonian Center for Astrophysics

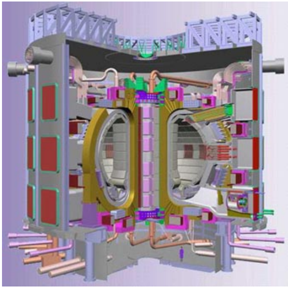
To MHD and beyond!

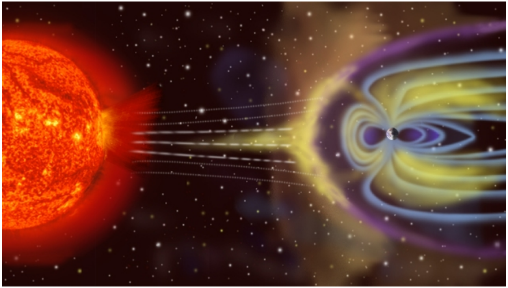
- What is MHD?
- The equations of MHD and their physical meaning
- Waves in MHD
  - Alfvén waves
  - Slow magnetosonic waves
  - Fast magnetosonic waves
- Beyond MHD
  - Extensions to MHD
  - Plasma kinetic theory
- Magnetic reconnection
- Final thoughts

What is MHD?

- **Fluid dynamics** studies how fluids behave in response to forces
  - How do rivers flow?
  - How do we breathe?
- **Electromagnetism** studies the effects of physical interactions between charged particles
  - What forces are exerted on free protons and electrons?
  - How does light work?
- **Magnetohydrodynamics** couples Maxwell's equations of electromagnetism with fluid dynamics to describe the large-scale behavior of conducting fluids such as plasmas
  - How does plasma behave in the solar atmosphere and wind?
  - How can we use magnetic fields to confine plasma?

MHD is important in solar physics, astrophysics, space plasma physics, and in laboratory plasma experiments





Left: The International Thermonuclear Experimental Reactor (a tokamak currently under construction in France)

Right: The solar wind interacting with Earth's magnetosphere

MHD at a glance (SI units)

Continuity Equation	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$
Momentum Equation	$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla p$
Ampere's law	$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$
Faraday's law	$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$
Ideal Ohm's law	$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$
Divergence constraint	$\nabla \cdot \mathbf{B} = 0$
Adiabatic Energy Equation	$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p}{\rho^\gamma} \right) = 0$

Definitions: **B**, magnetic field; **V**, plasma velocity; **J**, current density; **E**, electric field; ρ, mass density; p, plasma pressure; γ, ratio of specific heats (usually 5/3); t, time.

The MHD approximation

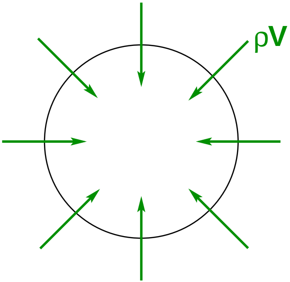
- Assume the plasma behaves like a fluid
  - Macroscopic behavior (long timescales, large distances)
  - Maxwellian particle distributions
- Ignore the most significant physics advances since 1860:
  - Relativity ($v^2 \ll c^2$)
  - Quantum mechanics
  - Displacement current in Ampere's law
- Assume the plasma is fully ionized
  - Limited applicability to weakly ionized plasmas like the photosphere and chromosphere
- Ignore resistivity, viscosity, thermal conduction, and radiative cooling in ideal MHD

Vector calculus refresher[1]

- The **gradient** of f (denoted by ∇f) is a vector pointing in the direction of the steepest slope of f. The magnitude of the gradient vector is the steepness of the slope.
- The **divergence** of $\mathbf{F}$ (denoted by $\nabla \cdot \mathbf{F}$) is the extent to which there is more of a quantity exiting a small region in space than entering it.
- The **curl** of $\mathbf{F}$ (denoted by $\nabla \times \mathbf{F}$) represents the swirliness of a vector field.

[1] Adapted partially from Wikipedia

The continuity equation describes conservation of mass



- The **continuity equation** written in conservative form is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

- The partial derivative $\partial \rho / \partial t$ refers to the change in density at a single point in space
- The divergence of the mass flux $\nabla \cdot (\rho \mathbf{V})$ says how much plasma goes in and out of the region
- Put sources and sinks of mass on right hand side

The second golden rule of astrophysics

"The density of wombats

times the velocity of wombats

gives the flux of wombats."

