Distribution of LRT for Testing the Equality of Several 2-Parameter Exponential Distributions

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Reader Aids—
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Abstract—Exact distribution of the likelihood-ratio-test (LRT) criterion for testing the equality of several 2-parameter exponential distributions is obtained for the first time in a computational closed form. This is then used to obtain the *s*-significance points of the LRT.

1. INTRODUCTION

There are several practical situations where data analysts are confronted with the problem of testing as to whether there are *s*-significant (*s*- implies statistically) differences among various groups (populations) when the underlying distribution is 2-parameter exponential [9, 15]. This paper addresses the problem of testing the equality of several 2-parameter exponential distributions. Paulson [14] considered the likelihood-ratio-test (LRT) criterion for testing the equality of the location parameters only, while Epstein & Tsao [3] considered the LRT for testing the equality of two 2-parameter exponential distributions and showed that they can be reduced to equivalent tests which are expressed in terms of *F*-distributions when the null hypothesis is true. However no such results are available for more than two populations. Jain, Rathie, Shah [8] obtained the distribution of LRT for testing the equality of several 2-parameter exponential distributions for the case of equal sample sizes but it is rather unwieldly and does not lend itself readily to practical use.

Hogg & Tanis [6] considered an iterative procedure (IP) suggested by Hogg [4, 5] for this problem. Mathai [10] obtained the non-null distribution of the LRT in the form of an integral and in a form not suited for *s*-power studies. Because the *s*-power functions of both the LRT and IP are not available in explicit form, Hsieh [7] used Monte Carlo simulation to approximate the *s*-power for selected sample sizes and alternatives. He concluded that overall, LRT has a higher *s*-power than IP.

This paper obtains the exact distribution of the LRT for the case of equal sample sizes in a computational form and presents a table of selected *s*-significance points of LRT. It is of considerable interest to obtain the exact distribution of LRT for the case of unequal sample sizes and its non-null distribution in a form suited for *s*-power studies.

2. PRELIMINARIES

Notation

p	number of samples
i	samples serial number, $i = 1, ..., p$
n	number of observations in a sample
x_{ij}	observation j is sample i; $j = 1, ..., n$; $i = 1, ..., p$
$\bar{x}_i$	mean of observations in sample i
$x_{1(i)}$	lowest observation in sample i
$\bar{x}_0$	mean of observations x_{ij}
$x_{(1)}$	smallest of the p lowest observations $x_{1(i)}$
λ	likelihood ratio
L	$\lambda^{1/n}$
L_0	$\lambda^{1/(pn)}$
H_0	null hypothesis
Π_i	product over i from 1 to p

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

Assumptions

1. p *s*-independent samples are available and sample i has been drawn from a 2-parameter exponential distribution with pdf:

$$\text{expc}(x; \theta_i, A_i) \equiv \theta_i^{-1} \exp[-(x - A_i)/\theta_i], \text{ for } x > A_i, \theta_i > 0$$

$$= 0, \text{ otherwise } (i = 1, ..., p) \qquad (2.1)$$

2. Each sample has the same number of observations.

The LRT for testing the hypothesis

$$H_0: \theta_1 = \theta_2 = ... = \theta_p \text{ and } A_1 = A_2 = ... = A_p \qquad (2.2)$$

against the general alternatives, was derived by Sukhatme [15] in the form —

$$\lambda = \Pi_i(\bar{x}_i - x_{(1)i})^n/(\bar{x}_0 - x_{(1)})^{pn}. \qquad (2.3)$$

Then Sukhatme [15] has shown that moment h of L_0 is —

$$E\{L_0^h\} = K \cdot p^h\{\Gamma(n - 1 + h/p)\}^p/\Gamma(pn + h - 1) \quad (2.4)$$

$$K \equiv \Gamma(pn - 1)/\{\Gamma(n - 1)\}^p. \quad (2.5)$$

It therefore follows from (2.5) that moment h of L is —

$$E\{L^h\} = K \cdot p^{ph}[\Gamma(n - 1 + h)]^p/\Gamma(pn + ph - 1). \quad (2.6)$$

3. EXACT DISTRIBUTION OF L

Notation

δ adjustment factor
m $n - \delta$
$B_r(\cdot)$ Bernoulli polynomial of degree r and order one
betf$(\cdot; c, d)$ beta Cdf, betf$(x; p, q) = \int_0^x y^{p-1}(1 - y)^{q-1} dy/B(p, q)$

Nomenclature

Mellin transform The Mellin integral transform of a function $f(x)$, defined only for $x > 0$, is —

$$M\{f(x)|s\} = E(x^{s-1}) = \int_0^\infty x^{s-1}f(x)dx$$

where s is any complex variable.

