QED Vacuum in Various External Field

Application of worldline instanton for Schwinger pair production

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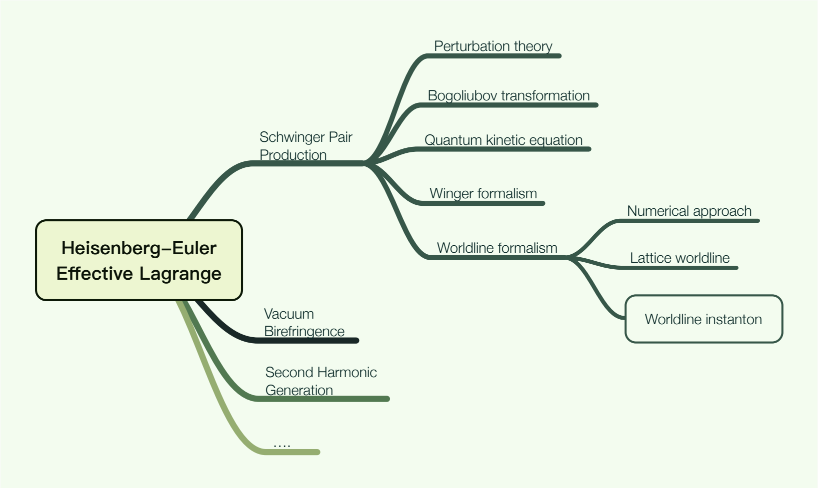
Outline

1. Schwinger pair production
2. Worldline formalism & Worldline instantons
3. 1D time-dependent field & $\mathcal{PT}$-symmetric QM
4. Field with $SO(3)$ Symmetry
5. Conclusion

Schwinger pair production

Schwinger Pair production



Worldline formalism & Worldline instantons

Vacuum pair spectrum

Vacuum-vacuum transition amplitude

$$\langle 0_{\mathrm{out}} | 0_{\mathrm{in}} \rangle = \mathrm{e}^{\mathrm{i}\Gamma[A]} \tag{1}$$

where $\mathrm{i}\Gamma[A]$ – sum of all the connected vacuum diagrams.

- $\mathrm{Re}(\Gamma)$: dispersive effects, such as vacuum birefringence;
- $\mathrm{Im}(\Gamma)$: absorptive effects, such as vacuum pair production.

The unitarity $\Rightarrow$ the total probability of producing particles

$$\sum_{n=1}^{\infty} P_n = 1 - \mathrm{e}^{-2\mathrm{Im}\Gamma[A]} \approx 2\mathrm{Im}\Gamma[A] \tag{2}$$

Essential problem

Calculating $\Gamma[A]$ in various background field.

Effective action & Worldline Formalism

For the one-loop graphs (Euclidean space)

$$\Gamma_{\mathrm{scal}}[A] = -\frac{1}{2}\mathrm{Tr}\ln\left[\frac{-(\partial + \mathrm{i}eA)^2 + m^2}{-\Box + m^2}\right] \tag{3}$$

$$\Gamma_{\mathrm{spin}}[A] = \ln\mathrm{Det}[\not{p} + e\not{A} - \mathrm{i}m] \tag{4}$$

Equivalence

Fermion = Scalar + fermionic factor $-2(-1)^2$ in the worldline summation

Schwinger proper time T

$$\Gamma[A] = \int_0^\infty \frac{\mathrm{d}T}{T}\mathrm{e}^{-m^2 T}\int_{x(T)=x(0)} \mathcal{D}x \exp\left[-\int_0^T \mathrm{d}\tau\left(\frac{\dot{x}^2}{4} + \mathrm{i}eA\cdot\dot{x}\right)\right] \tag{5}$$

Worldline Instantons

Weak field condition & asymptotic behavior of Bessel function

$$m\sqrt{\int_0^1 \mathrm{d}u\ \dot{x}^2} \gg 1, \quad \mathrm{K}_\nu \sim \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x} \tag{6}$$

