Improved Spherical and Hemispherical Scanning Algorithms

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Abstract—**A probe-corrected hemispherical-scanning algorithm has been developed which is applicable when the antenna under test radiates negligibly into its rear hemisphere. For a hundred-wavelengths diameter antenna, hemispherical scanning would be about three times more efficient computationally than prior full-sphere scanning algorithms. Improvements have also been made to full-sphere scanning, significantly increasing that algorithm's computational efficiency.**

I. Introduction

A THEORY OF HEMISPHERICAL scanning to obtain the far-field radiation pattern of a large class of antennas that radiate negligibly toward their rear hemisphere is developed. For such antennas, full sphere scanning is possible but undesirable, since the rear hemisphere data is essentially noise. Our theory of hemispherical scanning parallels full-sphere scanning [1]–[3] in that we transform discrete sets of probe-corrected near-field measurements into far-field patterns. However, only data points on a hemisphere, which is part of a sphere enclosing the test antenna, are used in the analysis. Consequently, the computational effort to obtain the far-field patterns is reduced proportionately.

Since spherical scanning is an integral part of hemispherical theory, we reexamine full-sphere scanning and adopt a number of improvements. Some conceptual and mathematical simplifications are introduced, leading to a significant reduction in computational effort. Combined with the savings inherent in hemispherical scanning and for antennas larger than 50 wavelengths across, we achieve a reduction in computational effort by a factor of between 2.75 and 3.5, depending on antenna diameter.

The outline of this paper is as follows. In Sections II and III we examine details of full-sphere scanning. We review the prior developments [2], [3] in Section II and outline current improvements in Section III. In Section IV we develop the theory of hemispherical scanning. In Section V we estimate the total computational savings incorporated in our procedures. In addition, Appendixes I and II provide some mathematical details and derivations of full-sphere scanning theory to complement the main text.

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II. Spherical Scanning Formalism—A Brief Review

A method of spherical scanning analysis was originally formulated by Jensen [1]. In his analysis, as well as in this work, multiple reflections between the probe and test antenna are assumed to be negligible. Following Jensen's formulation, Wacker [2] proposed the use of fast Fourier transforms (FFT's) and "circularly symmetric" probes to make the data processing effort tractable. Circular symmetry implies that the probe receiving pattern can be described in terms of cos χ and sin χ azimuthal dependence. As a consequence of these simplifications, the general mutual coupling equation between a test antenna and a probe, which is derived in a companion paper [4] by one of the authors, reduces to

$$W(\phi, \theta, \chi)=\sum_{n=1}^{N-1} \sum_{m=-n}^{n} \sum_{\mu=-1}^{1}{}' e^{im\phi}d^n_{m\mu}(\theta) \cdot e^{i\mu\chi} \sum_{s=1}^{2} Q^{smn}\hat{R}'_{s\mu n}, \quad (1)$$

where $W(\phi, \theta, \chi)$ is the signal received by the probe. Here, ϕ, θ, and χ denote the Euler angles [5] describing the angular orientation of the probe relative to the test antenna's coordinate system, the prime on the μ summation indicates that the term $\mu = 0$ is omitted, $N - 1$ is the number of significant spherical modes in the radiation pattern of the test antenna, the constants Q^{smn} denote the unknown modal expansion coefficients of the field radiated by the test antenna, and $\hat{R}'_{s\mu n}$ denotes known translated probe receiving coefficients. The $d^n_{m\mu}(\theta)$ are rotation coefficients expressible in terms of associated Legendre functions and are defined in Rose [5, ch. 4]. The functions containing the Euler angles in (1) express the effect of rotating the probe with respect to the test antenna.

The analysis of (1) to obtain the unknowns Q^{smn} from adequately sampled discrete data $W(\phi, \theta, \chi)$ proceeds as follows: we interchange the order of the n and m summations, so that (1) becomes

$$W(\phi, \theta, \chi)= \sum_{m=-N+1}^{N-1} \sum_{\mu=-1}^{1}{}' W^\mu_m(\theta)e^{i(m\phi+\mu\chi)}, \quad (2)$$

where, for each m and μ,

$$W^\mu_m(\theta)= \sum_{n=(1,|m|)}^{N-1} d^n_{m\mu}(\theta) \sum_{s=1}^{2} Q^{smn}\hat{R}'_{s\mu n}, \quad (3)$$

and $n = (1, |m|)$ selects the larger of 1, $|m|$ as the initial n value. The Fourier coefficients $W_m^{\mu}(\theta)$ in (2) and (3) are determined using known orthogonality relations for Fourier series. Then the orthogonality relation [5], [6]

$$\int_0^{\pi} d^{l}_{m\mu}(\theta) d^{n}_{m\mu}(\theta) \sin\theta \, d\theta = \frac{1}{n+1/2}\,\delta_{nl}$$

is used to obtain from (3) the sum

$$\sum_{s=1}^{2} \hat{R}'_{s\mu n} Q^{smn} = \left(n + \frac{1}{2}\right) \int_0^{\pi} W_m^{\mu}(\theta) d^{n}_{m\mu}(\theta) \sin\theta \, d\theta \quad (4)$$

for $\mu = \pm 1$, $m = -N + 1$ to $N - 1$, and $n \geqslant |m|$ over the range $n = 1, 2, \cdots, N - 1$. Assuming that the integral on the right-hand side has been evaluated for all indices, we then have two equations (for $\mu = \pm 1$) for each n, m to solve for $Q^{smn} (s = 1, 2)$. The far-field pattern $\bar{W}(\phi, \theta, \chi)$ of the test antenna is then obtained by substituting the now known Q^{smn} coefficients into (1) and replacing $\hat{R}'_{s\mu n}$ with $\bar{\hat{R}}'_{s\mu n}$, which are far-field ideal-dipole receiving coefficients expressed in the test antenna's coordinate system.

