

# A Method to Measure Magnetic Fields Accurately Using Ampere's Law

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**Abstract**—A proposed method to measure magnetic fields is described, and a simple demonstration of the new method has been performed. A derivation of the equation used to calculate the field is presented, and some of its implications are discussed. The objective of further experiments is to apply the technique to the measurement of the gyromagnetic ratio of the proton ( $\gamma_p$ ) to an accuracy near 1 part in  $10^8$ , using a precision solenoid whose departures from ideal are small.

## I. BACKGROUND

AMPERE'S LAW, in principle, can be used to calculate magnetic fields for ideal geometries of high symmetry. This present approach is a practical method of experimentally realizing the potential of this simple but powerful law. It will be demonstrated that mapping the magnetic field of a solenoid with a magnetic flux detector can replace the need to measure the physical dimensions of the solenoid in order to obtain the magnitude of the field. The precise mapping of magnetic fields is difficult wherever magnetic field gradients are large, since precision magnetometers do not work well in high field gradients. However, the flux through a detector (or a SQUID) can be measured with high precision in such fields. We will show mathematically that the line integral of the flux which is produced in a detector by a solenoid can be used to measure the magnetic field of the solenoid. This suggests to us a new measurement approach which should be useful. The possible sources of error should be entirely different from those found in previous measurements, and this should be of value in testing for unknown systematic effects. Also it is anticipated that this new method can achieve higher accuracy than previous techniques.

## II. SIMPLE EXAMPLE

First, let us consider an idealized experiment to clarify the basic approach. Ampere's law is valid for any closed path and in free space can be written

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 NI \quad (1)$$

where  $\mu_0$  is the permeability of free space, and  $N$  is the number of turns containing a current  $I$  which is enclosed by the line integral. Multiply and divide the left side of (1) by  $B_{\parallel}(0)$ , the component of  $\vec{B}$  parallel to the path at some

position  $l_0$  on the path. Equation (1) may then be rewritten in the form

$$B_{\parallel}(0) = \frac{\mu_0 NI}{\oint \frac{\vec{B} \cdot d\vec{l}}{B_{\parallel}(0)}} = \frac{\mu_0 NI}{L(0)}, \quad \text{where } L(0) \equiv \oint \frac{\vec{B} \cdot d\vec{l}}{B_{\parallel}(0)} \quad (2)$$

Although (2) is just a simple rearrangement of (1), it is useful because the length  $L(0)$  can be measured experimentally as follows. We have a solenoid of length  $S$  and diameter  $D$ . The magnetic profile of the solenoid along its axis can be measured from  $-\infty$  to  $+\infty$ . (The line integral is then closed at infinity where  $B = 0$  for finite current distributions.) This profile is measured along the axis by a magnetic field probe that is part of a laser interferometer system. The magnitude of the magnetic field is recorded at equal spacings along the solenoid axis.  $L(0)$  can then be computed by calculating the area under the magnetic profile curve. Because this profile is normalized to the value at  $l_0$ , the length obtained is independent of the calibration of the magnetic field probe. The probe, of course, must be linear over the range being used. For a perfect solenoid,  $L(0) = \sqrt{S^2 + D^2}$  if  $l_0$  is at the center of the solenoid. For a general imperfect solenoid,  $L(0)$  contains all of the required information about the diameter variations, pitch variations, return lead corrections, and the susceptibility corrections resulting from the solenoid structure.

This simple example is not practical because it is difficult to measure magnetic fields precisely in high field gradients. However, flux detectors can be used in high, magnetic field gradients, and they sense vector components of the field. Therefore, we extend the simple example above to the case where the flux through a detector coil is the quantity that is mapped along the axis of the solenoid.

