

# A Two-Parameter Family of Weights for Nonrecursive Digital Filters and Antennas

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**Abstract**—We derive analytically a two-parameter family of weights for use in finite duration nonrecursive digital filters and in finite aperture antennas. This family of weights is based on the Gegenbauer orthogonal polynomials, which are a generalization of both Legendre and Chebyshev polynomials. It is shown that one parameter controls the main lobewidth and the other parameter controls the sidelobe taper. For a fixed main lobewidth, it is observed that the Gegenbauer weights can achieve a dramatic decrease in sidelobes “far removed” from the main lobe in exchange for a “small” increase in the first sidelobe adjacent to the main lobe.

The Gegenbauer weights are derived first for discretely sampled apertures and filters. An appropriate limit is then taken to produce the Gegenbauer weighting function for continuously sampled apertures and filters. The continuous Gegenbauer weighting function contains the Kaiser-Bessel function as a special case. It is thus established that the Kaiser-Bessel function is implicitly based on Chebyshev polynomials of the second kind. Furthermore, the Dolph-Chebyshev/van der Maas weights are a limiting case of the discrete/continuous Gegenbauer weights.

## I. INTRODUCTION

THE choice of weights in the design of nonrecursive digital filters and antenna apertures is an important problem for which there is a large literature. In this paper we present the Gegenbauer weighting function, so named because it is based on the Gegenbauer orthogonal polynomials [1]. The Gegenbauer weights may be applied equally well to nonrecursive digital filters and both discrete and continuous antenna apertures. The resulting FIR filter coefficients can be used as a shading function for the spectrum analysis of sampled data to reduce sidelobe leakage. Our discussion in this paper will be restricted to the antenna form of the problem merely to avoid unnecessary complication in the presentation.

The Gegenbauer design is a two-parameter family of weighting functions. One parameter,  $z_0$ , is used to control the beamwidth. The other parameter,  $\mu$ , is used to achieve sidelobe taper. Both  $z_0$  and  $\mu$  may be varied continuously and independently of each other. The Gegenbauer design is especially useful in achieving dramatic decreases in distant sidelobes in exchange for “small” increases in the first sidelobe adjacent to the main lobe. Conversely, dramatic increases in distant sidelobes can be exchanged for “small” decreases in the first sidelobe. This will be clarified by the examples.

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The Gegenbauer weights are derived first for a finite discrete aperture. An appropriate limit then gives the Gegenbauer weighting function for a bounded continuous aperture. Many similarities between the Gegenbauer weights and the Dolph-Chebyshev/van der Maas weights [2], [3] will be evident from the derivation. In fact, these latter weights are limiting forms, as  $\mu \rightarrow 0$ , of Gegenbauer weights. Also, the Kaiser-Bessel weighting function [4, pp. 232–233] for the continuous aperture is the special case  $\mu = 1$  of the Gegenbauer design. This shows that the Kaiser-Bessel function is implicitly based upon Chebyshev polynomials of the second kind, a fact which seems to have escaped notice until now. This is interesting since, as is well known, the Dolph-Chebyshev/van der Maas weights are based on Chebyshev polynomials of the first kind.

One drawback to the van der Maas weighting function for the continuous aperture is that it has  $\delta$ -function spikes at the aperture endpoints. The Gegenbauer function does not have this feature; that is, the Gegenbauer weighting function for the continuous aperture is a bounded continuous real-valued function across the whole aperture. However, since the van der Maas function is a limiting case of the Gegenbauer function as  $\mu \rightarrow 0$ , the Gegenbauer function must approximate this behavior in the neighborhood of  $\mu = 0$ . The Taylor design [5] is an alternative way to overcome this  $\delta$ -function behavior of the van der Maas function, but it is unrelated to any of the Gegenbauer designs. The proof of this statement is self-evident from the examples presented later.

The Gegenbauer polynomials  $C_n^\mu(x)$  are defined here precisely as in Szegő [1] which is used as our standard both in function definition and notation, with only two exceptions. Szegő uses the notation  $P_n^{(\mu)}(x)$  instead of  $C_n^\mu(x)$  and refers to them as the ultraspherical polynomials. This paper will not attempt to recapitulate any of the known facts about the polynomials that can be referenced in Szegő. It suffices to say here only that  $C_n^\mu(x)$  is a real valued polynomial of degree precisely  $n$ , and that the system  $\{C_0^\mu(x), C_1^\mu(x), C_2^\mu(x), \dots\}$  is orthogonal on the real interval  $[-1, +1]$  with respect to the weight function  $(1 - x^2)^{\mu-1/2}$  provided  $\mu > -\frac{1}{2}$ ,  $\mu \neq 0$ . Moreover, by taking appropriate limits and using their hypergeometric functional form,  $C_n^\mu(x)$  can be defined for all real  $\mu$ . See [1, eq. (4.7.7)]. In particular, if  $T_n(x)$  and  $U_n(x)$  denote the Chebyshev polynomials of the first and second kinds, respectively, then [1, eq. (4.7.8), (4.7.17)]

$$C_0^\mu(x) = T_0(x), \quad \lim_{\mu \rightarrow 0} \frac{C_n^\mu(x)}{\mu} = \frac{2}{n} T_n(x), \quad n \geq 1 \quad (1)$$

and [1, eq. (4.7.2)]

$$C_n^1(x) = U_n(x), \quad n \geq 0. \quad (2)$$

The derivation of formulas more general than are perhaps necessary in the antenna application is relegated to the Appendix. Special cases of these formulas will be extracted as needed and used without comment in the main body of this paper; however, every effort will be made to motivate the discussion.

## II. GEGENBAUER WEIGHTS FOR A DISCRETE APERTURE

The Gegenbauer design for a finite discrete aperture is derived for a single frequency half-wavelength equispaced linear array of omnidirectional elements. Other than the steering factor, we will always assume the aperture (discrete or continuous) is symmetrically weighted about the geometric center of the array. The array axis is taken to be the  $x$ -axis and all angles are measured from a line normal to the array axis.

Let  $N$  be the number of elements in the array (hence  $N \geq 2$ ), and let the positions of these elements be  $x_k = k\lambda/2$ ,  $k = 1, 2, \dots, N$ , where  $\lambda$  is the wavelength of the design frequency. (In the Appendix,  $\lambda$  denotes an arbitrary real variable, not frequency.) If the array is steered to look in the direction  $\theta_l$ ,  $-\pi/2 \leq \theta_l \leq \pi/2$ , and if the array receives a plane wave of wavelength  $\lambda$  from the arrival direction  $\theta_a$ ,  $-\pi/2 \leq \theta_a \leq \pi/2$ , then the complex transfer function of a linear beamformer is given by

$$F(u) \triangleq \sum_{k=1}^N w_k \exp(-i\pi k u) \quad (3)$$

where

$$u \triangleq \sin \theta_a - \sin \theta_l \quad (4)$$

and  $\{w_k\}_1^N$  are the individual element weights. Symmetrical weighting is assumed, so  $w_{N-k+1} = w_k$  for all  $k$ . Positive weighting is desirable, but not necessary.

The Dolph-Chebyshev design proceeds as follows for a design specification of  $-S$  dB peak sidelobe level. Let

$$z_0 \triangleq \frac{1}{2} \{ [r + \sqrt{r^2 - 1}]^{1/n} + [r - \sqrt{r^2 - 1}]^{1/n} \}, \quad (5)$$

$$r \triangleq 10^{S/20}$$

and  $n \triangleq N - 1$ . Notice that  $z_0 > 1$  if and only if the peak sidelobe level is lower than the level of the maximum response axis, or MRA. From (A20) of the Appendix, the expansion

$$T_n(z_0 \cos u) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{k,n}(z_0) \cos [(n-2k)u] \quad (6)$$

clearly exists, where the prime on the summation means that  $\frac{1}{2}$  the last term in the sum is taken if  $n$  is even, and all of it is taken if  $n$  is odd. From (A21) we have explicitly

$$c_{k,n}(z_0) = n(n-k-1)! \sum_{m=0}^k \frac{(m)_{k-m} (z_0^2 - 1)^m z_0^{n-2m}}{m! (k-m)! (n-k-m)!} \quad (7)$$

The coefficients  $c_{k,n}(z_0)$  were first given in this form by van der Maas [3], who derived them using a method different from that in the Appendix. By inspection, notice that  $c_{k,n}(z_0) > 0$  for all  $k$  whenever  $z_0 > 1$ . The coefficients  $c_{k,n}(z_0)$  yield the

element weights  $\{w_k\}_1^N$  when we define for  $N$  even:

$$w_{N-k+1} = w_k \triangleq \frac{1}{2} c_{t(k), N-1}(z_0) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} k = 1, 2, \dots, \frac{N}{2} \quad (8)$$

$$t(k) \triangleq \frac{N}{2} - k$$

for  $N$  odd:

$$w_{n-k+1} = w_k \triangleq \frac{1}{2} c_{t(k), N-1}(z_0) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} k = 1, 2, \dots, \frac{N+1}{2}. \quad (9)$$

$$t(k) \triangleq \frac{N+1}{2} - k$$

Thus, the complex transfer function (3) is given explicitly for these weights by

$$F(u) = e^{i\pi(N+1)u/2} T_{N-1}(z_0 \cos(\frac{1}{2}\pi u)); \quad (10)$$

the maximum response occurs for  $u = 0$ ,

$$F(0) = T_{N-1}(z_0); \quad (11)$$

and the smallest positive value of  $u$  such that  $F(u) = 0$  is given by

$$u_0 = \frac{2}{\pi} \arccos \left( \frac{1}{z_0} \cos \left( \frac{\pi}{2(N-1)} \right) \right) \quad (12)$$

The half beamwidth as measured to the first null from the MRA is precisely  $u_0$ .