To evaluate the integral in (4), the procedure [2] has been to express $d^{n}_{m\mu}(\theta)$ as [6]

$$d^{n}_{m\mu}(\theta) = i^{\mu - m} \sum_{m'=-n}^{n} \Delta^{n}_{m'\mu} \Delta^{n}_{m'm} e^{im'\theta}, \quad (5)$$

where $\Delta^{n}_{m'\kappa} = d^{n}_{m'\kappa}(\pi/2)$ for $\kappa = m, \mu$. (Note that Edmonds [6] follows a slightly different convention for $d^{n}_{m\mu}(\theta)$, but (5) still holds if the $\Delta^{n}_{m'\kappa}$ are consistently redefined.) Then from (3) we can express $W_m^{\mu}(\theta)$ as the finite Fourier series

$$W_m^{\mu}(\theta) = \sum_{m''=1-N}^{N-1} W^{m\mu}_{m''} e^{im''\theta}. \quad (6)$$

The Fourier coefficients $W^{m\mu}_{m''}$ are obtained from the measured data. The integral (4) is converted into a finite double series containing terms that can be integrated analytically and then summed to give the desired result. In the next section a new method of evaluating this integral is developed.

III. Some New Developments in Full-Sphere Scanning

Our overview of spherical-scanning up to this point has followed the development suggested by Wacker [2]. A part of his procedure was to substitute (5) and (6) directly into (4) to evaluate the integral. However, a computational simplification is achieved by first expressing $d^{n}_{m,\pm 1}(\theta) \sin\theta$ in terms of $d^{n}_{m0}(\theta)$, where the latter is proportional to the associated Legendre function $P_n^m(\cos\theta)$. Thus, making use of the Fano–Racah recursion relations [7] for $d^{n}_{m\mu}(\theta)$ and recursion relations between contiguous Legendre functions [8], we obtain

$$d^{n}_{m,\pm 1}(\theta) \sin\theta = \frac{-1}{\sqrt{n(n+1)}} \cdot \left[m d^{n}_{m0}(\theta) \pm \sin\theta \frac{d}{d\theta} d^{n}_{m0}(\theta) \right], \quad (7)$$

and

$$(2n+1)\sin\theta \frac{d}{d\theta} d^{n}_{m0}(\theta) = n\sqrt{(n-m+1)(n+m+1)} \cdot d^{n+1}_{m0}(\theta) - (n+1)\sqrt{(n+m)(n-m)}\, d^{n-1}_{m0}(\theta). \quad (8)$$

Consequently, substituting (7) and (8) into (4), the integral becomes

$$I_n^{m\mu} = n\sqrt{(n+1)^2 - m^2}\, V^{m\mu}_{n+1} - (n+1)\sqrt{n^2 - m^2}\, V^{m\mu}_{n-1} + m\mu(2n+1) V_n^{m\mu}, \quad (9)$$

where $I_n^{m\mu}$ is the right-hand side of (4) multiplied by $\sqrt{n(n+1)}\, i^{\mu - m}$, and

$$V_n^{m\mu} \equiv \frac{1}{2} i^{-m-1} \int_0^{\pi} W_m^{\mu}(\theta) d^{n}_{m0}(\theta)\, d\theta. \quad (10)$$

The middle term in (9) is zero unless $m < n$.

The procedure used to evaluate the $V_n^{m\mu}$ integrals is the same as that outlined in the last paragraph of Section II. However, we use the complex conjugate of (5) in (10) to simplify our final result (note that $d^{n}_{m\mu}(\theta)$ is a real function [5]—see Appendix II). Moreover, since $\Delta^{n}_{m'0}$ is zero when $n - m'$ is odd, the series representation of $d^{n}_{m0}(\theta)$ via (5) reduces to $n + 1$ terms instead of $2n + 1$. For a given μ and $m \neq 0$, and $n = |m|$, two integrals of the type in (10) need to be evaluated (three for $m = 0$). But for subsequent n in the range $|m| < n \leqslant N - 1$ only $V^{m\mu}_{n+1}$ is evaluated. Consequently, $I_n^{m\mu}$ for $n > (1, |m|)$ is obtained by computing half as many terms as in (4). A similar simplification occurs in the computation of the far-field pattern $\bar{W}(\phi, \theta, \chi)$ using (1) with (5) and (7).

To obtain $d^{n}_{m0}(\theta)$ via (5), we wish to calculate only nonzero products of $\Delta^{n}_{m'0}\Delta^{n}_{m'm}$. A three-term backwards recursion relationship specifying the nonzero products can be derived using linear combinations of formulas in [9] or [10]. The initial conditions valid for fixed $n \geqq m \geqq 0$ are

$$\Delta^{n}_{n+2,0}\Delta^{n}_{n+2,m} = 0$$

and

$$\Delta^{n}_{n,0}\Delta^{n}_{n,m} = \frac{(-)^m}{2^{2n}} \frac{(2n)!}{n!\sqrt{(n+m)!(n-m)!}}.$$

Starting with the earlier three-term recursion relations one can show

$$\Delta^{n}_{m'-2,0}\Delta^{n}_{m'-2,m} = \frac{2[n(n+1) - m'^2 - 2m^2]\Delta^{n}_{m'0}\Delta^{n}_{m'm} - [n(n+1) - (m'+1)(m'+2)]\Delta^{n}_{m'+2,0}\Delta^{n}_{m'+2,m}}{n(n+1) - (m'-1)(m'-2)} \quad (11)$$

for all $2 \leqq m' \leqq n$. For $m' > 0$ the magnitudes of the terms calculated in (11) increase as m' decreases, so the numerical stability of (11) is guaranteed by the theory of three-term recursion relations [11].