Stationary phase method:

$$\Gamma[A] \approx \sqrt{\frac{2\pi}{m}} \int_{x(1)=x(0)} \mathcal{D}x \left(m\sqrt{\int_0^1 \mathrm{d}u\ \dot{x}^2} \right)^{-1/4} \exp\left(-S\right) \tag{7}$$

with

$$S = m\sqrt{\int_0^1 \mathrm{d}u\ \dot{x}^2} + \mathrm{i}e\int_0^1 \mathrm{d}u\ A \cdot \dot{x} \tag{8}$$

The saddle point of the phase: worldline instanton solution

$$\ddot{x}_\mu = \frac{\mathrm{i}em}{a} F_{\mu\nu}\dot{x}_\nu, \quad a = \sqrt{\int_0^1 \mathrm{d}u\ \dot{x}^2} \tag{9}$$

1D time-dependent field & $\mathcal{PT}$-symmetric QM

1D time-dependent Field

$A_1 = E_0 f(t)$, cyclicly spatial coordinate, conservation law

$$\dot{x}_1 + \frac{\mathrm{i}ea}{m} A_1(x_4) = \mathrm{const} \tag{10}$$

Worldline instanton "action"

$$S_0 = \frac{m}{a} \int_0^1 \mathrm{d}u (\dot{x}_4)^2 = 2\frac{m}{a} \int_{x_4^-}^{x_4^+} \mathrm{d}x_4 \; \dot{x}_4 \tag{11}$$

where

$$\dot{x}_4 = a\sqrt{1 + \frac{e^2 A_1^2}{m^2}} \Longrightarrow u = \int \frac{\mathrm{d}x_4}{a\sqrt{1 + \frac{e^2 A_1^2}{m^2}}} \tag{12}$$

and $x_4^{\pm}$ – tuning points, which can be found from the following equation

$$\frac{\omega^2 f^2}{\gamma^2} + 1 = 0, \quad \gamma = \frac{m\omega}{eE_0} \tag{13}$$

1D time-dependent Field

Worldline instanton "action"

$$S_0 = \frac{m^2 \pi}{e E_0} g(\gamma) \tag{14}$$

where $g(\gamma)$ – reduction factor

$$g(\gamma) = \frac{2\omega}{\pi\gamma} \int_{x_4^-}^{x_4^+} \mathrm{d}x_4 \sqrt{1 + \frac{\omega^2 f^2}{\gamma^2}} \tag{15}$$

WKB: worldline instantons on the complex plane

- Bohr–Sommerfeld quantization:

$$\oint p \, \mathrm{d}x = 2\pi\hbar \left(n + \frac{1}{2} \right)$$

- $\mathcal{PT}$-symmetric quantum mechanics:

Complex worldline instantons $\Rightarrow$ Stokes phenomenon

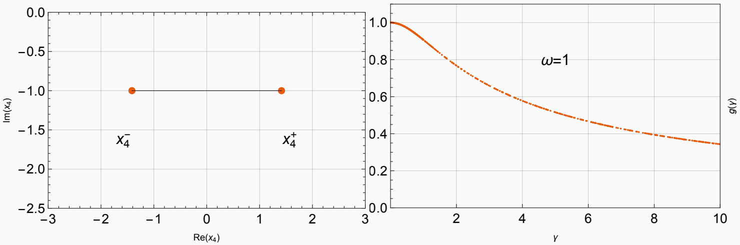
Standing Wave at Nodes

$A_1 = \frac{E_0}{\omega}\sin(\omega t + \phi)$, $0 \leq \phi < \pi$,

$$\text{Tuning points: } x_4^{\pm} = -i\frac{\phi}{\omega} \pm \frac{1}{\omega}\,\text{arcsinh}\left(\frac{\gamma}{\omega}\right) \tag{16}$$

$$g(\gamma) = -\frac{4\mathrm{i}}{\pi\gamma}\mathbb{E}\left(\mathrm{i}\,\text{arcsinh}\,(\gamma)\,\middle|\,-\frac{1}{\gamma^2}\right) \tag{17}$$

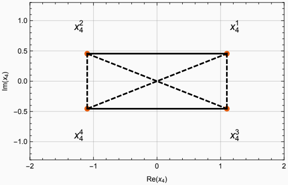
Eq.(45) in *Brezin, E., and C. Itzykson, 1970, Phys. Rev. D 2, 1191.*