The Gegenbauer design proceeds in an analogous fashion. We replace the old constant  $z_0$  by a new variable  $z_\mu$  which will be defined later (30); however, for  $\mu = 0$ ,  $z_\mu$  is still defined by (5). Now, in the expansion

$$C_n^\mu(z_\mu \cos u) = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{k,n}(z_\mu) \cos [(n-2k)u] \quad (13)$$

the coefficients  $b_{k,n}(z_\mu)$  depend on  $z_\mu$  and are given explicitly by

$$b_{k,n}(z_\mu) = 2(\mu)_{n-k} \sum_{m=0}^k \frac{(\mu+m)_{k-m} (z_\mu^2 - 1)^m z_\mu^{n-2m}}{m! (k-m)! (n-k-m)!} \quad (14)$$

Both of these identities are special cases of (A18) and (A19) of the Appendix. Note that  $b_{k,n}(z_0) > 0$  for all  $k$ , provided that  $z_\mu > 1$  and  $\mu > 0$ . Note also that, by (1), (14) reduces to (7) in the limit as  $\mu \rightarrow 0$ . For numerical computation, the following form is preferred to (14). Let  $A = 1 - z_\mu^{-2}$ , so that  $0 < A < 1$  when  $z_\mu > 1$ , and then compute the right-hand side of

$$\frac{b_{k,n}(z_\mu)}{2\mu z_\mu^n} = \frac{1}{n-k} \binom{\mu+n-k-1}{n-k-1} \sum_{m=0}^k \binom{\mu+k-1}{k-m} \binom{n-k}{m} A^m. \quad (15)$$

The binomial coefficients are defined here for any real number  $\alpha$  and any nonnegative integer  $p$  by

$$\binom{\alpha}{0} \triangleq 1, \binom{\alpha}{p} \triangleq \frac{\alpha(\alpha-1)\cdots(\alpha-p+1)}{p!}, \quad p \geq 1 \quad (16)$$

although they are best computed recursively using

$$\binom{\alpha}{p} = \frac{\alpha - p + 1}{p} \binom{\alpha}{p-1}, \quad p \geq 1 \quad (17)$$

to avoid floating point overflow at some intermediate point in the computation.

It should be pointed out that (15) can be evaluated numerically for all  $\mu$  since, for fixed  $n$  and  $k$ , (15) is a polynomial in  $\mu$ . However, (15) is correct only if  $\mu \neq -1, -2, -3, \dots$ . If coefficients are required for, say,  $\mu = -5$ , both sides of (13) must first be divided by  $\mu + 5$  and the limit taken as  $\mu + 5 \rightarrow 0$ . Consequently, in (15), the factor  $\mu + 5$  must be divided out algebraically before numerical computation begins.

The coefficients  $b_{k,n}(z_\mu)$  yield element weights  $\{w_k\}_1^N$  when we define

for  $N$  even:

$$\left. \begin{aligned} w_{N-k+1} &= w_k \triangleq \frac{1}{2} b_{t(k), N-1}(z_\mu) \\ t(k) &\triangleq \frac{N}{2} - k \end{aligned} \right\} k = 1, 2, \dots, \frac{N}{2} \quad (18)$$

for  $N$  odd:

$$\left. \begin{aligned} w_{N-k+1} &= w_k \triangleq \frac{1}{2} b_{t(k), N-1}(z_\mu) \\ t(k) &\triangleq \frac{N+1}{2} - k \end{aligned} \right\} k = 1, 2, \dots, \frac{N+1}{2} \quad (19)$$

With these weights, the complex transfer function (3) is given explicitly by

$$F(u) = e^{i\pi(N+1)u/2} C_{N-1}^\mu(z_\mu \cos(\frac{1}{2}\pi u)). \quad (20)$$

The maximum response of  $F(u)$  should occur for  $u = 0$ , and is

$$F(0) = C_{N-1}^\mu(z_\mu). \quad (21)$$

(For a discussion of unusual situations when the MRA might not occur at  $u = 0$ , see below in this section.)

The smallest positive value of  $u$  satisfying  $F(u) = 0$  is given by

$$u_\mu \triangleq \frac{2}{\pi} \arccos\left(\frac{1}{z_\mu} x_{N-1}^{(\mu)}\right) \quad (22)$$

where  $x_{N-1}^{(\mu)}$  is the largest zero of the Gegenbauer polynomial  $C_{N-1}^\mu(x)$ . Thus, for  $\mu > -1/2$ ,  $x_{N-1}^{(\mu)}$  must lie in the open interval  $(-1, +1)$ . In fact, it must be very near  $+1$  for values of  $\mu$  of interest in this application. An explicit analytic expression for  $x_{N-1}^{(\mu)}$  is not known except in certain special cases (e.g., the Chebyshev polynomials) and so must be solved for numerically. This minor difficulty is readily overcome using Newton-Raphson iteration. Recall  $n = N - 1$ . Since [1, eq. (4.7.14)]

$$\frac{d}{dx} C_n^\mu(x) = 2\mu C_{n-1}^{\mu+1}(x) \quad (23)$$

the Newton-Raphson iteration is

$$y_{k+1} = y_k - \frac{C_n^\mu(y_k)}{2\mu C_{n-1}^{\mu+1}(y_k)}, \quad k = 1, 2, \dots \quad (24)$$

$$y_1 \triangleq x_n^{(0)} = \cos\left(\frac{\pi}{2n}\right).$$

The ratio in (24) is perhaps best evaluated by computing two different sequences

$$\{C_p^\mu(y_k)\}_{p=1}^n \quad \text{and} \quad \{C_p^{\mu+1}(y_k)\}_{p=1}^n \quad (25)$$

numerically from the fundamental recursion [1, eq. (4.7.17)]

$$pC_p^\alpha(x) = 2(p+\alpha-1)x C_{p-1}^\alpha(x) - (p+2\alpha-2)C_{p-2}^\alpha(x), \quad p = 2, 3, 4, \dots, n \quad (26)$$

$$C_0^\alpha(x) = 1, \quad C_1^\alpha(x) = 2\alpha x.$$

The recursion (26) is valid for  $\alpha \neq 0, -1, -2, -3, \dots$ . This method may have weaknesses whenever  $\mu$  is very close to 0 (say,  $|\mu| \leq 10^{-4}$ ) because of the division by  $\mu$  in (24); however,  $\mu$  would normally be taken either equal to 0 (to give the Dolph-Chebyshev design) or else sufficiently different from 0 to affect sidelobe levels appreciably. This latter stipulation seems to require  $|\mu| > 10^{-4}$ . In the antenna application, then, computation of the Newton-Raphson iteration step from the recursion (26) seems perfectly safe whenever a special precaution is taken for  $\mu = 0$ . In practice this author has never seen the iteration require more than four steps, and he has never seen it converge to the wrong point. If, however, it should ever happen to converge to the wrong point, the Newton-Raphson iteration can be restarted with the new initial point  $y_1 = 1$ . Also, the inequality [1, eq. (6.21.3)]

$$\frac{\partial}{\partial \mu} x_n^{(\mu)} < 0 \quad \text{for all } \mu \quad (27)$$

implies that

$$x_n^{(\mu)} < x_n^{(0)} = \cos\left(\frac{\pi}{2n}\right) < x_n^{(\mu)}, \quad \mu > 0, \quad (28)$$

which can serve as a check. Incidentally, inequality (27) holds for *all* the positive zeros  $C_n^\mu(x)$ , not merely the largest one.