This completes our overview of the theory of full-sphere scanning. In the next section we present a discrete data-point formulation for hemispherical scanning. There, part of the $W_m^\mu(\theta)$ data is shifted and then combined with an unshifted part so that the complete θ range $0 \leqslant \theta \leqslant \pi$ contains nontrivial data. Our improved full-sphere algorithm is utilized to process this combined data, resulting in combined far-field Fourier-transform coefficients. From this it should be obvious that a discrete data-point algorithm for full-sphere scanning falls out simply from the subsequent presentation.

IV. Hemispherical Scanning

We have pointed out in the Introduction that there exists a class of antennas that radiate primarily into one hemisphere. All the mathematical formulas discussed up until now apply well to this class of antennas. However, we can take advantage of the fact that radiation into the back hemisphere is essentially negligible. One can rearrange data according to the m index so the full sphere is covered. This is accomplished by using the symmetry relation [6], $d^n_{-m,\mu}(\theta + \pi) = (-)^{n+m} d^n_{m\mu}(\theta)$, which allows us to think of terms with negative m indexes as data appearing in the back hemisphere. Consequently, the positive and negative m-index data can be combined in a single equation to cover the whole sphere,

$$W_m^\mu(\theta) + W_{-m}^\mu(\theta+\pi) = \sum_{n=(1,|m|)}^{N-1} d^n_{m\mu}(\theta) \cdot \sum_{s=1}^{2} (Q^{smn} + (-)^{n+m} Q^{s,-m,n}) \hat{R}'_{s\mu n}.$$

Now we can carry out the full-sphere algorithm just as before using the sum on the left as the known data to obtain a corresponding sum of far-field Fourier coefficients. Since different m-index quantities are not coupled by the near-to far-field transformation, the individual far-field quantities $\bar{W}_m^\mu(\theta)$ and $\bar{W}_{-m}^\mu(\theta + \pi)$ are readily discernible from their sum according to which hemisphere θ is in. Therefore, restricting m to just positive integer values (thereby reducing the parameter range by one half) will in fact produce the far-field quantity $\bar{W}_m^\mu(\theta)$ for all m, from which the far-field pattern $\bar{W}(\phi, \theta, \chi)$ is obtained.

If we combine the improvements to full-sphere scanning analysis presented in the previous section with the ideas outlined in the paragraph above, we can analyze hemispherical data in an efficient manner. This procedure applies provided the axis of the hemisphere coincides with the polar axis of the measurement sphere.[1] The details will now be presented.

Since we have assumed, via (1), that the signal received by the probe is virtually bandlimited, our algorithms admit either a continuous or discrete formulation. We present the discrete formulation for ease of computer implementation.

The range on the spherical angles for the measured data is $0 \leqq \theta \leqq \pi$, $0 \leqq \phi < 2\pi$. However, by assumption, data in the range $\pi/2 \leqq \theta \leqq \pi$ is negligible in comparison with data in the range $0 \leqq \theta < \pi/2$. Now we assume that the received signal, $W(\phi, \theta, \chi)$, is known at discrete lattice points θ_κ and ϕ_l on the surface of the measurement hemisphere for two probe rotation angles, χ, separated from each other by 90°. We now have sufficient information to evaluate the unknown Q^{smn} coefficients in (1). For each θ_κ on the hemisphere we can write discrete Fourier transforms (DFT's) of the received signal in ϕ and χ as

$$W_m^\mu(\kappa\Delta\theta) = \frac{1}{4M} \sum_{l=0}^{2M-1} \{ W(l\Delta\phi, \kappa\Delta\theta, \chi=0°) - i\mu W(l\Delta\phi, \kappa\Delta\theta, \chi=90°)\} e^{-iml\Delta\phi} \quad (12)$$

where $\Delta\theta = \pi/N$, $\Delta\phi = \pi/M$, and $M \leqq N$, depending on how rapidly the radiation pattern varies with ϕ, $\kappa = 0, 1, \cdots, N/2 - 1$, $m = -M + 1, -M + 2, \cdots, M - 1$, and $\mu = \pm 1$.

To Fourier transform the data in θ we must specify $W_m^\mu(\kappa\Delta\theta)$ over the range $-\pi \leqq \kappa\Delta\theta \leqq \pi$. We define a new spherical coordinate system such that $-\pi \leqq \theta \leqq \pi$, $0 \leqq \phi < \pi$ and use the fact that coordinates θ, $\phi + \pi$ in the old system transform to $-\theta$, ϕ in the new system. Hence, $W(\phi, -\theta, \chi) = W(\phi + \pi, \theta, \chi + \pi)$. Consequently, using (12), we can show that