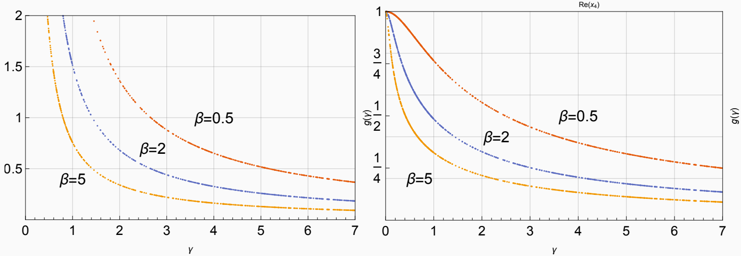


Two more cases

Lorentzian: $A_1 = E_0/(1+\beta^2 t^2)$

Linear chirp: $A_1 = E_0 \sin(\omega t + \beta t^2)$





Left: Lorentzian pulse; Right: Linear chirp pulse.

Field with $SO(3)$ Symmetry

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Imaginary Lorentz equation on the manifold

$$\ddot{x}^\mu + \dot{x}^\nu \dot{x}^\alpha \Gamma^\mu_{\nu\alpha} = \frac{\mathrm{i}ea}{m} g^{\mu\alpha} F_{\alpha\beta} \dot{x}^\beta \tag{18}$$

On the 3-sphere

$$\begin{gathered} \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \quad \Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r\sin^2(\theta), \\ \Gamma^3_{23} = \cot(\theta), \quad \Gamma^2_{33} = -\cos(\theta)\sin(\theta) \end{gathered} \tag{19}$$

For the field with $SO(3)$ symmetry $A_0 = f(r)$

$$k_0 = \dot{x}_4 + \frac{eaf}{m}, \quad k_1 = r^2 \dot{\phi} \sin^2(\theta) \tag{20}$$

$$k_2 = k_1^2 \csc^2(\theta) + r^4 \dot{\theta}^2, \quad k_0^2 + \frac{2\mathcal{E}}{m} = a^2 \tag{21}$$

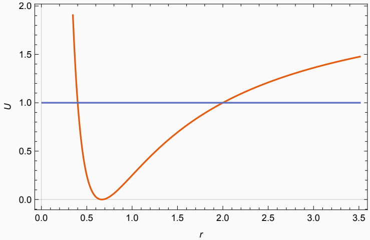
$$\frac{m}{2}\dot{r}^2 + \frac{mk_2}{2r^2} + \frac{aef(aef - 2mk_0)}{2m} = \mathcal{E} \tag{22}$$

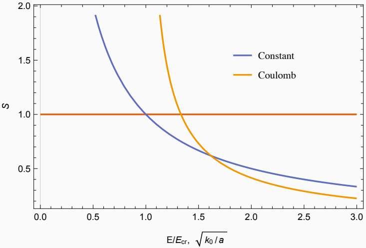
Coulomb Field

$A_0 = \alpha/r,\ \alpha = -eZ,\ k_1 = k_2 = 0,\ k_0 > a$

$$S_0 = \frac{m}{a}\int_0^1 \dot{r}^2\, \mathrm{d}u = 2\sqrt{2m}\int_{r^-}^{r^+} \mathrm{d}r\sqrt{\mathcal{E} - U} \tag{23}$$

$$S_0 = 2\pi e^2 Z\left(\frac{k_0/a}{\sqrt{k_0^2/a^2 - 1}} - 1\right), \quad T = \frac{2\pi e^2 k_0 Z}{a^2 m\left(k_0^2/a^2 - 1\right)^{3/2}} \tag{24}$$





Conclusion

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1. If the turning points are complex, the worldline instanton action could be real, if and only if turning points are $\mathcal{PT}$-symmetric, i.e. $-(x^+)^* = x^-$. Constant magnetic field, anti $\mathcal{PT}$-symmetric, worldline instanton action pure imaginary.
2. If the complex turning points are multiple, then the worldline instanton action bouncing between two $\mathcal{PT}$-symmetric could be real.
3. Critical field in the Coulomb field is higher than the Schwinger limit.