The reason for all this concern over calculation of the half-beam width (22) is simply to be able to make fair comparisons between sidelobe levels of different Gegenbauer designs, that is, different values of  $\mu$ . It is well known that the sidelobe levels in Dolph-Chebyshev beam patterns are sensitive functions of the beamwidth, and there is every reason to expect similar behavior in the Gegenbauer designs. Therefore, as  $\mu$  is varied it is helpful to maintain a fixed beamwidth; specifically, we always require  $u_\mu = u_0$  for all  $\mu$ . This in turn, from (22) and (12), gives

$$\frac{1}{z_\mu} x_{N-1}^{(\mu)} = \frac{1}{z_0} \cos\left(\frac{\pi}{2(N-1)}\right) \quad (29)$$

or, converting convenience into a definition,

$$z_\mu \triangleq z_0 x_{N-1}^{(\mu)} \sec\left(\frac{\pi}{2(N-1)}\right). \quad (30)$$

From (30) it is now clear that computing the largest zero,  $x_{N-1}^\mu$ , of  $C_{N-1}^\mu(x)$  is of considerable importance.

With the definition (30), all Gegenbauer designs with different values of  $\mu$  and fixed  $z_0$  have the same beamwidth as measured to the first null off the MRA. Thus, the beamwidth is varied simply by changing the value of  $z_0$  in exactly the same way as in Dolph-Chebyshev, i.e., (5).

An interesting consequence of (30) is that  $z_\mu$  might not always

be greater than 1 for all  $\mu \geq 0$ . This observation follows immediately from the derivative (27). Hence, for some critical positive value of  $\mu$ , say  $\mu^*$ , we have  $z_{\mu^*} = 1$ . In (15) the number  $A$  is negative for  $\mu > \mu^*$ , so the positivity of the weights cannot be guaranteed without direct calculation because (15) is an alternating series for  $\mu > \mu^*$ . At the critical point  $\mu^*$ ,  $A = 0$  and the sum in (15) collapses to a single term. Simplifying gives

$$b_{k,n}(z_{\mu^*}) = 2 \binom{n-k+\mu^*-1}{n-k} \binom{k+\mu^*-1}{k} \quad (31)$$

which can be found also in Szegő [1, eq. (4.9.19)]. The weights for the critical case  $\mu = \mu^*$  can now be varied merely by changing  $\mu^*$ . In particular, for  $\mu^* = 1$ , (31) gives the uniformly weighted array; that is,  $\omega_k = 1$  for all  $k$ . The beamwidth obtained from the weights (31) depends on (and only on) the critical value  $\mu^*$  because  $\mu^*$  implicitly depends on  $z_0$ .

Since Gegenbauer designs have the two parameters  $z_0$  and  $\mu$ , with  $z_0$  controlling main lobewidth, the parameter  $\mu$  must control sidelobe behavior. From (20) and (30) we see that sidelobes occur for  $u$  satisfying

$$|z_\mu \cos(\frac{1}{2} \pi u)| \leq \cos(\frac{1}{2} \pi u_0) < 1. \quad (32)$$

In the sidelobe region, then, we can define

$$\cos \phi = z_\mu \cos(\frac{1}{2} \pi u), \quad 0 < \phi < \pi.$$

For the moment let us suppose  $0 < \mu < 1$ . Then, from Szegő [1, eq. (7.33.5)]

$$(\sin \phi)^\mu |C_n^\mu(\cos \phi)| < 2^{1-\mu} n^{\mu-1} / \Gamma(\mu) \quad (33)$$

so the transfer function  $F(u)$  must satisfy

$$|F(u)| < (1 - z_\mu^2 \cos^2(\frac{1}{2} \pi u))^{-\mu/2} 2^{1-\mu} n^{\mu-1} / \Gamma(\mu) \quad (34)$$

throughout the sidelobe region defined by (32). For  $\mu$  outside the  $(0, 1)$  interval, but excluding  $\mu = 0, -1, -2, \dots$ , the sharpness of the inequality (34) is lost. A special case of a result given in Szegő [1, eq. (8.21.14) with  $p = 1$ ] implies that

$$|F(u)| \leq (1 - z_\mu^2 \cos^2(\frac{1}{2} \pi u))^{-\mu/2} 2^{1-\mu} \binom{n+\mu-1}{n} + O(n^{\mu-2}) \quad (35)$$

throughout the sidelobe region defined by (32). For  $\mu$  outside (35) is asymptotic to  $n^{\mu-1} / \Gamma(\mu)$  as  $n \rightarrow \infty$ , so the leading term of the right-hand side of (35) is asymptotic to the right-hand side of (34). For fixed  $\mu$ , the right-hand side of (35) appears to be an excellent envelope for the sidelobes of the Gegenbauer designs.

For  $\mu > 0$ , it is clear from (35) that the sidelobe envelope must steadily decay as  $u$  approaches endfire, i.e.,  $u = 1$ . Since [use (26)]

$$C_n^\mu(0) = \begin{cases} 0, & \text{if } n \text{ odd} \\ (-1)^m \binom{\mu+m-1}{m}, & \text{if } n = 2m \end{cases}$$

we have

$$|F(1)| = |C_n^\mu(0)| \approx 2^{1-\mu} n^{\mu-1} / \Gamma(\mu) \quad (36)$$

approximately, for  $n$  even. Contrasting this approximation with

the inequality (35) leads to the conclusion that (36) is an excellent approximation to the sidelobe envelope for both even and odd  $n$ . Thus, we utilize (36) for all  $n$ . Applying results proved below in another context [specifically, set  $v = 0$  in (54) and (55)] gives an approximation for the maximum response

$$|F(0)| \approx \frac{2^{1-\mu} n^{2\mu-1}}{\Gamma(\mu)} \sqrt{\frac{\pi}{2}} \frac{I_{\mu-1/2}(\tau)}{\tau^{\mu-1/2}} \quad (37)$$

with  $r$  defined by (5) and

$$\tau \triangleq [(\operatorname{arccosh} r)^2 + \pi^2/4 - j_{\mu-1/2}^2]^{1/2} \quad (38)$$

where  $j_{\mu-1/2}$  is the smallest positive zero of the Bessel function  $J_{\mu-1/2}(x)$  of the first kind and order  $\mu - 1/2$ , and  $I_{\mu-1/2}(x)$  is the modified Bessel function of the first kind and order  $\mu - 1/2$ . Therefore, we have the relative level

$$\frac{|F(1)|}{|F(0)|} \approx n^{-\mu} \sqrt{\frac{2}{\pi}} \left\{ \frac{I_{\mu-1/2}(\tau)}{\tau^{\mu-1/2}} \right\}^{-1}. \quad (39)$$

This result happens to be exact for  $\mu = 0$ , the Dolph-Chebyshev case, as can be easily verified. Evidently this result also implies that the sidelobe height at endfire is a function of  $n$ , even when  $\mu$  and the beamwidth parameter  $z_0$  are fixed. In other words, the sidelobe tapering effect of a given value of  $\mu$  depends on  $n$ , unless  $\mu = 0$ . Numerical examples bear out the  $n^{-\mu}$  dependence in (39).

An important observation based on (35) and (39) is that for  $\mu < 0$  the sidelobes may well steadily increase as  $\mu$  approaches endfire. That this is in fact the case is borne out by the examples given later.

It should be emphasized that although the Gegenbauer weights must be positive if  $0 \leq \mu \leq \mu^*$ , they might not necessarily be positive if  $\mu < 0$  or if  $\mu > \mu^*$ . For  $\mu < 0$  it can happen that all are positive, or that some are negative. Only numerical computation can show which is the case. If some of the weights are negative, it becomes a possibility that the maximum response might not occur for  $u = 0$ .

For the Gegenbauer weights it is readily shown that a sufficient condition for the MRA to be at  $u = 0$  is that  $C_n^\mu(x)$  attain its maximum over the interval  $[-1, 1]$  at  $x = 1$ . By a well-known result [1, eq. (7.33.1)] the maximum of  $C_n^\mu(x)$  occurs at  $x = 1$  if and only if  $\mu > 0$ . Thus, a sufficient condition for  $u = 0$  to be the MRA is that  $\mu \geq 0$ . For  $\mu < 0$  the MRA depends on the size of  $z_\mu$  and must be verified numerically. From [1, eq. (7.33.1)] the maximum of  $C_n^\mu(x)$  occurs at or near  $x = 0$  when  $\mu < 0$ ; therefore, if the MRA is not at  $u = 0$ , then the MRA must be at or near endfire. This observation is rendered quite reasonable when considered in the light of the examples presented later. This author has never experienced a case where the MRA was not  $u = 0$  for  $\mu > -1/2$  and reasonable values of  $z_\mu$ .