$$W_m^\mu(-\theta) = (-)^{m+1} W_m^\mu(\theta). \quad (13)$$

Equation (13) specifies how data in the range $-\pi/2 < \theta < 0$ is defined in terms of data in the range $0 \leqq \theta < \pi/2$. This effectively doubles the range on κ over that in (12). Now by assumption data outside the range $-\pi/2 < \kappa\Delta\theta < \pi/2$ is equal to zero. We obtain nontrivial data over the full range on κ and in the process halve the number of θ-FFT's that need to be summed by doing the following: we shift the data $W_{-m}^\mu(\kappa\Delta\theta)$ by π radians and combine it with $W_m^\mu(\kappa\Delta\theta)$ (which is unshifted). In this combination, for a given κ specifying an angle θ, either $W_m^\mu(\kappa\Delta\theta)$ or $W_{-m}^\mu[(\kappa + N)\Delta\theta]$ will be equal to zero. Taking the DFT of this sum we obtain for $0 \leqq m'' \leqq N$, $0 \leqq m \leqq M - 1$, and $\mu = \pm 1$

$$\overline{(W_{m''}^{m,\mu} + (-)^{m''} W_{m''}^{-m,\mu})} = \frac{1}{2N} \sum_{\kappa=0}^{2N-1} \{ W_m^\mu(\kappa\Delta\theta) + W_{-m}^\mu[(\kappa+N)\Delta\theta]\} e^{-im''\kappa\Delta\theta}. \quad (14)$$

We simultaneously evaluate these DFT's by pairs, combining an even and odd m DFT into a single DFT, as outlined in Appendix I.

The evaluation of the $V_n^{m\mu}$ integrals ((10)) requires that we obtain $W_{m''}^{m\mu}$, for $-N + 1 \leqq m'' \leqq N - 1$. We extend the range on m'' from that in (14), using the Fourier transform of (13), to obtain

$$W_{2N-m''}^{m\mu} = W_{-m''}^{m\mu} = (-)^{m+1} W_{m''}^{m\mu}. \quad (15)$$

[1] For the case where the radiation pattern is concentrated near the equator, one can utilize the technique [12] of setting the Fourier coefficients $W_m^\mu(\theta)$ to zero for m odd as though the data in the interval $-\pi/2 < \phi < \pi/2$ were replicated in the interval $\pi/2 < \phi < 3\pi/2$.

The solid line over the two Fourier coefficients in (14) designates that the quantities cannot be separately resolved. However, we can use the combined quantities in the series representations of two $V_n^{m\mu}$ integrals

$$\overline{(V_n^{m,\mu}-(-)^n V_n^{-m,\mu})}=\sum_{m'=0}^{n}\epsilon_{m'}\Delta_{m'm}^{n}\Delta_{m'0}^{n}\overline{(G_{m'}^{m\mu}-(-)^{m'}G_{m'}^{-m\mu})},\quad(16)$$

where

$$\overline{(G_{m'}^{m\mu}-(-)^{m'}G_{m'}^{-m\mu})}=\sum_{m''=0}^{N-1}{}'\,\epsilon_{m'}[1-(-)^{m''+m'}]\cdot\left[\frac{1}{m''-m'}+\frac{(-)^m}{m''+m'}\right]\overline{(W_{m''}^{m,\mu}+(-)^{m''}W_{m''}^{-m,\mu})}.\quad(17)$$

In (16), the ranges $0 \leqq m < M$, $m \leqq n < N$ apply with $\mu = \pm 1$. The prime on the m'' summation in (17) indicates that terms of the form 0/0 are omitted, and $\epsilon_\kappa = \{^{1,\kappa\neq 0}_{1/2,\kappa=0}$. Equation (16) represents the double series expansion for evaluating the integral expression (10) when its positive and negative m-index versions are combined as indicated. In arriving at (17), we combined duplicate terms and terms that cancel each other.

The individual coefficients on the left-hand sides of (14), (16), and (17) could be separated by formulating a second equation involving the differences rather than the sums of the indicated quantities. This would double the number of equations and FFT's needed and it would produce full-sphere scanning. Equally, one could recover full-sphere scanning by either deleting the first or second term in the overhead line quantities and by having m take both positive and negative values.

The sum in (17) can be rearranged so that m'' goes from $-N+1$ to $N-1$, with the result that $\epsilon_{m''}$ and $(-)^m/(m''+m')$ are eliminated from the summation. Using such a formulation, Larsen [3] noticed that this inner sum can be interpreted as a lagged product, thereby permitting its efficient computation as a $4N$-term FFT and permitting an extension of the scheme in Appendix I so as to halve the number of DFT computations. Here, we present an alternate formulation using $2N$-term DFT's. Although an extension of Appendix I would not apply here, our formulation does have an advantage in the case of equatorial scanning when odd m-index DFT's are omitted. Otherwise, the computational efforts involved are similar. Accordingly, we define $Y^{m\mu}(\kappa)$ for $0 \leqq \kappa \leqq 2N-1$ as the DFT of $\epsilon_{m''}\overline{(W_{m''}^{m,\mu}+(-)^{m''}W_{m''}^{-m,\mu})}$, when $0 \leqq m'' \leqq N-1$, with zero fill for $N \leqq m'' < 2N$. Breaking the right-hand side of (17) in two, we have the exact expressions

$$\sum_{m''=0}^{N-1}{}'\,\epsilon_{m''}\overline{(W_{m''}^{m,\mu}+(-)^{m''}W_{m''}^{-m,\mu})}\frac{1-(-)^{m''-m'}}{m''-m'}=\frac{1}{2N}\sum_{\kappa=0}^{2N-1}Y^{m\mu}(\kappa)B_1(-\kappa)e^{-i(\pi/N)m'\kappa}$$

