

Underlying Scientific Principles

A convolution coefficient matrix, $[C]$, can be given as $[C] = ([J]^T[w][J])^{-1}[J]^T[w]$, where $[J]$ is the Jacobian matrix and $[w]$ is the weight matrix. $[w]$ is a diagonal matrix with its diagonal elements equal to the weights at the respective kernel nodes. $[J]$ (and therefore $[C]$ also) depends (1) on the format and the order of the used multidimensional polynomial, (2) on shape and size of the kernel, and (3) on the manner the polynomial terms and the kernel nodes are traversed when assembling $[J]$.

We consider a D -dimensional space having a right-handed Cartesian coordinate system, in which coordinates of a point can be given as (x_1, x_2, \dots, x_D) . In this space, the following D -dimensional polynomial of order $P_1 \times P_2 \times \dots \times P_D$ is considered:

$$f(x_1, x_2, \dots, x_D) = \sum_{i_1=0}^{P_1} \sum_{i_2=0}^{P_2} \dots \sum_{i_D=0}^{P_D} \left(a_{i_1 i_2 \dots i_D} \prod_{d=1}^D x_d^{i_d} \right) \quad (E1)$$

The above format is chosen because it allows excellent filtering at all the places inside a kernel, as demonstrated for $D = 2$ by Nikitas and Pappa-Louisi [1]. Consequently, it can be used to filter at internal and at near-boundary locations, both, of a sampled domain. When assembling $[J]$, the innermost summation of (E1) is expanded very first, thereafter the second innermost summation, and so on. The outermost summation is expanded at last. As a result, when moving from left towards right on the right hand side of the expanded form of the polynomial, i_D varies most frequently, thereafter i_{D-1} , and so on. As an example, the expanded polynomial takes the following form when $D = 2$, $P_1 = 2$, and $P_2 = 3$:

$$\begin{aligned} f(x_1, x_2) = & \underbrace{a_{00} + a_{01}x_2 + a_{02}x_2^2 + a_{03}x_2^3}_{i_1=0} + \underbrace{a_{10}x_1 + a_{11}x_1x_2 + a_{12}x_1x_2^2 + a_{13}x_1x_2^3}_{i_1=1} \\ & + \underbrace{a_{20}x_1^2 + a_{21}x_1^2x_2 + a_{22}x_1^2x_2^2 + a_{23}x_1^2x_2^3}_{i_1=2} \end{aligned} \quad (E2)$$

When nodes of a rectangular kernel of size $K_1 \times K_2 \times \dots \times K_D$ are traversed in the column-major fashion, the coordinate x_D varies most frequently, thereafter the coordinate x_{D-1} , and so on. The coordinate x_1 varies least frequently. Such a traversal path is depicted in Fig.1 by means of grey-color arrows, for $D = 2$, $K_1 = 7$ and $K_2 = 5$. As evident, first the node (1) is considered, then the node (2), and so on. The node (35) is considered in the end.

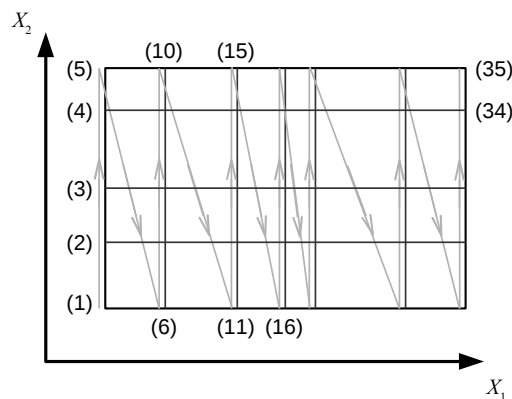


Fig.1 Traversal path inside a two-dimensional kernel of size 7×5.