It would be interesting to know how much energy is contained in the main lobe of a Gegenbauer design. From (20) and (30), this requires a tractable form for the integral

$$\int_0^{u_0} [C_n^\mu(z_\mu \cos(\frac{1}{2} \pi u))]^2 du \quad (40)$$

which we do not have. On the other hand, the total "weighted" energy contained in all of the sidelobes is the smallest possible

for  $\mu > -1/2$ . Specifically, if  $\pi_{n-1}$  denotes a polynomial of degree at most  $n-1$ , then [1, eq. (4.7.15)] for  $\mu > -1/2$

$$\begin{aligned} \min_{\pi_{n-1}} \int_{-1}^1 (1-x^2)^{\mu-1/2} [x^n - \pi_{n-1}(x)]^2 dx \\ = \frac{2^{1-2\mu} \pi}{[\Gamma(\mu)]^2} \frac{\Gamma(n+2\mu)}{(n+\mu)\Gamma(n+1)}, \quad (\mu \neq 0). \end{aligned} \quad (41)$$

Furthermore, if  $\tilde{\pi}_{n-1}(x)$  is the minimizing polynomial, then [1, eq. (4.7.9)]

$$C_n^\mu(x) = 2^n \binom{n+\mu-1}{n} (x^n - \tilde{\pi}_{n-1}(x)).$$

Substituting  $x = z_\mu \cos(\pi u/2)$  thus establishes our claim. However, a problem with this formulation is that part of the main lobe energy is included in the total weighted sidelobe energy. The reason is that the  $x$ -interval  $[x_n^{(\mu)}, +1]$  is transformed [use (12) and (30)] to the  $u$ -interval

$$\left[ \frac{2}{\pi} \arccos\left(\frac{1}{z_0 x_n^{(\mu)}}\right) \cos\left(\frac{\pi}{2n}\right), u_0 \right] \quad (42)$$

which is a subset of the main lobe region. For the Dolph-Chebyshev case  $\mu = 0$ , this  $u$ -interval goes from the first null up to the point on the main lobe equal to the overall sidelobe level and, so, is not considerable. For larger values of  $\mu$ , this  $u$ -interval grows larger because of (27) and thus contributes progressively more significant portions to the weighted sidelobe energy estimate.

### III. GEGENBAUER WEIGHTS FOR A CONTINUOUS APERTURE

The Gegenbauer weights derived for the discrete finite aperture have a limiting form as  $n \rightarrow \infty$  with total aperture length  $2L$  held constant. This is essentially the high-frequency limit of the weights as functions of design frequency. The limiting form is a continuous real-valued function defined on the whole aperture and must be nonnegative if  $0 < \mu < \mu^*$ . The case  $\mu = 0$  develops  $\delta$ -function spikes at the aperture endpoints; i.e., the case  $\mu = 0$  gives the van der Maas function. For  $\mu > \mu^*$  the limit is still continuous, but we cannot guarantee by simple inspection that it is nonnegative across the entire aperture. For  $\mu < 0$ , the integral (60) below diverges.

Let the continuous aperture be taken to be the closed interval  $[-L, L]$  on the  $x$ -axis. Rewriting (3) gives

$$F(u) = \int_1^N W_0(x) \exp(-i\pi x u) dx \quad (43)$$

where

$$W_0(x) \triangleq \sum_{k=1}^N w_k \delta(x-k). \quad (44)$$

(The integral in (43) includes *all* of the impulses at 1 and  $N$ .) Scaling the interval  $[1, N]$  to the given aperture  $[-L, L]$  and using the fact that the weights  $\{w_k\}_1^N$  are symmetric gives

$$G(v) = 2 \int_0^L W(\xi) \cos(\xi v) d\xi \quad (45)$$

where

$$n = N - 1 \quad (46)$$

$$x = \frac{n\xi}{2L} + \frac{n+2}{2} \quad (47)$$

$$W(\xi) = \frac{n}{2L} \sum_{k=1}^N w_k \delta\left[\xi - \left(k - 1 - \frac{n}{2}\right) \frac{2L}{n}\right] \quad (48)$$

$$v = \frac{\pi n u}{2L} \quad (49)$$

$$G(v) = e^{i\pi(n+2)u/2} F(u). \quad (50)$$

From (20), for the Gegenbauer weights,

$$e^{i2L(n+2)v/n} G(v) = C_n^\mu\left(z_\mu \cos\left(\frac{Lv}{n}\right)\right). \quad (51)$$

In order to take the limit in (51) as  $n \rightarrow \infty$ , we need to establish the asymptotic behavior

$$z_\mu \cong \sec\left(\frac{L\tau'}{n}\right), \quad n \rightarrow \infty \quad (52)$$

$$\tau' \triangleq \frac{\tau}{L} \quad (53)$$

where  $\tau$  is defined by (38). The proof uses the asymptotic results

$$z_0 = \cosh\left(\frac{1}{n} \operatorname{arccosh} r\right) \quad (54)$$

$$\cong 1 + \frac{(\operatorname{arccosh} r)^2}{2n^2}, \quad n \rightarrow \infty$$

$$\cong \sec\left(\frac{1}{n} \operatorname{arccosh} r\right), \quad n \rightarrow \infty \quad (55)$$

and

$$x_n^{(\mu)} \cong \cos\left(\frac{j_{\mu-1/2}}{n}\right), \quad n \rightarrow \infty. \quad (56)$$

Apparently (54) was first given in [6]; it follows directly from the definition of the Chebyshev polynomials and the fact that  $r > 1$ . On the other hand, (56) follows from the Mehler-Heine result, (A2) of the Appendix, by specializing it to the Gegenbauer polynomials using [1, eq. (4.7.1)]. Now, from (30),

$$z_\mu \cong \sec\left(\frac{1}{n} \operatorname{arccosh} r\right) \cos\left(\frac{j_{\mu-1/2}}{2}\right) \sec\left(\frac{\pi}{2n}\right), \quad n \rightarrow \infty$$

$$\cong \sec\left(\frac{L\tau'}{n}\right), \quad n \rightarrow \infty$$

with  $\tau'$  defined by (53). We point out that if  $\tau'$  is pure imaginary, then the hyperbolic secant can replace the secant in (52). The possibility of imaginary  $\tau'$  does not affect the validity of the following argument.

Finally, from (51), normalizing by the factor  $n^{1-2\mu}/(2\mu)$  to

keep  $G(v)$  bounded gives

$$\begin{aligned}
 H(v) &\triangleq \lim_{n \rightarrow \infty} \frac{n^{1-2\mu}}{2\mu} e^{i2L(n+2)v/n} G(v) \\
 &= \lim_{n \rightarrow \infty} \frac{n^{1-2\mu}}{2\mu} C_n^\mu \left( \frac{\cos \frac{Lv}{n}}{\cos \frac{L\tau'}{n}} \right) \\
 &= \frac{\sqrt{\pi/2}}{2^\mu \Gamma(\mu+1)} \frac{J_{\mu-1/2}(L\sqrt{v^2-\tau'^2})}{(L\sqrt{v^2-\tau'^2})^{\mu-1/2}}
 \end{aligned} \tag{57}$$

where (58) is merely (A6) of the Appendix. Thus, (58) gives the beam pattern of the continuous Gegenbauer weighting function on the interval  $[-L, L]$ . The first null of  $H(v)$  is

$$v_0 = \frac{1}{L} [(\operatorname{arccosh} r)^2 + \pi^2/4]^{1/2} \tag{59}$$

which is derived from (58) by using (53). Note that  $v_0$  is independent of  $\mu$  because of (30).

The beam pattern (58) is easier to derive than the continuous Gegenbauer weighting function. Although one can find the Fourier transform of (58) as a special case of Sonine's second finite integral, (A1), the assertion that this transform is indeed the limit of the Gegenbauer weights for a discrete aperture requires a separate proof. Conceivably the Gegenbauer weights might diverge even though the limit (58) exists. This in fact happens only for  $\mu \leq 0$ . The proof constitutes about half the attention of the Appendix; see especially (A8), (A22), (A26), (A27), and (A29). The final answer can be found by specializing (A29), using (A25), to yield

$$\begin{aligned}
 H(v) &= \frac{1}{2^\mu \Gamma(\mu+1) (L\tau')^{\mu-1}} \int_0^1 (\sqrt{1-\xi^2})^{\mu-1} \\
 &\quad I_{\mu-1}(L\tau' \sqrt{1-\xi^2}) \cos Lv\xi d\xi.
 \end{aligned} \tag{60}$$

The continuous Gegenbauer weighting function on the aperture is obvious on setting  $\xi = L\xi$ . The continuous Gegenbauer function depends on the parameter  $\mu$ , which we must restrict to  $\mu \geq 0$  for the integral to converge [see (A23)]. It also depends on the beamwidth parameter  $z_0$  through the variable  $\tau'$  defined by (53).