and

$$\sum_{m''=0}^{N-1}{}'\,\epsilon_{m''}\overline{(W_{m''}^{m,\mu}+(-)^{m''}W_{m''}^{-m,\mu})}\frac{1-(-)^{m''+m'}}{m''+m'}=\frac{1}{2N}\sum_{\kappa=0}^{2N-1}Y^{m\mu}(-\kappa)B_2(\kappa)e^{-i(\pi/N)m'\kappa}$$

where

$$B_1(\kappa)=\sum_{m''=1-N}^{N-1}{}'\,\frac{1-(-)^{m''}}{m''}e^{i(\pi/N)m''\kappa};$$

$$B_2(\kappa)=\sum_{m''=1}^{2N-1}\frac{1-(-)^{m''}}{m''}e^{i(\pi/N)m''\kappa}.$$

Now from (6) and (15) we have

$$W_m^{\mu}(\kappa\Delta\theta)=\sum_{m''=0}^{N-1}\epsilon_{m''}W_{m''}^{m\mu}e^{i(\pi/N)m''\kappa}+(-)^{m+1}\sum_{m''=0}^{N-1}\epsilon_{m''}W_{m''}^{m\mu}e^{-i(\pi/N)m''\kappa}\quad(18)$$

so that we only need to evaluate one of $Y^{m\mu}(\kappa)$ and $Y^{m\mu}(-\kappa)$. Note from (16) that we require the evaluation of $G_{m'}^{m\mu}$ just for $0 \leqq m' \leqq N-1$. Consequently, one can realize a modest (on the order of $50/\log_2 N$ percent) computational savings by carrying out each DFT as two N-term FFT's, rather than as one $2N$-term FFT, by substituting $\kappa = 2p+q$. Hence, combining the above results we obtain

$$Y^{m\mu}(2p+q)=\sum_{m''=0}^{N-1}\{\epsilon_{m''}\overline{(W_{m''}^{m,\mu}+(-)^{m''}W_{m''}^{-m,\mu})}\cdot e^{i(\pi/N)qm''}\}e^{i(2\pi/N)m''p},\quad\begin{cases}p=0,1,\cdots,N-1\\ q=0,1\end{cases}\quad(19)$$

and

$$\overline{(G_{m'}^{m\mu}-(-)^{m'}G_{m'}^{-m,\mu})}=\frac{1}{2N}\sum_{q=0}^{1}e^{-i(\pi/N)qm'}\cdot\left[\sum_{p=0}^{N-1}A^{m\mu}(2p+q)e^{-i(2\pi/N)m'p}\right],\quad 0\leqq m'\leqq N-1\quad(20)$$

where

$$A^{m\mu}(\kappa)=Y^{m\mu}(\kappa)\{B_1(-\kappa)+B_2(\kappa)\}-\{W_m^{\mu}(\kappa\Delta\theta)+W_{-m}^{\mu}[(\kappa+N)\Delta\theta]\}B_2(\kappa),\qquad \kappa=2p+q.$$

We now complete the evaluation of the $\overline{V_n^{m\mu}-(-)^nV_n^{-m\mu}}$ coefficients in (16) obtaining the $\overline{G_{m'}^{m\mu}-(-)^{m'}G_{m'}^{-m\mu}}$ coefficients as shown above and making use of (11) for the *in situ* computation of the deltas as they are needed. Then, from (9), we obtain

$$\begin{aligned}\overline{(I_n^{m,\mu}+(-)^n I_n^{-m,\mu})} = {} & n\sqrt{(n+1)^2-m^2} \\ & \cdot \overline{(V_{n+1}^{m\mu}+(-)^n V_{n+1}^{-m,\mu})} \\ & -(n+1)\sqrt{n^2-m^2} \\ & \cdot \overline{(V_{n-1}^{m,\mu}+(-)^n V_{n-1}^{-m,\mu})} \\ & + m\mu(2n+1) \\ & \cdot \overline{(V_n^{m,\mu}-(-)^n V_n^{-m,\mu})}.\end{aligned} \tag{21}$$

Now from (4) and (9) we obtain the expression $\sqrt{n(n+1)}\, i^{\mu-m} \sum_{s=1}^{2} Q^{smn} \hat{R}'_{s\mu n} = I_n^{m\mu}$. Considering $\mu = \pm 1$ this can be solved for $Q^{smn}(s = 1, 2)$. We define a corresponding far-field quantity $S_n^{m\mu} = i^{\mu-m} \sum_{s=1}^{2} Q^{smn} \bar{\hat{R}}'_{s\mu n}/\sqrt{n(n+1)}$. Since the probe and test antenna are normally polarization matched in the vertical (y) direction at $\theta = \phi = \chi = 0$, we choose $\bar{\hat{R}}'_{s\mu n}$ corresponding to a y-directed ideal dipole; thus, it follows from [4] that $\bar{\hat{R}}'_{s\mu n} = 1/4\ \mu^s\ \sqrt{n(n+1)}\ i^{1-n}$. Consequently, upon solving $S_n^{m\mu}$ in terms of $I_n^{m\mu}$ and combining results to match (21), we obtain the matrix equation

$$\begin{bmatrix} \overline{(S_n^{m,1}+(-)^n S_n^{-m,1})} \\ \overline{(S_n^{m,-1}+(-)^n S_n^{-m,-1})} \end{bmatrix} = \frac{1}{\hat{D}} \begin{bmatrix} \hat{R}'_{2-1n}-\hat{R}'_{1-1n} & \hat{R}'_{21n}-\hat{R}'_{11n} \\ \hat{R}'_{2-1n}+\hat{R}'_{1-1n} & \hat{R}'_{21n}+\hat{R}'_{11n} \end{bmatrix} \cdot \begin{bmatrix} \overline{(I_n^{m,1}+(-)^n I_n^{-m,1})} \\ \overline{(I_n^{m,-1}+(-)^n I_n^{-m,-1})} \end{bmatrix} \tag{22}$$

where the elements of the two column matrices correspond to $\mu = \pm 1$ and

$$\hat{D} = 4 i^{n-1}\sqrt{n(n+1)}\,(\hat{R}'_{11n}\hat{R}'_{2-1n} - \hat{R}'_{1-1n}\hat{R}'_{21n}).$$