With the $[C]$ available, the polynomial coefficients can be readily calculated as $\{a\} = [C]\{q\}$, where $\{a\}$ is a column vector constituted of the coefficients $a_{i_1 i_2 \dots i_D}$ in such a manner that when one traverses through its elements from up to down, i_D varies most frequently, thereafter i_{D-1} , and so on; and $\{q\}$ represents a column vector constituted of $^{(k)}q$, which is the sampled value at $(k)^{\text{th}}$ kernel node. For the aforementioned two-dimensional polynomial and kernel (whose order and size are $P_1 \times P_2 = 2 \times 3$ and $K_1 \times K_2 = 7 \times 5$, respectively),

$$\{a\} = \left[\underbrace{a_{00} \ a_{01} \ a_{02} \ a_{03}}_{i_1=0} \ \underbrace{a_{10} \ a_{11} \ a_{12} \ a_{13}}_{i_1=1} \ \underbrace{a_{20} \ a_{21} \ a_{22} \ a_{23}}_{i_1=2} \right]^T \quad (\text{E3}), \text{ and}$$

$$\{q\} = \left[{}^{(1)}q \quad {}^{(2)}q \quad {}^{(3)}q \quad \dots \quad {}^{(35)}q \right]^T \quad (\text{E4})$$

With the knowledge of $\{a\}$, the filtered value or any smoothed partial derivative of order $m_1 + m_2 + \dots + m_D$ can be easily calculated at any point inside the kernel, after using (E1) or the following relationship:

$$\frac{\partial^{m_1+m_2+\dots+m_D} f}{\partial x_1^{m_1} \partial x_2^{m_2} \dots \partial x_D^{m_D}} = \begin{cases} \sum_{i_1=m_1}^{P_1} \sum_{i_2=m_2}^{P_2} \dots \sum_{i_D=m_D}^{P_D} \left(a_{i_1 i_2 \dots i_D} \prod_{d=1}^D \frac{i_d! x_d^{i_d-m_d}}{(i_d-m_d)!} \right), & \text{if } 0 \leq m_d \leq P_d \text{ for all } d; \\ 0, & \text{if } m_d > P_d \text{ for any } d; \end{cases} \quad (\text{E5}),$$

which is obtained after differentiating (E1), analytically. Here, note that when $(x_d, i_d) = (0, 0)$ for any d , (E1) contains an indeterminate term 0^0 . Similarly, when $(x_d, i_d) = (0, m_d)$ for any d , (E5) contains the term 0^0 . It can be shown after expanding (E1) and (E5) that all such 0^0 terms are equal to 1.

Computation of $[C]$ with Arbitrary Accuracy

Any $[C]$ is contaminated by the truncation errors present in the floating point computations. These errors decrease with increasing bits-precision of the used floating point numbers; and therefore they affect increasingly farther significant digits of the $[C]$ elements. It, therefore, implies that if the bits-precision is sufficiently high, it will not affect a given number of initial significant digits. Whether some desired initial significant digits are affected, or not, is recognized after comparing $[C]$ calculated with two different bits-precisions. Only the desired number of initial significant digits are kept in the elements of $[C]$ when comparing, while discarding the farther digits. If the two $[C]$'s match, it implies that the $[C]$ elements are accurate until the desired number of significant digits.

Some elements of $[C]$ may be genuinely zero, but they may appear non-zero due to the truncation errors. These fictitious elements must be recognized and subsequently set to zero, as they remain non-zero in many cases, even where high bits-precision floating-point numbers are used. As a result, any two $[C]$'s may never match. Orders of these elements keep decreasing with increasing bits-precision. The genuine elements, on the other hand, tend to converge to their true values, eventually making their orders also converge to some fixed values. This convergent nature of the orders of the genuine elements with increasing bits-precision offers an efficient way to distinguishing them from the fictitious elements.

A $[C]$ with its genuine elements kept only until the desired significant digits and its fictitious elements set to zero would exactly match a similar $[C]$ calculated at a different bits-precision, provided that the bits-precision is sufficiently high in both the cases. This is the desired $[C]$.

REFERENCES

1. J. Nikitas and A. Pappa-Louisi, "Comments on the two-dimensional smoothing data," *Analytica Chimica Acta*, Vol. 415, pp. 117–125 (2000).