## III. FLUX DETECTOR USE

Let  $\Phi(z)$  be the flux through the detector coil produced by the current in the solenoid. This detector coil can be used to map the "flux profile" of the solenoid along the beam direction of a laser interferometer that is used to measure the position of the detector coil. Thus the length  $L(0)$  which is defined as follows

$$L(0) \equiv \int_{-\infty}^{\infty} \frac{\Phi(z) dz}{\Phi(0)} \quad (3)$$

can be empirically measured.  $\Phi(0)$  is the flux through the detector at  $z = 0$ . In the appendix we show that the magni-

Manuscript received June 19, 1978; revised July 24, 1978.  
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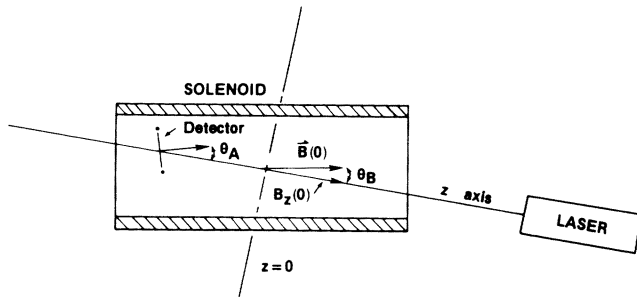


Fig. 1. Schematic drawing of the proposed experimental configuration. A solenoid produces a magnetic field  $\vec{B}(0)$ . A laser beam defines the  $z$  axis along which a profile of the field in the solenoid is measured with a flux detector.  $\theta_B$  is the angle between the magnetic field and the  $z$  axis.  $\theta_A$  is the angle between the normal vector of the flux detector and the  $z$  axis.

tude of the magnetic field (averaged over the detector coil) at the center of the solenoid ( $z = 0$ ) is given by

$$|\vec{B}(0)| = \frac{\mu_0 NI}{L(0)} \{1 - \frac{1}{2}\theta_A\theta_B + \frac{1}{2}[\theta_B]^2 + \dots\} \quad (4)$$

where  $\theta_A$  is the angle between the laser and the normal vector of the detector coil, and  $\theta_B$  is the angle between the laser and the magnetic field (see Fig. 1). The correction terms in (4) were calculated assuming we have a solenoid with axial symmetry. If  $\theta_A = 0$  and the coil detector lies in one plane, then (4) is valid for all current sources including asymmetric ones.

The higher order terms in (4) must also be examined, and the primary concern is whether a correction factor for the detector coil need be applied. For a thin-film SQUID detector that is small ( $\sim 1$  mm) and which is confined to one plane, these higher order terms are certainly negligible. For a detector which has a larger area or which has a complicated geometry, such as the coil used in our preliminary experiment, these terms must be evaluated to ensure that this uncertainty is negligible.

If either the plane of the detector coil is orthogonal to the laser beam ( $\theta_A = 0$ ) or the magnetic field being measured is along the laser beam direction ( $\theta_B = 0$ ), and if the field is uniform over the detector coil at  $z = 0$ , then  $L(0) = L(0)$  as used in (2). In a practical experiment both  $\theta_A$  and  $\theta_B$  should be small ( $\sim 10^{-4}$ ), so that the term  $\frac{1}{2}\theta_A\theta_B$  can be made negligible.

We feel that the detector coil and the laser can be adequately aligned to permit a 0.01-ppm measurement. A remaining critical question is: Can a detector coil be developed that has sufficient precision and linearity to measure  $L(0)$  to this level? A thin-film SQUID detector can be fabricated in the proper shape and has, in fact, sufficient resolution [1]. We will therefore need to develop a SQUID system with sufficient linearity and dynamic range to make the experiment feasible.

For a complete measurement of  $L(0)$  we must estimate the flux profile out to infinity. Alternative approaches may be considered. 1) The dipole term can be estimated four-dimensional measurements. 2) A double solenoid geometry can be used in which the  $1/r^3$  and  $1/r^5$  terms in the field are

nearly zero. This is accomplished by having two concentric solenoids of equal area turns but of different diameters. The current flows in opposite directions in the two solenoids. The resulting field is uniform at the center and drops off very fast outside the solenoids.