The momentum equation is analogous to $ma = \mathbf{F}$

- The **momentum equation** is

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \mathbf{J} \times \mathbf{B} - \nabla p$$

  Additional forces like gravity go on the right hand side.[2]

- The **total derivative** represents how much a quantity is changing as you follow a parcel of plasma:

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$$

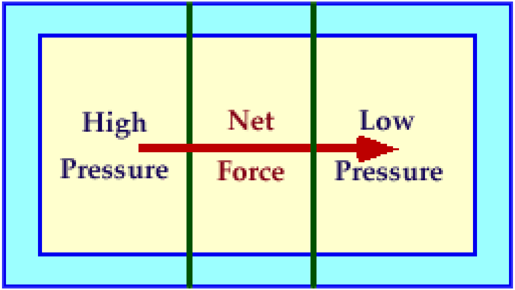
- Forces must cancel each other out in a static equilibrium:

$$\mathbf{J} \times \mathbf{B} = \nabla p$$

  When $\mathbf{J} \times \mathbf{B} = 0$, the plasma is **force-free**.

[2] If you neglect gravity, it may be your downfall! (I had to drop at least one pun in.)

The pressure gradient force $-\nabla p$ pushes plasma from regions of high pressure to low plasma pressure



The Lorentz force term includes two components

- The current density is given by the relative drift between ions and electrons:

$$\mathbf{J} = ne\left(\mathbf{V}_i - \mathbf{V}_e\right)$$

  $\rightarrow \mathbf{J} \times \mathbf{B}$ is analogous to $\mathbf{F} = q\mathbf{V} \times \mathbf{B}$.

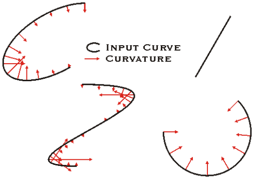
- Using vector identities and Ampere's law ($\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$), we rewrite the Lorentz force term $\mathbf{J} \times \mathbf{B}$ as:

$$\mathbf{J} \times \mathbf{B} = \frac{(\mathbf{B} \cdot \nabla)\,\mathbf{B}}{\mu_0} - \nabla\left(\frac{B^2}{2\mu_0}\right)$$

  However: the Lorentz force is orthogonal to $\mathbf{B}$, but these two terms are not.

The Lorentz force can be decomposed into two terms with forces othogonal to $\mathbf{B}$ using field line curvature

- The curvature vector $\boldsymbol{\kappa}$ points toward the center of curvature and gives the rate at which the tangent vector turns:

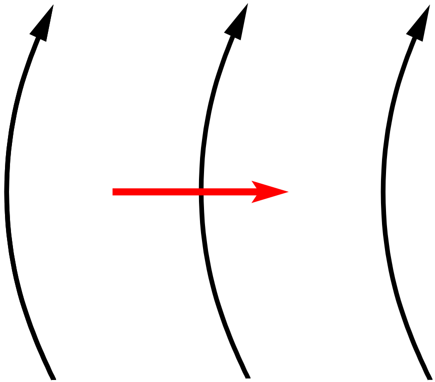


- We can then write the Lorentz force as

$$\underbrace{\mathbf{J} \times \mathbf{B}}_{\text{Lorentz force}} = \underbrace{\boldsymbol{\kappa}\frac{B^2}{\mu_0}}_{\text{magnetic tension}} - \underbrace{\nabla_\perp \left(\frac{B^2}{2\mu_0}\right)}_{\text{magnetic pressure}} \quad (1)$$

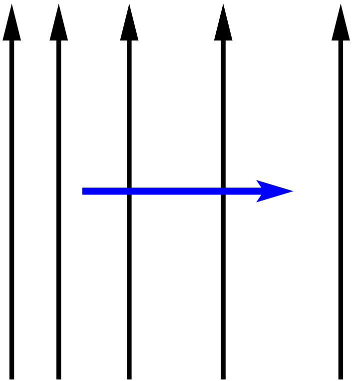
where all terms are orthogonal to $\mathbf{B}$. The operator $\nabla_\perp$ takes the gradient only in the direction orthogonal to $\mathbf{B}$.