The Kaiser-Bessel window is a special case of (60), as is easily seen by setting  $\mu = 1$ . Since the Gegenbauer polynomials  $C_n^\mu(x)$  for  $\mu = 1$  are, from (2), the Chebyshev polynomials of the second kind, it is clear that Kaiser-Bessel must be their continuous analog. Also, our claim that the van der Maas weighting function is a limiting case of (60) as  $\mu \rightarrow 0$  can be seen from

$$\lim_{\mu \rightarrow 0^+} x^{\mu-1} I_{\mu-1}(x) = \frac{I_1(x)}{x} + 2\delta(x). \tag{61}$$

Substituting  $L\tau' \sqrt{1-\xi^2}$  for  $x$  in (61) and then substituting in (60) yields the van der Maas function. The result (61) was pointed out to the author by A. H. Nuttall in a private communication [7] while the present paper was being drafted.

Note that the beam pattern function (58) is a well-defined function of  $v$  for all real and complex values of  $\mu$  (in fact, it is an entire function of  $v$  for all  $\mu$ ) so that it can be computed and inspected in the absence of any corresponding weighting function. In particular, for negative  $\mu$  the beam pattern function (58) grows with increasing  $v$  just as might be expected from the discrete aperture case. However, the beam pattern (58) for  $\mu \leq 0$  is not realizable as the cosine transform of a continuous function on the closed interval, or aperture,  $[-L, L]$ .

#### IV. EXAMPLES

The five examples presented here are for the discrete aperture with 100 elements at a half wavelength spacing and steered broadside. The half beamwidth, measured from the MRA to the first null, is  $2.565588^\circ$  and is the same for all five examples. This is accomplished by defining  $z_\mu$  as in (30) and computing it in the manner described in detail in Section II, (23)-(28). The remaining free parameter,  $\mu$ , we take equal to 0.4, 0.2, 0.0, -0.2, -0.4, successively. The Gegenbauer weights are computed in the suggested form (15), and the resulting beam patterns for these five values of  $\mu$  are given in Figs. 1-5, respectively. The independent variable in these patterns is the angle  $\theta_a$ , not  $u$ ; the vertical axis is  $20 \log_{10} |F(\sin \theta_a)|$ .

Perhaps the most prominent feature of these five beam patterns is that the sidelobe structure for a fixed positive value of  $\mu$  is "reciprocal" to that for  $-\mu$ . Consider  $\mu = \pm 0.4$ , for instance. If the reader takes a Xerox of both beam patterns and turns one of them upside down on top of the other (literally) and holds the pair up to the light, then it will be abundantly clear what "reciprocal" means in this context. The cause of this attractive matching of sidelobe envelopes is that the bound (35) is, in fact, very reflective of true sidelobe taper. Thus, for positive  $\mu$  the sidelobes decay, while for negative  $\mu$  the sidelobes grow. For  $\mu = 0$  the sidelobes neither grow nor decay; they remain constant. The case  $\mu = 0$  is, of course, the Dolph-Chebyshev design. The author has not undertaken any further studies to determine the accuracy of the sidelobe envelope factor.

Another important feature is that the first sidelobe *alone* seems to be extremely important in determining the possible size of the remaining sidelobes. Although this is not a rigorous statement, it does seem to be borne out by these examples. For  $\mu = 0.2$  the first sidelobe is increased by about 1 dB to -29 dB, the second sidelobe seems unchanged at -30 dB, and all the remaining sidelobes are uniformly (and progressively) lower than the -30 dB Dolph-Chebyshev case ( $\mu = 0$ ) with the last sidelobe depressed about 34 dB. Similar but "reciprocal" remarks hold for the  $\mu = -0.2$  case. For  $\mu = 0.4$  ( $\mu = -0.4$ ) the second sidelobe is slightly higher (lower) than -30 dB, but the point made here is still substantially true.

The weights for the cases  $\mu = 0.4, 0.2$ , and  $0.0$  are all positive. For the cases  $-0.2$  and  $-0.4$ , the only negative weights corresponded to the elements adjacent to the end elements.

All five examples have 49 sidelobes on either side of the MRA. This can be attributed to the fact that the Gegenbauer polynomial  $C_n^\mu(x)$  has all its  $n$  zeros in the open interval  $(-1, +1)$  when  $\mu > -1/2$ . Thus, from (20),  $F(u)$  must have  $N-1 = 99$  zeros in the open  $u$  interval  $(0, 2)$ . By Rolle's theorem of elementary calculus,  $F(u)$  must have 98 points (i.e., sidelobe peaks) interior to  $(0, 2)$  where  $|F'(u)| = 0$ . Since  $|F(u)|$  is an even func-

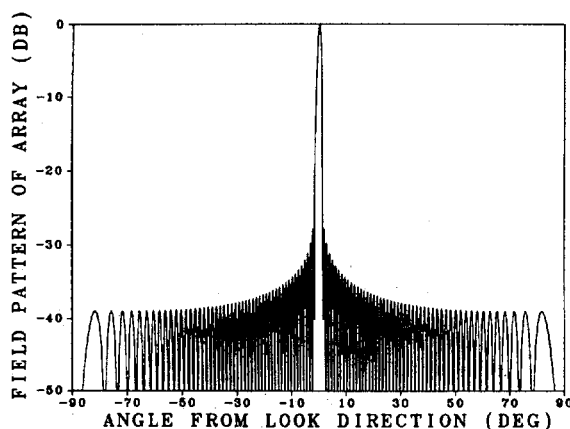


Fig. 1. Gegenbauer 100 element array;  $\mu = 0.4$ ; first null =  $2.565588^\circ$ .

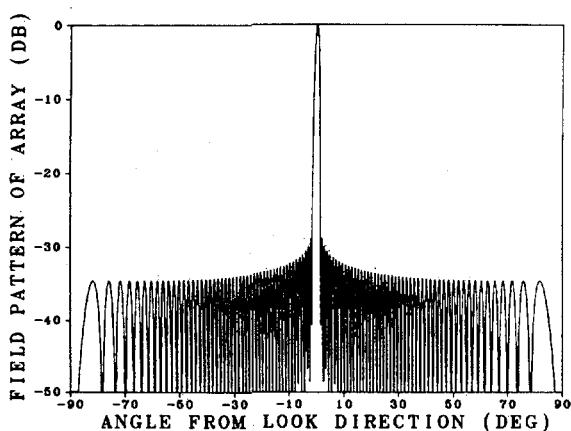


Fig. 2. Gegenbauer 100 element array;  $\mu = 0.2$ , first null =  $2.565588^\circ$ .

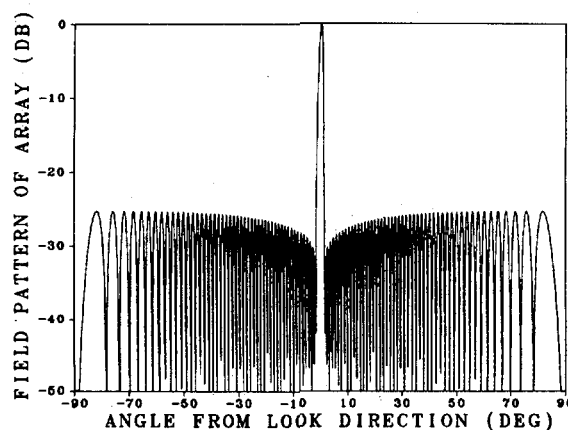


Fig. 4. Gegenbauer 100 element array;  $\mu = -0.2$ ; first null =  $2.565588^\circ$ .

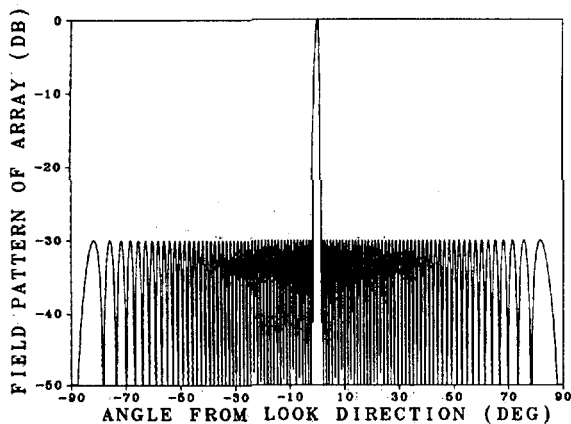


Fig. 3. Gegenbauer 100 element array;  $\mu = 0$ ; first null =  $2.565588^\circ$ . (This is classic Dolph-Chebyshev.)