$\mathbf{O}(t)$ $f(t)$ is $\mathbf{O}(t)$ if the function $f(t)$ is bounded by some constant multiple of t, for large t.

Using the Mellin transform of the moment function of L in (2.7), pdf$\{L\}$ is [16]:

$$f(\ell) = K \cdot (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \ell^{-h-1}p^{ph}[\{\Gamma(n - 1 + h)\}/ \Gamma\{p(n + h)-1\}]dh \quad (3.1)$$

Define:

$$t \equiv m + h \quad (3.2)$$

Equation (3.3) results from (3.1) and (3.2):

$$f(\ell) = K \cdot p^{-pm}\ell^{m-1}(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \ell^{-t}\phi(t)dt \quad (3.3)$$

$$\phi(t) \equiv p^{pt}\{\Gamma(t + \delta - 1)\}^p/\Gamma\{p(t + \delta)-1\} \quad (3.4)$$

Use the asymptotic expansion for the logarithm of the gamma function [1, pp 204]. Then —

$$\phi(t) = K_1 \cdot t^{-\nu}[1 + q_1/t + q_2/t^2 + \ldots] \quad (3.5)$$

$$K_1 \equiv (2\pi)^{(p-1)/2} \cdot p^{3/2-p\delta} \quad (3.6)$$

$$\nu \equiv 3(p - 1)/2. \quad (3.7)$$

The coefficients q_r are recursively determined using (3.8):

$$q_r = \sum_{k=1}^{r} kA_kq_{r-k}/r, \; q_0 = 1 \quad (3.8)$$

$$A_r \equiv (-1)^r[p^{-r}B_{r+1}(p\delta - 1) - pB_{r+1}(\delta - 1)]/r(r + 1). \quad (3.9)$$

Equation (3.5) shows that:

$$\phi(t)/K_1 = \mathbf{O}(t^{-\nu}) \quad (3.10)$$

with real part of t tending to infinity; $\phi(t)$ has therefore the following exact representation as a factorial series [12, 13]:

$$\phi(t) = K_1 \cdot \sum_{k=0}^{\infty} R_k\{\Gamma(t + \alpha)/\Gamma(t + \alpha + \nu + k)\}, \; R_0 = 1 \quad (3.11)$$

where α is a convergence factor chosen such that $R_1 = 0$ and the coefficients R_k are obtained using the following recurrence relations [11]:

$$\sum_{j=0}^{k} R_{k-j}d_{k-jj} = q_k \; (k = 1, 2, \ldots) \quad 3.12)$$

$$d_{ir} = \sum_{k=1}^{r} kC_{ik}d_{ir-k}/r, \; d_{i0} = 1 \quad (3.13)$$

$$c_{ir} = (-1)^{r-1}[B_{r+1}(\alpha) - B_{r+1}(\alpha + a + i)]/r(r + 1). \quad 3.14)$$

Using (3.11) in (3.3) and noting that term by term integration is valid since a factorial series is uniformly convergent in a half-plane [2], pdf $\{L\}$ is [11]:

$$f(\ell) = K \cdot (2\pi)^{(p-1)/2} p^{3/2-pm} \sum_{i=0}^{\infty} R_i\ell^{m+\alpha-1}(1 - \ell)^{\nu+i-1}/ \Gamma(\nu + i). \quad (3.15)$$

We now proceed to choose the convergence factors δ and α. Using the asympotic expansion for the logarithm of the gamma distribution, we write:

$$K \cdot (2\pi)^{(p-1)/2} p^{3/2-pn} = m^\nu[1 + T_1/m + T_2/m^2 + \ldots]. \quad (3.16)$$