The magnetic tension force wants to straighten magnetic field lines



- The tension force is directed radially inward with respect to magnetic field line curvature

Regions of high magnetic pressure exert a force towards regions of low magnetic pressure



- The magnetic pressure is given by $p_B \equiv \frac{B^2}{2\mu_0}$

The ratio of the plasma pressure to the magnetic pressure is an important dimensionless number

- Define plasma β as

$$\beta \equiv \frac{\text{plasma pressure}}{\text{magnetic pressure}} \equiv \frac{p}{B^2/2\mu_0}$$

- If $\beta \ll 1$ then the magnetic field dominates
  - Solar corona
- If $\beta \gg 1$ then plasma pressure forces dominate
  - Solar interior
- If $\beta \sim 1$ then pressure/magnetic forces are both important
  - Solar chromosphere
  - Parts of the solar wind and interstellar medium
  - Some laboratory plasma experiments

Faraday's law tells us how the magnetic field varies with time

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

But how do we get the electric field?

Ohm's law provides the electric field

- The ideal MHD Ohm's law is given by

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$$

- In ideal MHD, the magnetic field is *frozen-in* to the plasma. If two parcels of plasma are connected by a magnetic field line at one time, then they will be connected by a magnetic field line at all other times.
- For resistive MHD, Ohm's law becomes

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$$

  where η is the resistivity. Resistivity allows the frozen-in condition to be broken.
- Can also include the Hall effect which is important on short length scales.

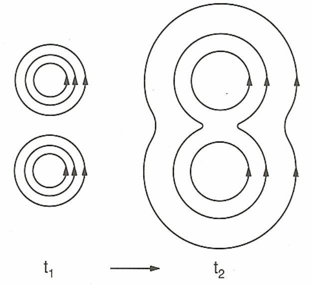
With Ohm's law we can rewrite Faraday's law as the induction equation

- Using the resistive Ohm's law:

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{\nabla \times (\mathbf{V} \times \mathbf{B})}_{\text{convection}} + \underbrace{\frac{\eta}{\mu_0} \nabla^2 \mathbf{B}}_{\text{diffusion}}$$

  Diffusion is usually represented by a second order spatial derivative.

- An example of resistive diffusion:



Thermal conduction is a common extension to MHD

- Heat diffuses much more quickly along magnetic field lines than perpendicular to them
  - Makes it more difficult to simulate plasmas
- The temperature along magnetic field lines is usually approximately constant
  - Exceptions: rapid heating events, rapid magnetic connectivity changes

Waves

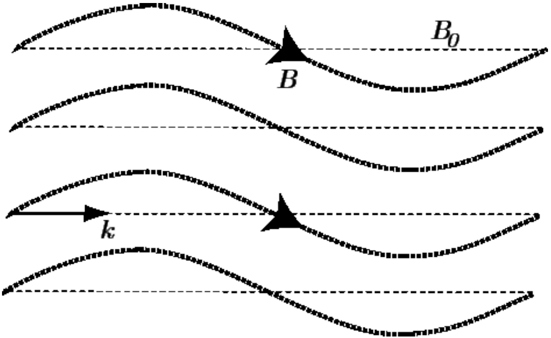
- There are three primary waves that arise from MHD:
  - Alfvén wave
  - Slow magnetosonic wave
  - Fast magnetosonic wave
- There are two important speeds
  - The sound speed is given by

$$V_S \equiv \sqrt{\frac{\gamma p}{\rho}}$$

  - The Alfvén speed is given by

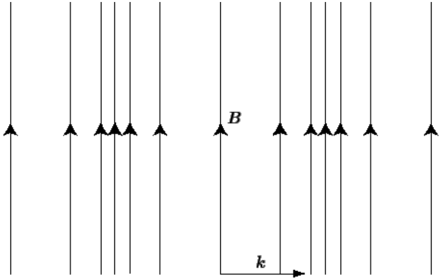
$$V_A \equiv \frac{B}{\sqrt{\mu_0 \rho}}$$

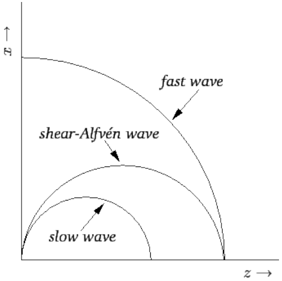
Alfvén Waves



- Alfvén waves propagate at the Alfvén speed: $V_A \equiv \frac{B}{\sqrt{\mu_0 \rho}}$
- The restoring force is magnetic tension
- This is a shear wave with no compression involved
- Disturbances propagate parallel to **B**

Slow and Fast Magnetosonic Waves





- *Left:* The restoring forces for magnetosonic waves propagating perpendicular to **B** are given by gas and magnetic pressure gradients. This shows a compressional wave.
- *Right:* The phase velocity of MHD waves are a function of angle when **B** is in the z direction and β is small.
- Sound waves are magnetosonic waves propagating along **B**

How useful is MHD?