A derivation of the full-sphere scanning equivalent of (22) is presented in Appendix II. Equation (22) follows by adding together the matrix equations for $+m$ and $-m$ indexes and using the expression for $\bar{\hat{R}}'_{s\mu n}$.

The computational effort in obtaining the far field is minimized by reusing the modal summation coefficients generated by (11). There results

$$\overline{(\bar{V}_{m''}^{m,\mu}+(-)^{m''}\bar{V}_{m''}^{-m,\mu})} = \sum_{n=(1,m,m'')}^{N-1} \Delta_{m''m}^{n}\Delta_{m''0}^{n} \cdot \overline{(S_n^{m,\mu}+(-)^n S_n^{-m,\mu})}. \tag{23}$$

The notation $n = (1, m, m'')$ denotes that the largest of 1, m, or m'' is used as the starting point for the summation. Again, the ranges $0 \leqq m \leqq M - 1$, $0 \leqq m'' \leqq N - 1$, and $\mu = \pm 1$ apply in (23). The modal summation coefficients used in (23) correspond to the Fourier transform of $d_{m0}^{n}(\theta)$. Consequently, the far-field Fourier transform coefficients $\bar{W}_{m''}^{m\mu}$ are obtained by combining the Fourier transform of (7) with the summation (23). In terms of (7), the operations equivalent to division by $\sin\theta$ and differentiation with respect to θ are respectively a backwards recursion relation and multiplication by the Fourier transform variable m''. The backwards recursion starts with the initial conditions

$$\overline{(\bar{U}_{N-1}^{m,\mu}-\bar{U}_{N-1}^{-m,\mu})} = 0,$$

$$\overline{(\bar{U}_{N-2}^{m,\mu}+\bar{U}_{N-2}^{-m,\mu})} = -2m\mu\overline{(\bar{V}_{N-1}^{m,\mu}-\bar{V}_{N-1}^{-m,\mu})}$$

where we have assumed N even. The backwards recursion relation, effecting division by $\sin\theta$ for $m'' = N - 3, N - 4, \cdots, 0$, is given by

$$\overline{(\bar{U}_{m''}^{m,\mu}+(-)^{m''}\bar{U}_{m''}^{-m,\mu})} = \overline{(\bar{U}_{m''+2}^{m,\mu}+(-)^{m''}\bar{U}_{m''+2}^{-m,\mu})} - 2m\mu\overline{(\bar{V}_{m''+1}^{m,\mu}-(-)^{m''}\bar{V}_{m''+1}^{-m,\mu})}.$$

The far-field Fourier transform coefficients are then given by

$$\overline{(\bar{W}_{m''}^{m,\mu}+(-)^{m''}\bar{W}_{m''}^{-m,\mu})} = \overline{(\bar{U}_{m''}^{m,\mu}+(-)^{m''}\bar{U}_{m''}^{-m,\mu})} - m''\overline{(\bar{V}_{m''}^{m,\mu}+(-)^{m''}\bar{V}_{m''}^{-m,\mu})}.$$

We extend the range on m'' to negative integers using (15), obtaining

$$\overline{(\bar{W}_{-m''}^{m,\mu}+(-)^{m''}\bar{W}_{-m''}^{-m,\mu})} = (-)^{m+1}\overline{(\bar{W}_{m''}^{m,\mu}+(-)^{m''}\bar{W}_{m''}^{-m,\mu})}$$

This permits us to evaluate the DFT

$$\bar{W}_m^{\mu}(\kappa\Delta\theta) + \bar{W}_{-m}^{\mu}[(\kappa+N)\Delta\theta] = \sum_{m''=-N+1}^{N-1} \overline{(\bar{W}_{m''}^{m,\mu}+(-)^{m''}\bar{W}_{m''}^{-m,\mu})}e^{im''\kappa\Delta\theta}. \tag{24}$$

This DFT is just the inverse of (14), only this time using far-field quantities. As in (14), odd and even m Fourier coefficients can be transformed simultaneously, thus halving the number of θ-FFT's actually carried out. Note that the coefficients on the left-hand side of (24) are resolvable, since the $+m$ index coefficients are in the range $-\pi/2 < \theta < \pi/2$, while the $-m$ index coefficients are in the range $\pi/2 < \theta < 3\pi/2$.

Finally, we obtain the far field of the test antenna from

$$\bar{W}(\phi, \theta, \chi) = \sum_{m=-M+1}^{M-1} e^{im\phi} \sum_{\mu=-1}^{1}{}' \ \bar{W}_m^{\mu}(\theta)e^{i\mu\chi}, \tag{25}$$

where $\phi = l\Delta\phi$, $l = 0, 1, \cdots, 2M - 1$; $\theta = \kappa\Delta\theta$, $\kappa = 0, 1, \cdots, N/2$; and $\chi = 0, \pi/2$. $\bar{W}(\phi, \theta, \chi)$ corresponds to a vector component of the far-zone electric field radiated by the test antenna; the particular vector component is determined by the angle χ. Thus, since a y-directed ideal dipole was chosen as the far-field receiving probe, $\bar{W}(\phi, \theta, \chi = \pi/2)$ corresponds to the θ component while $-\bar{W}(\phi, \theta, \chi = 0)$ corresponds to the ϕ component.