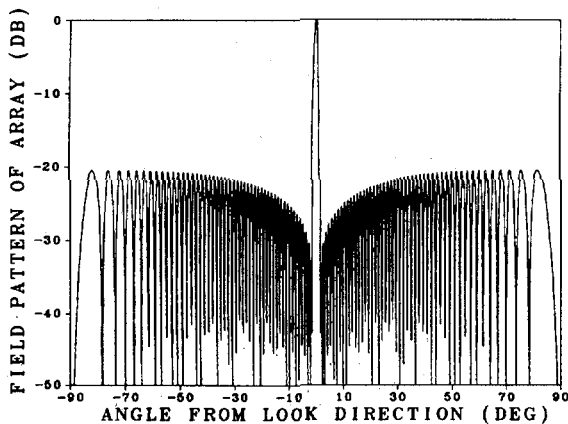


Fig. 5. Gegenbauer 100 element array;  $\mu = -0.4$ ; first null =  $2.565588^\circ$ .

tion of  $u$ , half of these sidelobes must be on each side of the MRA.

All five examples exhibit a plateau in the decay, or growth, of sidelobes at sufficiently great distances from the MRA. This feature is also an artifact of the sidelobe envelope factor (35).

Taken together, these examples indicate that the ratio (39) is, on a log plot, roughly linear in  $\mu$  for fixed  $n$  and beamwidth parameter  $z_0$ . Whether this linearity is true only for reasonably

small values of  $\mu$  has not been determined. A careful mathematical proof of approximate  $\mu$  linearity of the logarithm of (39) would be nice to have.

## V. DISCUSSION AND SUMMARY

The Gegenbauer weighting functions for the discrete and continuous aperture, as well as for nonrecursive digital filters, permits the designer to maintain a fixed specified beamwidth as

defined via (30) while scanning continuously in  $\mu$  to discriminate against spatially distributed noise sources and/or extraneous signals by tapering the sidelobes. The required weights can be calculated quickly and accurately by the analytic formulas provided here; hence, it might be possible to choose  $\mu$  adaptively to achieve some objective such as maximizing signal-to-noise ratio. The beam patterns for negative  $\mu$  are particularly interesting in that it may be possible to discriminate against noise sources that lie nearby (in bearing) the desired signal source, and thereby enhance tracking capability.

One advantage of the Gegenbauer weights is that they are derived for a discrete aperture exactly, and the continuous aperture weighting function is then discovered as their limit. If only a continuous aperture function is defined, then it must be sampled at a finite set of points in any application to a discrete aperture. How this sampling is best done is not commonly discussed, and it leaves a certain ambiguity in the discrete aperture weights. The discrete Gegenbauer weights given by (18) and (19) above do not have this problem.

When steering a Gegenbauer array design, no different problems should arise than what is normally expected in the usual Dolph-Chebyshev design. Gegenbauer designs can be steered nearly to endfire before encountering the first grating lobe.

A difference beam pattern can be constructed from the Gegenbauer weights in the usual way of changing the signs of the weights on one-half of the array. If this is done, the difference beam pattern is proportional to  $|C_n^\mu(z_\mu \sin(\pi u/2))|$ . This is easy to show from the constructions (18)-(20). The result is a beam pattern with a null at  $u = 0$ .

All the nulls of the Gegenbauer beam pattern seem to shift strictly away from the MRA as  $\mu$  increases. This effect is evident in the examples. It is quite possible to use this effect to deliberately control null placement to cancel localized noise sources. A mathematical proof that the nulls must shift in this manner requires knowledge of the relative size of the derivatives (with respect to  $\mu$ ) of all of the zeros of  $C_n^\mu(x)$ . Although this information is not known to the author, it is not really necessary to have it in order to utilize the null shifting effect in practice.

The Gegenbauer weights for discrete and continuous apertures was derived by the author between March and May 1981. The mathematical results contained in the Appendix first appeared in [11].

## APPENDIX

### MATHEMATICAL DERIVATIONS AND RESULTS

Sonine's second finite integral [8, p. 376] may be written

$$\int_0^{\pi/2} J_\mu(x \sin \theta) J_\lambda(y \cos \theta) \sin^{\mu+1} \theta \cos^{\lambda+1} \theta d\theta \\ = \frac{x^\mu y^\lambda J_{\mu+\lambda+1}(\sqrt{x^2+y^2})}{(\sqrt{x^2+y^2})^{\mu+\lambda+1}} \quad (\text{A1})$$

for all complex  $x$  and  $y$ , and is valid when both  $\text{Re}(\mu) > -1$  and  $\text{Re}(\lambda) > -1$ . At least three proofs of this result are known. One involves expanding the integral in powers of  $x$  and  $y$ ; another involves integration over subsets of the surface of the unit sphere in  $R^3$ . Both are given in [8]. The third proof using the generalized Laguerre polynomials  $L_n^{(\alpha)}(x)$  is mentioned in [12].

For the case of real  $\mu$  and  $\lambda$ , a fourth proof is given here that depends in an essential way on the identity (A7). In this connection, the particular form of the coefficients  $a_{k,n}(y)$  is important; that is, the easily derived identity (A10) does not seem to be all useful, but the identity (A8) is exactly what is needed. It facilitates the investigation of the limiting form (A27) of  $a_{k,n}(y)$  as  $n$  tends to infinity. The identity (A8) is apparently new; however, the special case of  $y = 1$  was known to Gegenbauer.

Equation (A8) is interesting in another regard as well. A simple inspection suffices to prove that  $a_{k,n}(y) > 0$  for all  $n$  and  $k$  whenever  $y > 1$  and  $\mu \geq \lambda > 0$ . The coefficients remain positive in the two limiting cases  $\mu > 0, \lambda = 0$  and  $\mu = \lambda = 0$ , as can be seen from (A18)-(A21). In fact, it was only this positivity result that the author originally sought.

The result (A3) of the Mehler-Heine type is apparently new. It is needed to prove (A1) by our methods. It has additional interest in that it duplicates the result given by Szegő (A2) simply by setting  $y = 0$ . Mathematically, however, (A2) and (A3) are equivalent. The special cases (A4a) and (A4b) involving Chebyshev polynomials are particularly striking.

Let  $\alpha$  and  $\beta$  be arbitrary real numbers. For any complex number  $x$ , the Mehler-Heine theorem states that

$$\lim_{n \rightarrow \infty} n^{-\alpha} P_n^{(\alpha, \beta)} \left( \cos \frac{x}{n} \right) = (x/2)^{-\alpha} J_\alpha(x) \quad (\text{A2})$$

where  $J_\alpha(x)$  is the Bessel function of the first kind of order  $\alpha$  [1, eq. (1.71.1)], [2, sect. 3.1(8)]. A straightforward proof of (A2) can be found in Szegő [1, Theorem 8.1.1]. Szegő's proof can be readily modified to show that

$$\lim_{n \rightarrow \infty} n^{-\alpha} P_n^{(\alpha, \beta)} \left( \frac{\cos \frac{x}{n}}{\cos \frac{y}{n}} \right) = \left( \frac{1}{2} \sqrt{x^2 - y^2} \right)^{-\alpha} J_\alpha(\sqrt{x^2 - y^2}) \quad (\text{A3})$$

for all complex  $x$  and  $y$ . Like the Mehler-Heine result, this formula holds uniformly for  $x$  and  $y$  in every bounded region of the complex plane. The special case  $\alpha = \beta = -1/2$  gives the interesting result

$$\lim_{n \rightarrow \infty} T_n \left( \frac{\cos \frac{x}{n}}{\cos \frac{y}{n}} \right) = \cos \sqrt{x^2 - y^2} \quad (\text{A4a})$$

where  $T_n(x)$  is the Chebyshev polynomial of the first kind [1, eq. (4.1.7)], while the special case  $\alpha = \beta = 1/2$  gives

$$\lim_{n \rightarrow \infty} n^{-1} U_n \left( \frac{\cos \frac{x}{n}}{\cos \frac{y}{n}} \right) = \frac{\sin \sqrt{x^2 - y^2}}{\sqrt{x^2 - y^2}} \quad (\text{A4b})$$

where  $U_n(x)$  is the Chebyshev polynomial of the second kind [1, eq. (4.1.7)]. These follow from (A3) by using Stirling's



formula and the well-known results [1, eq. (1.71.2)]

$$J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z, \quad J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z. \quad (\text{A5})$$

We will need another special case of the general result; specifically, for  $\mu > -1$ ,

$$\lim_{n \rightarrow \infty} \frac{n^{1-2\mu}}{2\mu} C_n^\mu \left( \frac{\cos \frac{x}{n}}{\cos \frac{y}{n}} \right) = \sqrt{\pi/2} \frac{J_{\mu-1/2}(\sqrt{x^2-y^2})}{2^\mu \Gamma(\mu+1) (\sqrt{x^2-y^2})^{\mu-1/2}} \quad (\text{A6})$$

where  $C_n^\mu(x)$  are the ultraspherical, or Gegenbauer, polynomials [1, eq. (4.7.1)]. (Szegő uses the notation  $P_n^{(\mu)}(x)$  instead of  $C_n^\mu(x)$ .)