We choose δ such that $T_1 = 0$ and this gives —

$$\delta = 13(1 + p)/18p. \quad (3.17)$$

Now we choose α such that $R_1 = 0$ and this gives —

$$\alpha = (1 - \nu)/2. \quad (3.18)$$

TABLE 1
Percentage Points of $L \equiv \lambda^{1/n}$

n	$p = 2$ 1%	$p = 2$ 5%	$p = 3$ 1%	$p = 3$ 5%	$p = 4$ 1%	$p = 4$ 5%	$p = 5$ 1%	$p = 5$ 5%
6	.3157	.4519	.1879	.2860	.1186	.1894	.7705	.1281
7	.3835	.5168	.2480	.3520	.1685	.2491	.1173	.1794
8	.4405	.5684	.3025	.4084	.2168	.3031	.1586	.2285
9	.4885	.6105	.3511	.4566	.2620	.3514	.1990	.2741
10	.5294	.6452	.3943	.4981	.3036	.3943	.2376	.3159
15	.6653	.7552	.5494	.6385	.4636	.5486	.3954	.4751
20	.7409	.8134	.6430	.7184	.5670	.6421	.5039	.5772
25	.7889	.8493	.7049	.7696	.6379	.7039	.5808	.6468
30	.8219	.8736	.7486	.8051	.6891	.7477	.6376	.6970
35	.8460	.8912	.7812	.8311	.7278	.7803	.6811	.7349
40	.8644	.9045	.8063	.8511	.7580	.8054	.7153	.7644
50	.8905	.9232	.8425	.8795	.8020	.8417	.7658	.8074
60	.9082	.9358	.8673	.8988	.8325	.8666	.8012	.8371
70	.9210	.9449	.8854	.9128	.8549	.8848	.8273	.8589
80	.9306	.9517	.8891	.9234	.8720	.8986	.8473	.8756
90	.9382	.9570	.9099	.9317	.8855	.9094	.8632	.8887
100	.9443	.9613	.9186	.9384	.8964	.9182	.8761	.8994

$p \equiv$ number of samples
$n \equiv$ size of each sample
$\lambda \equiv$ likelihood ratio

From (3.15) the Cdf $\{L\}$ is:

$$F(\ell) = K \cdot (2\pi)^{(p-1)/2} p^{3/2-pn} \cdot \sum_{i=0}^{\infty} R_i' \, \text{betd}(\ell; m + \alpha, \nu + i) \quad (3.19)$$

$$R_i' \equiv R_i\{\Gamma(m + \alpha)/\Gamma(m + \alpha + \nu + i)\}. \quad (3.20)$$

The distribution of L given in (3.19) is in a computational form and can be used to compute the exact percentage points of the test statistic. This distribution is very useful in life tests and accident data. The representation of the distribution of L is computationally very convenient because of the stable recurrence relations given in (3.8) and (3.12).

4. NUMERICAL COMPUTATIONS

The 0.05 and 0.01 *s*-significance points of L were computed for $p = 2(1)5$ and various values of n. A DEC-system was used and the values are correct to 4 significant figures in table 1. For $n \le 20$, the number of terms of the series (3.19) required for 5 figure accuracy varied from 15 to 20 and increased with higher values of p while for higher values of n, the number of terms varied from 10 to 15 depending upon p. It has been verified that in each case, the total integral over 0 to 1 of the series (3.19) approach the theoretical value 1. The computations were checked using the following Box approximation [1, pp 203] which holds for large values of n:

$$\Pr(-2n\varrho \, \ell\text{n}\, L \le z) = \text{csqf}(z; f) + w_2[\text{csqf}(z, f + 4) - \text{csqf}(z, f)] + \mathbf{O}(n^{-3})$$

$$f \equiv 3(p - 1)$$

$$\varrho \equiv 1 - 13(1 + p)/18\,pn$$

$$w_2 \equiv \frac{1}{6}\,[B_2\{np(1 - \varrho)\text{-}1\} - p^3 B_2\{n(1 - \varrho)\text{-}1\}]/(\varrho np)^2$$

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