V. Operation Count Estimation

By halving the number of terms summed both to evaluate integrals and to calculate far-field Fourier coefficients, our full-sphere scanning algorithm saves about 4/3 N^3 arithmetic operations over the operation count of prior formulations, where N is the order of the first negligible spherical mode (for the time being, we take $M = N$). Here, we are simply counting the number of terms summed in four similar summations in two different algorithms and taking the difference. However, our reformulation of these summations introduces additional steps partially offsetting the operation count reduction by about 14 N^2 operations, a negligible amount compared to the N^3 savings. Here, in order to maintain a consistent estimate that accounts for timing differences between real and complex arithmetic on a computer, we use a biased operation count to tabulate a particular arithmetic step. Thus, one arithmetic operation is defined as a real-complex multiplication followed by a complex addition, one real multiplication and addition is about five-eights of an arithmetic operation, while one complex multiplication and addition is about one and one-quarter operations. These last-mentioned estimates are consistent with timed results on a large-scale main-frame computer.

We establish a biased operation count for both old and new delta-coefficient recursion relations by timing these two different algorithms on a large-scale main-frame computer and comparing the results with the time required by our previously defined standard arithmetic operation. From this, we find that the number of arithmetic operations needed for calculating all of the so-called delta coefficients, the real multiplication coefficients in the summations mentioned above, can be estimated to go from 5/8 N^3 operations in the old algorithm to 1/4 N^3 operations in our new algorithm, resulting in a savings of 3/8 N^3 operations. If we assume that an N-term FFT requires $N \log_2 N$ operations for its execution,[2] then reformulating the full-sphere algorithm results in a savings of about 41/24 $N^3 - 5N^2$ operations. An overall estimate of the number of arithmetic operations in our full-sphere scanning algorithm, neglecting operation counts of order N or fewer, is 19/12 $N^3 + 32N^2(\log_2 N + 2\ 1/4)$ operations, while a similar estimate for hemispherical scanning would be 11/12 $N^3 + 12N^2(\log_2 N + 3\ 1/2)$ operations.

In the preceding, we took $M = N$. If in actuality $M \leqq N/2$, then to first order the corresponding operation counts are obtained by multiplying the preceding results by 1.5 M/N. In the case of hemispherical scanning with odd m Fourier coefficients set to zero (radiation out of the equator of the measurement sphere), the above $M = N$ operation count for full-sphere scanning is halved.

VI. Conclusion

Hemispherical scanning has been formulated within a compact set of equations. In doing this, we have developed a significantly more efficient algorithm for both hemispherical and full-sphere scanning. For example, with $N = 360$, we estimate earlier full-sphere algorithms would require nearly one-and-two-thirds times as many operational steps as our new full-sphere algorithm and nearly three times as many steps as our hemispherical algorithm. When back-hemisphere radiation is negligible, it will be nearly one and three-quarters times more efficient to use hemispherical scanning than the new full-sphere algorithm. Finally, our new formulations using the FFT for computing the lagged product within the spherical algorithm remain exact; i.e., no aliasing errors are introduced.

Appendix I

Simultaneous Evaluation by Pairs of θ Fourier Transforms

We wish to obtain the DFT of $W_m^\mu(\theta)$, taking advantage of the fact that as function of θ this is an even function or an odd function, depending on whether m is odd or even, respectively. This character of the function that we wish to Fourier transform is evident from (13), while from (15) we see that the Fourier transform coefficients of an even or odd function are themselves even or odd. We make use of these properties in order to cut in half the total number of separate θ FFT's that must be processed by the spherical-scanning algorithms. To keep our presentation simple, the full-sphere scanning case is treated here. The required modifications to (14) in order to apply this technique to single-hemisphere scanning should be obvious.

We define

$$\left.\begin{aligned} X_l^\mu(\kappa\Delta\theta) &= W_{2l}^\mu(\kappa\Delta\theta) + W_{2l+1}^\mu(\kappa\Delta\theta) \\ X_l^\mu([N_\theta-\kappa]\Delta\theta) &= -W_{2l}^\mu(\kappa\Delta\theta) + W_{2l+1}^\mu(\kappa\Delta\theta) \end{aligned}\right\} \begin{aligned} &\kappa = 0, 1, 2, \cdots, \frac{N_\theta}{2} \\ &l = 0, 1, 2, \cdots, \frac{N_\phi}{2} - 1. \end{aligned} \tag{26}$$

The DFT of (26) is given by

$$A_l^\mu(m'') = \frac{1}{N_\theta} \sum_{\kappa=0}^{N_\theta - 1} X_l^\mu(\kappa\Delta\theta) e^{-im''\kappa\Delta\theta}. \tag{27}$$

Separating the even-odd Fourier coefficients that are combined

[2] When N is a power of 2, an N-term FFT requires 1/2 $N(\log_2 N - 5/2)$ complex multiplications and $N \log_2 N$ complex additions for its execution [13].

together in (27) results in

$$\left.\begin{aligned} W_{m''}^{2l,\mu} &= 1/2[A_l^{\mu}(m'') - A_l^{\mu}(N_\theta - m'')] \\ W_{m''}^{2l+1,\mu} &= 1/2[A_l^{\mu}(m'') + A_l^{\mu}(N_\theta - m'')] \end{aligned}\right\} \begin{aligned} m'' &= 0, 1, 2, \cdots, \frac{N_\theta}{2} \\ l &= 0, 1, 2, \cdots, \frac{N_\phi}{2} - 1. \end{aligned}$$

Two θ DFT's are combined together in (27), so that the total number of θ FFT operations is halved with negligible additional effort. With minor modifications, this same algorithm is applied in reverse to halve the number of inverse θ Fourier transforms needed to compute the far field.