We derive Sonine's second finite integral by finding an alternate form for the left-hand side of (A6). This requires the following result. For  $\mu \geq \lambda > 0$ , the coefficients  $a_{k,n}(y)$  in the expansion

$$C_n^\mu(xy) = \sum_{k=0}^{\lfloor n/2 \rfloor} a_{k,n}(y) C_n^{\lambda-2k}(x), \quad n=0, 1, 2, \dots \quad (\text{A7})$$

are given explicitly by

$$a_{k,n}(y) = (n-2k+\lambda) (\mu)_{n-k} \sum_{m=0}^k \frac{(\mu-\lambda+m)_{k-m} (y^2-1)^m y^{n-2m}}{m! (k-m)! (\lambda)_{n-k-m+1}} \quad (\text{A8})$$

where we take  $0^0 = 1$  and  $(0)_0 = 1$  whenever they occur. Setting  $y = 1$  in (8) gives

$$a_{k,n}(1) = \frac{(n-2k+\lambda) (\mu)_{n-k} (\mu-\lambda)_k}{k! (\lambda)_{n-k+1}} \quad (\text{A9})$$

which is due to Gegenbauer [1, eq. (4.10.27)]. Furthermore, for real  $y > 1$  and  $\mu \geq \lambda > 0$ , the coefficients  $a_{k,n}(y)$  are all positive as can be seen by inspection in (A8).

The formula (A.8) is derived as follows. Let  $\mu \geq \lambda > 0$ . In the expression [1, eq. (4.7.31)]

$$C_n^\mu(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{(\mu)_{n-m}}{m! (n-2m)!} (2x)^{n-2m}$$

we replace  $x$  with  $xy$ , substitute

$$\frac{(2x)^{n-2m}}{(n-2m)!} = \sum_{s=0}^{\lfloor (n-2m)/2 \rfloor} \frac{(n-2m+\lambda-2s)}{s! (\lambda)_{n-2m-s+1}} C_{n-2m-2s}^\lambda(x),$$

and collect terms to get

$$a_{k,n}(y) = \sum_{m=0}^k (-1)^m \frac{(n-2k+\lambda) (\mu)_{n-m} y^{n-2m}}{m! (k-m)! (\lambda)_{n-m-k+1}} \quad (\text{A10})$$

$$= y^{n-2k} (n-2k+\lambda) Q_k(2y^2-1) \quad (\text{A11})$$

where  $Q_k$  is a polynomial defined for general complex argument  $u$  by

$$Q_k(u) = \sum_{m=0}^k \frac{(-1)^m (\mu)_{n-m}}{m! (k-m)! (\lambda)_{n-m-k+1}} \cdot \left(\frac{u+1}{2}\right)^{k-m} \quad (\text{A12})$$

For arbitrary  $\alpha$  and  $\beta$ , the Jacobi polynomial of degree  $k \geq 0$  can be written

$$P_k^{(\alpha,\beta)}(u) = \sum_{m=0}^k (-1)^m \frac{(k+\alpha+\beta+1)_{k-m} (k-m+\beta+1)_m}{m! (k-m)!} \cdot \left(\frac{u+1}{2}\right)^{k-m} \quad (\text{A13})$$

which follows from [1, eq. (4.21.2)] using the identity [1, eq. (4.1.3)]. Setting  $\alpha = \mu - \lambda - 1$  and  $\beta = \lambda + n - 2k$  in (A13) shows that

$$Q_k(u) = \frac{(\mu)_{n-k}}{(\lambda)_{n-k+1}} P_k^{(\mu-\lambda-1, \lambda+n-2k)}(u) \quad (\text{A14})$$

Expanding the Jacobi polynomial in (A14) using [1, eq. (4.3.2)]

$$P_k^{(\alpha,\beta)}(u) = \sum_{m=0}^k \frac{(1+\alpha)_k (1+\beta)_k}{m! (k-m)! (1+\alpha)_m (1+\beta)_{k-m}} \cdot \left(\frac{u-1}{2}\right)^m \left(\frac{u+1}{2}\right)^{k-m} \quad (\text{A15})$$

and substituting  $u = 2y^2 - 1$  gives

$$Q_k(2y^2-1) = (\mu)_{n-k} \sum_{m=0}^k \frac{(\mu-\lambda+m)_{k-m} (y^2-1)^m y^{2k-2m}}{m! (k-m)! (\lambda)_{n-k-m+1}} \quad (\text{A16})$$

Thus, (A16) and (A11) establish (A8).

Two limiting cases of (A7) are easily derived from [1, eq. (4.7.8)]

$$\lim_{\lambda \rightarrow 0} \frac{n}{2\lambda} C_n^\lambda(x) = T_n(x), \quad n \geq 1 \quad (\text{A17})$$

and are worth recording. Thus, for  $\mu > 0$ ,

$$C_n^\mu(xy) = \sum_{k=0}^{\lfloor n/2 \rfloor} b_{k,n}(y) T_{n-2k}(x), \quad n=0, 1, 2, \dots \quad (\text{A18})$$

where

$$b_{k,n}(y) = 2(\mu)_{n-k} \sum_{m=0}^k \frac{(\mu+m)_{k-m} (y^2-1)^m y^{n-2m}}{m! (k-m)! (n-k-m)!} \quad (\text{A19})$$

and

$$T_n(xy) = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{k,n}(y) T_{n-2k}(x), \quad n = 1, 2, 3, \dots \tag{A20}$$

where

$$c_{k,n}(y) = n(n-k-1)! \sum_{m=0}^k \frac{(m)_{k-m} (y^2 - 1)^m y^{n-2m}}{m! (k-m)! (n-k-m)!} \tag{A21}$$

The notation  $\Sigma'$  means that 1/2 of the last term in the sum is taken if  $n$  is even, and all of it is taken if  $n$  is odd. Note that inspection shows that  $y > 1$  implies that  $b_{k,n}(y)$  and  $c_{k,n}(y)$  are positive.

Sonine's second finite integral is now derived from (A6). Fix  $x$  and  $y$ . Let  $N = \lfloor n/2 \rfloor$ . From (A7)

$$\begin{aligned} \frac{n^{1-2\mu}}{2\mu} C_n^\mu \left( \frac{\cos \frac{x}{n}}{\cos \frac{y}{n}} \right) &= \frac{1}{1+N} \sum_{k=0}^N \left\{ \frac{\lambda n^{1-2\mu} (1+N)}{\mu(n-2k)^{1-2\lambda}} a_{k,n} \left( \frac{1}{\cos \frac{y}{n}} \right) \right\} \left\{ \frac{(n-2k)^{1-2\lambda}}{2\lambda} \right. \\ &\cdot \left. C_{n-2k}^\lambda \left( \cos \frac{x}{n} \right) \right\} = \int_0^1 f_n(1-\zeta) g_n(1-\zeta) d\zeta \tag{A22} \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \sum_{m=0}^k \frac{\left( \zeta + \frac{\lambda}{n} \right) (\mu+1)_{n-k-1} n^{-2(\mu-\lambda-1)} \left( \frac{1}{n} + \frac{N}{n} \right) (\mu-\lambda+m)_{k-m} \sin^{2m} \frac{y}{n}}{\zeta^{1-2\lambda} \cos^n \frac{y}{n} m! (k-m)! (\lambda+1)_{n-k-m}} \tag{A26}$$

where we have defined for  $0 \leq \zeta \leq 1$

$$f_n(1-\zeta) = \sum_{k=0}^N \frac{\lambda n^{1-2\mu} (1+N)}{\mu(n-2k)^{1-2\lambda}} a_{k,n} \left( \frac{1}{\cos \frac{y}{n}} \right) \chi_{E_k}(1-\zeta)$$

$$g_n(1-\zeta) = \sum_{k=0}^N \frac{(n-2k)^{1-2\lambda}}{2\lambda} C_{n-2k}^\lambda \left( \cos \frac{x}{n} \right) \chi_{E_k}(1-\zeta)$$

and  $\chi_{E_k}$  is the characteristic (indicator) function of the interval

$$E_k = \begin{cases} \left[ \frac{k}{N+1}, \frac{k+1}{N+1} \right), & k = 0, 1, \dots, N-1 \\ \left[ \frac{k}{N+1}, \frac{k+1}{N+1} \right], & k = N. \end{cases}$$

It can be verified that  $\chi_{E_k}(2k/n) = 1$  for  $k = 0, 1, \dots, N$ .