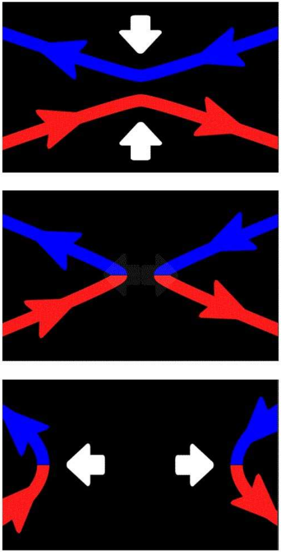
- MHD is appropriate for large-scale behavior
- MHD is usually good predictor of stability
- MHD is often inappropriate when there are non-Maxwellian distribution functions
  - Collisionless plasmas
  - Situations with lots of energetic, non-thermal particles
- MHD is a reasonable approximation for most solar physics applications, but effects beyond MHD are often important
- MHD is a mediocre description of laboratory plasmas

There are two general approaches to going beyond MHD

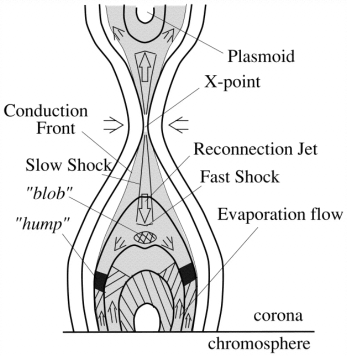
- **Extended MHD**
  - Keep the fluid approximation
  - Add more terms to the equations to include more effects
- **Kinetic theory**
  - Abandon the fluid approximation
  - Keep track of particle distribution functions

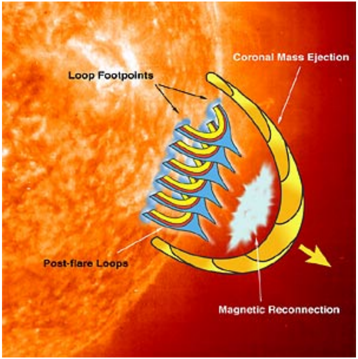
- Or . . . take both approaches simultaneously!

Magnetic Reconnection is the breaking and rejoining of magnetic field lines in a highly conducting plasma



Solar flares and CMEs are powered by magnetic reconnection





- Explosive release of magnetic energy
- Bidirectional Alfvénic jets
- Very efficient particle acceleration
- Flux ropes escape as coronal mass ejections (CMEs)

Magnetic reconnection is a fundamental process in laboratory and astrophysical plasmas

- Classical theories based on resistive diffusion predict slow reconnection (weeks to months. . .)
- Fast reconnection allows magnetic energy to be explosively converted into kinetic and thermal energy
- Collisionless or non-fluid effects are (probably) needed to explain why fast reconnection occurs in flares (tens of seconds to minutes!)

Summary

- MHD describes the macroscopic behavior of plasmas
- Each term in the MHD equations represents a different physical effect
- There are three types of MHD waves: Alfvén waves, fast magnetosonic waves, and slow magnetosonic waves
- Physics beyond MHD is often needed to describe plasma behavior
- Magnetic reconnection is the breaking and rejoining of magnetic field lines in a highly conducting plasma
  - Releases magnetic energy during solar flares and CMEs
  - Degrades confinement in laboratory plasmas

Useful references

- *The Physics of Plasmas* by T.J.M. Boyd and J.J. Sanderson. One of the most understandable introductions to plasma physics that I've found.
- *Magnetohydrodynamics of the Sun* by Eric Priest. Very useful resource for the mathematical properties of MHD as applied to solar physics.
- *Principles of Magnetohydrodynamics* by Hans Goedbloed and Stefaan Poedts. Good introduction to MHD with a broad focus on applications.
- *Introduction to Plasma Physics and Controlled Fusion* by Francis Chen. A beginning graduate level introduction to plasma physics with less emphasis on MHD.