Appendix II

The Near-Field to Far-Field Fourier Coefficients Transformation

Here, we wish to eliminate the unknown modal expansion coefficients Q^{smn} from

$$\sum_{s=1}^{2} \hat{R}'_{s\mu n} Q^{smn} = \frac{1}{\sqrt{n(n+1)}} i^{m-\mu} I_n^{m\mu}$$

and

$$S_n^{m\mu} = \frac{1}{\sqrt{n(n+1)}} i^{\mu-m} \sum_{s=1}^{2} \bar{\bar{R}}'_{s\mu n} Q^{smn}$$

to obtain $S_n^{m\mu}$ in terms of $I_n^{m\mu}$. Here, $\hat{R}'_{s\mu n}$ denotes known translated probe-receiving coefficients, while $\bar{\bar{R}}'_{s\mu n}$ denotes known translated ideal dipole receiving coefficients. The values of $I_n^{m\mu}$ are assumed to have already been calculated, and we wish to solve for the $S_n^{m\mu}$ coefficients, which in turn are summed to obtain the radiated far-field Fourier expansion coefficients.

Efficient processing dictates that four cases of the above equations be considered simultaneously, corresponding to $\mu = \pm 1$ and $m = \pm |m|$. These four cases are most expediently handled using matrix analysis. By inspection we solve the first equation above for the unknown modal expansion coefficients, obtaining

$$\begin{bmatrix} i^{-|m|}Q^{1|m|n} & i^{|m|}Q^{1-|m|n} \\ i^{-|m|}Q^{2|m|n} & i^{|m|}Q^{2-|m|n} \end{bmatrix} = \frac{1}{\sqrt{n(n+1)}D} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \cdot \begin{bmatrix} \hat{R}'_{2-1n} & \hat{R}'_{21n} \\ \hat{R}'_{1-1n} & \hat{R}'_{11n} \end{bmatrix} \begin{bmatrix} I_n^{|m|,1} & I_n^{-|m|,1} \\ I_n^{|m|,-1} & I_n^{-|m|,-1} \end{bmatrix} \tag{28}$$

where

$$D \equiv \hat{R}'_{11n}\hat{R}'_{2-1n} - \hat{R}'_{1-1n}\hat{R}'_{21n}.$$

We next write our expression for the $S_n^{m\mu}$ coefficients as the matrix equation

$$\begin{bmatrix} S_n^{|m|,1} & S_n^{-|m|,1} \\ S_n^{|m|,-1} & S_n^{-|m|,-1} \end{bmatrix} = \frac{1}{\sqrt{n(n+1)}} \begin{bmatrix} \bar{\bar{R}}'_{11n} & -\bar{\bar{R}}'_{21n} \\ -\bar{\bar{R}}'_{1-1n} & \bar{\bar{R}}'_{2-1n} \end{bmatrix} \cdot \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} i^{-|m|}Q^{1|m|n} & i^{|m|}Q^{1-|m|n} \\ i^{-|m|}Q^{2|m|n} & i^{|m|}Q^{2-|m|n} \end{bmatrix}. \tag{29}$$

Now, we recognize the 2×2 matrix on the extreme right in (29) as that given by (28); consequently, we can substitute (28) into (29) and so eliminate the unknown Q^{snn} coefficients from our expression for $S_n^{m\mu}$. However, an elimination of the 2×2 matrix containing the Q^{smn} coefficients results in equal treatment for the positive and negative $|m|$ index quantities. Accordingly, we can collapse the resulting $S_n^{m\mu}$ and $I_n^{m\mu}$ matrices, obtaining

$$\begin{bmatrix} S_n^{m,1} \\ S_n^{m,-1} \end{bmatrix} = \frac{1}{n(n+1)D} \begin{bmatrix} \bar{\bar{R}}'_{11n} & -\bar{\bar{R}}'_{21n} \\ -\bar{\bar{R}}'_{1-1n} & \bar{\bar{R}}'_{2-1n} \end{bmatrix} \cdot \begin{bmatrix} \hat{R}'_{2-1n} & \hat{R}'_{21n} \\ \hat{R}'_{1-1n} & \hat{R}'_{11n} \end{bmatrix} \begin{bmatrix} I_n^{m,1} \\ I_n^{m,-1} \end{bmatrix}. \tag{30}$$

Equation (30) is valid for determining the signal received by an arbitrary ($\mu = \pm 1$ only) antenna with translated receiving coefficients $\bar{\bar{R}}'_{s\mu n}$, physically located an arbitrary distance beyond the measurement sphere. Notice that (30) reduces to $S_n^{m\mu} = I_n^{m\mu}/[n(n+1)]$ when $\bar{\bar{R}}'_{s\mu n} = \hat{R}'_{s\mu n}$. In the event that the $\bar{\bar{R}}'_{s\mu n}$ coefficients correspond to an ideal dipole located at infinity, (30) reduces to a particularly simple form (cf. (22)).

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