Assume for the moment that both  $|f_n(\zeta)|$  and  $|g_n(\zeta)|$  are bounded above by integrable functions of  $\zeta$ . To do this, it will be seen that we must restrict attention to  $\lambda > -1/2$ ,  $\mu > -1/2$ ,  $\mu > \lambda$ , so that the integral [1, eq. (1.7.4)]

$$\int_0^1 \zeta^{2\lambda} (1-\zeta^2)^{\mu-\lambda-1} d\zeta = \frac{\Gamma(\lambda + \frac{1}{2}) \Gamma(\mu - \lambda)}{2\Gamma(\mu + \frac{1}{2})} \tag{A23}$$

will be finite. If  $f = \lim f_n$  and  $g = \lim g_n$ , the bounded convergence theorem [9, p. 110] implies

$$\lim_{n \rightarrow \infty} \frac{n^{1-2\mu}}{2\mu} C_n^\mu \left( \frac{\cos \frac{x}{n}}{\cos \frac{y}{n}} \right) = \int_0^1 f(1-\zeta) g(1-\zeta) d\zeta. \tag{A24}$$

Let  $\zeta$  in  $(0, 1)$  be rational. Then  $1-\zeta = 2k/n$  for sufficiently large  $k$  and  $n$ , so that

$$\begin{aligned} g(1-\zeta) &= \lim_{n \rightarrow \infty} g_n(1-\zeta) \\ &= \lim_{\substack{n \rightarrow \infty \\ 1-\zeta = 2k/n}} \frac{(n-2k)^{1-2\lambda}}{2\lambda} C_{n-2k}^\lambda \left( \cos \frac{x}{n} \right) \\ &= \sqrt{\pi/2} \frac{J_{\lambda-1/2}(\zeta x)}{2^\lambda \Gamma(\lambda+1) (\zeta x)^{\lambda-1/2}} \tag{A25} \end{aligned}$$

with the last step following immediate from (A6). Thus, (A25) holds for all  $\zeta$  in  $[0, 1]$  by continuity. Similarly, from (A8) and for all  $\zeta$  rational in  $(0, 1)$ ,

$$\begin{aligned} f(1-\zeta) &= \lim_{n \rightarrow \infty} f_n(1-\zeta) \\ &= \lim_{\substack{n \rightarrow \infty \\ 1-\zeta = 2k/n}} \frac{\lambda n^{1-2\mu} (1+N)}{\mu(n-2k)^{1-2\lambda}} a_{k,n} \left( \frac{1}{\cos \frac{y}{n}} \right) \end{aligned}$$

Interchange the limit and the summation, and evaluate the limit of the  $m$ th term (convert Pochhammer symbols to gamma functions, apply Stirling's formula, and use  $k(n-k) = (1-\zeta^2)n^2/4$ ) to obtain

$$\begin{aligned} f(1-\zeta) &= \sum_{m=0}^{\infty} \frac{\zeta^{2\lambda} (1-\zeta^2)^{\mu-\lambda-1}}{2^{2\mu-2\lambda-1}} \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)} \\ &\cdot \frac{(\frac{1}{2}y\sqrt{1-\zeta^2})^{2m}}{m! \Gamma(\mu-\lambda+m)} \\ &= \frac{\zeta^{2\lambda} (1-\zeta^2)^{\mu-\lambda-1}}{2^{2\mu-2\lambda-1}} \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)} \\ &\cdot \frac{I_{\mu-\lambda-1}(y\sqrt{1-\zeta^2})}{(\frac{1}{2}y\sqrt{1-\zeta^2})^{\mu-\lambda-1}} \tag{A27} \end{aligned}$$

where  $I_\nu(z)$  denotes the modified Bessel function of the first order  $\nu$  (see [8, sect. 3.7(2)]). We must require  $\mu > \lambda$  in (A27) to have convergence. Continuity again assures that (A27) holds for all  $\zeta$  in  $(0, 1)$ . Now, interchanging the limit and the sum was valid because an upper bound for the total sum can be found. Since the absolute value of the  $m$ th term in (A26) is bounded by

$$B \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)} \frac{(\frac{1}{2}|y|\sqrt{1-\zeta^2})^{2m}}{m! \Gamma(\mu-\lambda+m)}$$

where

$$B = \frac{\xi + \frac{\lambda}{n}}{\xi^{1-2\lambda}} \frac{n^{-2(\mu-\lambda-1)}}{n^{2m} |\cos^n \frac{y}{n}|} \left( \frac{1-\xi^2}{4} \right)^{-m}$$

$$\cdot \frac{\Gamma(k+\mu-\lambda)}{\Gamma(k-m+1)} \frac{\Gamma(n-k+\mu)}{\Gamma(n-k+\lambda+1-m)}$$

$$\cong \xi^{2\lambda} \left( \frac{1-\xi^2}{4} \right)^{\mu-\lambda-1}, \quad n \rightarrow \infty,$$

the total sum in (A26) is bounded by

$$F(\xi) = L \xi^{2\lambda} (1-\xi^2)^{\mu-\lambda-1} \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)}$$

$$\cdot \sum_{m=0}^{\infty} \frac{(\frac{1}{2}|y|\sqrt{1-\xi^2})^{2m}}{m! \Gamma(\mu-\lambda+m)} < \infty \quad (\text{A28})$$

for some constant  $L$  independent of  $\xi$ . The series in (A28) is a continuous function of  $\xi$  on  $[0, 1]$  if  $\mu > \lambda$ . Hence, from (A23),  $F(\xi)$  is an integrable function that bounds  $|f_n(\xi)|$  for all  $n$ .

From (A24), (A25), and (A27) we have

$$\lim_{n \rightarrow \infty} \frac{n^{1-2\mu}}{2\mu} C_n^\mu \left( \frac{\cos \frac{x}{n}}{\cos \frac{y}{n}} \right)$$

$$= \frac{\sqrt{\pi/2}}{2^\mu \Gamma(\mu+1) x^{\lambda-1/2} y^{\mu-\lambda-1}}$$

$$\cdot \int_0^1 \xi^{\lambda+1/2} (\sqrt{1-\xi^2})^{\mu-\lambda-1}$$

$$\cdot J_{\lambda-1/2}(\xi x) I_{\mu-\lambda-1}(y\sqrt{1-\xi^2}) d\xi$$

$$= \frac{\sqrt{\pi/2}}{2^\mu \Gamma(\mu+1)} \frac{J_{\mu-1/2}(\sqrt{x^2-y^2})}{(\sqrt{x^2-y^2})^{\mu-1/2}} \quad (\text{A29})$$

with the last equation from (A6). Substituting  $\xi = \sin \theta$  and  $y = iy'$  in the last two formulas, and setting

$$\mu' = \lambda - \frac{1}{2} > -1 \quad \text{and} \quad \lambda' = \mu - \lambda - 1 > -1 \quad (\text{A30})$$

yields Sonine's second finite integral (A1). The only thing left to prove is that  $|g_n(\xi)|$  is bounded by an integrable function on  $[0, 1]$ . Szegő's argument [1, p. 192] in the proof of (A2) can be modified easily to show  $|g_n(\xi)|$  is bounded by a constant.

The proof of (A1) presented here was intentionally restricted to real  $\mu$  and  $\lambda$ . However, it is not hard to see from (A23) and

(A30) that the proof can be carried out for complex  $\mu$  and  $\lambda$ , provided appropriate remarks are made in appropriate places about the complex case. If such remarks are made, our derivation proves (A1) for  $\text{Re}(\mu) > -1$  and  $\text{Re}(\lambda) > -1$ . Divergence of (A23) is seen to be the cause of the restrictions on  $\mu$  and  $\lambda$ .

The material contained in this Appendix was first documented in [11].

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