#### IV. PRELIMINARY EXPERIMENT

We measured the mutual inductance between a detector coil and the precision solenoid used in our present  $\gamma_p$  work. A laser interferometer was used to measure the position of the detector along the axis of the solenoid. Our results calculated using (4) agree with the known value of the field to the accuracy of the inductance measurements (about 1 part in  $10^4$ ). The detector coil was then rotated  $28^\circ$  off perpendicular, and we obtained the same results for the field. We returned the detector coil to the perpendicular, displaced it 2-cm off axis and repeated the experiment with no significant change in results. All three "flux profiles" were slightly different, but the integrals all gave the same value within the expected uncertainty.

#### V. FUTURE PLANS

At this same conference six years ago, CPEM 72, we presented a new technique for accurately measuring solenoid dimensions [2]. We are just now completing a  $\gamma_p$  experiment at the 0.2-ppm level using that technique. Our present results are 10 times more accurate than any other  $\gamma_p$  determination. We feel that further improvements using the method introduced at CPEM 72 are definitely possible, but a very large effort will be required. We hope that this latest approach using Ampere's law will reduce the time required to achieve an even higher accuracy.

#### APPENDIX

To derive (4) we start with the definition of the flux to give us

$$\int_{-\infty}^{\infty} \Phi(z) dz = \int_{-\infty}^{\infty} \int_s (\vec{B} \cdot d\vec{a}) dz \quad (5)$$

where  $s$  is the surface of the detector coil and  $\vec{B}$  is the field produced by the current  $I$ . If  $\vec{B}$  and  $d\vec{a}$  are resolved into components perpendicular to and parallel to the path direction, (5) can be written

$$\int_{-\infty}^{\infty} \Phi(z) dz = \int_{-\infty}^{\infty} \left\{ \int_s B_{\parallel} da_{\parallel} + \int_s B_{\perp} da_{\perp} \right\} dz.$$

Since neither the orientation relative to  $dz$  nor the shape of the detector coil is changed in translation,  $da_{\parallel}$  and  $da_{\perp}$  are independent of  $dz$ , so the order of integration can be changed as follows:

$$\int_s \int_{-\infty}^{\infty} B_{\parallel} dz da_{\parallel} + \int_s \int_{-\infty}^{\infty} B_{\perp} dz da_{\perp}.$$

Now,  $\int_{-\infty}^{\infty} B_{\parallel} dz = \mu_0 NI$  by Ampere's law, and, if the path  $z$  is coaxial with the solenoid, then  $\int_{-\infty}^{\infty} B_{\perp} dz = 0$ . However, in general, this second term is not zero, and as a result this

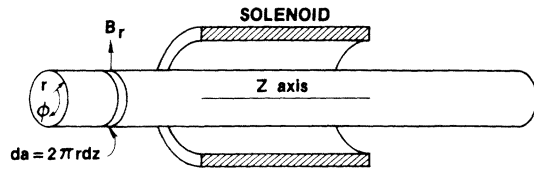


Fig. 2. Drawing of the surface used for the evaluation of the surface integral of (6). The long cylindrical surface has no net magnetic flux through its surface. The  $\int_{-\infty}^{\infty} B_{\perp} dz = 0$  for each generator of the surface.

term requires the most discussion and some computer analysis in order to give a general result.

First, we will prove the special case where  $\int_{-\infty}^{\infty} B_{\perp} dz = 0$ . If we have a perfect solenoid, then there is no component of the field in the  $\phi$  direction (see Fig. 2). Draw a long cylindrical surface whose axis coincides with that of the solenoid. One of Maxwell's equations ( $\vec{\nabla} \cdot \vec{B} = 0$ ) tells us that the net flux passing through this cylindrical surface is zero.

$$\begin{aligned} \int_{\text{Vol}} \vec{\nabla} \cdot \vec{B} d^3x &= \int_{s_1} \vec{B} \cdot d\vec{a} = \int_{-\infty}^{\infty} 2\pi r B_r dz \\ &= 2\pi r \int_{-\infty}^{\infty} B_r dz = 0. \end{aligned} \quad (6)$$

From the symmetry in  $\phi$ ,  $B_r$  is the same as  $B_{\perp}$  for any generator of the cylinder; therefore,  $\int_{-\infty}^{\infty} B_{\perp} dz = 0$  for this special case. This is an important special case which shows that we can displace our path of integration off the axis without error.

We can now rewrite (5) as follows:

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi(z) dz &= \mu_0 NI \int_s da_{\parallel} + \int_s \int_{-\infty}^{\infty} B_{\perp} dz da_{\perp} \\ &\simeq \mu_0 NI A_{\parallel} + A_{\perp} \int_{-\infty}^{\infty} B_{\perp} dz \end{aligned} \quad (7)$$

where  $A_{\parallel}(A_{\perp})$  is the area of the detector coil parallel (perpendicular) to the  $z$  axis. The approximation that we can take  $\int_{-\infty}^{\infty} B_{\perp} dz$  out of the surface integral in (7) assumes that the  $\int_{-\infty}^{\infty} B_{\perp} dz$  is independent of radial displacements. For perfect solenoids this is a valid assumption, and for small angles  $\theta_B \ll 1$  we compute the  $\int_{-\infty}^{\infty} B_{\perp} dz$  and find as expected that it is nearly independent of small axial displacements. For asymmetric fields we can compute the correction

from (7) without this approximation. Using our definition of  $L(0)$  and (7), we obtain

$$\Phi(0) = \frac{\mu_0 NI (A_{\parallel} + \epsilon A_{\perp})}{L(0)} \quad (8)$$

where

$$\epsilon \equiv \frac{\int_{-\infty}^{\infty} B_{\perp} dz}{\int_{-\infty}^{\infty} B_{\parallel} dz} = \frac{\int_{-\infty}^{\infty} B_{\perp} dz}{\mu_0 NI}.$$

$\Phi(0)$  can also be written

$$\Phi(0) = \overline{B_{\parallel}(0)} A_{\parallel} + \overline{B_{\perp}(0)} A_{\perp} \quad (9)$$

where  $\overline{B_{\parallel}(0)}$  and  $\overline{B_{\perp}(0)}$  are the components of the magnetic field averaged over the detector. Using (8) and (9) and the small angle approximation

$$\overline{B_{\parallel}(0)} \approx \frac{\mu_0 NI}{L(0)} \frac{1 + \epsilon \theta_A}{1 + \theta_B \theta_A}. \quad (10)$$

In (6) we showed that for a perfect solenoid,  $\epsilon$  is not dependent on radial displacements. For the special case of a current loop we performed an expansion of the field and showed that  $\epsilon = \frac{1}{2}\theta_B(1 + \dots)$ . For a general perfect solenoid, we used a computer to calculate  $\epsilon$  approximately and to find for small angles  $\epsilon \simeq \frac{1}{2}\theta_B(1 + q)$  where  $q$  is less than  $10^{-3}$ . We also showed that any cross term between a displacement and  $\theta_B$  is small. Thus, assuming  $\epsilon \simeq \frac{1}{2}\theta_B$  for a perfect solenoid, we can calculate the magnitude of  $|\vec{B}(0)|$  and obtain (4).

In practice,  $\epsilon$  can be estimated by changing  $\theta_A$  and measuring a new  $L(0)$ . Changing  $\theta_A$  is also a good way to empirically adjust  $\theta_B$  until it is small. We must also be careful that the return lead (or any other nonsymmetrical source) is properly taken into account when evaluating  $\epsilon$ , and we must examine the case where the  $z$  axis does not intersect the solenoid axis.

#### ACKNOWLEDGMENT

E. Wicklund helped with the theory and with the preliminary experiment. Discussions with Drs. L. Holdeman and M. Cage were also very useful.

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