

# Green's Function in the Spectral Domain for Biaxial and Uniaxial Anisotropic Planar Dielectric Structures

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**Abstract**—Green's function solutions of the biaxial and uniaxial anisotropic layered-medium planar-structure is formulated in terms of Maxwell's equations. Diagonalized biaxial and uniaxial permittivity tensors in the coordinate system of interest are treated. The Green's function is found in the double Fourier transformed domain for three longitudinal-to-an-axis coupled electric-magnetic field sets applied to a simple layered structure. The approach is applicable to structures having discontinuities in two orthogonal planar directions such as patch radiators or resonators. Spectral Green's function is usable in method-of-moment calculations assisted by Galerkin's method.

## I. INTRODUCTION

THE EFFECT OF unintentional dielectric substrate anisotropy on planar microwave and millimeter wave circuit performance may be important. Additionally, an extra degree of control over resonator or radiator behavior may be desirable by utilizing, for example, electro-optic or piezo-electric effects which depend on dielectric anisotropy. Radiation patterns could be adjustable using such effects for a single radiating patch antenna in contrast to a patch on an isotropic substrate [1]–[7]. Input impedance of the radiating patch could be similarly varied. Mutual coupling between radiating elements in a phased array antenna could also be controlled by altering the tensor permittivity  $\hat{\epsilon}$  elements, causing mutual coupling impedance values to be changed in a way to reduce them or to use them to constructively affect the radiation pattern. Resonators could be made with variable  $Q$  and resonant frequency, the energy storage depending on the field configuration which would be a function of  $\hat{\epsilon}$ . Such variable  $Q$  action could be envisioned for closed resonators possessing conducting walls as well as open dielectric resonators.

Spectral Green's function  $\hat{G}$  solution of an anisotropic layered planar structure is formulated here in terms of Maxwell's equations. The differential equation formulation is general and is applicable to frequency dispersive nonradiative and radiative problems having conductive discontinuities in two orthogonal planar directions. The anisotropic medium is taken as biaxial or uniaxial, and diagonalized in the coordinate system used. Fundamental differential equations are developed in three different coupled field component sets longitudinal-to-an-axis. The spectral Green's function is found for an impulse surface current vector source for a structure simple enough to illustrate the analytical procedure. Any one of the three field sets can reduce the biaxial case to the uniaxial case, but the specific set choices employed felicitously demonstrate the reduction procedure using the differential equations. This is because the coupling coefficients in the second order differential equations become zero, allowing the field components longitudinal-to-an-axis to generate a field solution transverse elec-

tric (TE) or transverse magnetic (TM) to the axis which is the optic axis. For the general biaxial case, the three field set formulations, although providing different representations for  $\hat{G}$ , provide the same final  $\hat{G}$  solution which is given explicitly in the paper for use at the interface of the structure. Extension of the method to structures with more layers or more conductive regions is readily accomplished although the problem does become considerably more involved.  $\hat{G}$  is usable in Fourier-transform domain moment method calculations employing Galerkin's technique to find, for example the actual currents on a patch antenna, and from that determining the radiation field pattern, input impedance, and mutual coupling impedance of coupled rectangular microstrip antennas [8]. Similarly,  $\hat{G}$  is usable in resonant frequency determination for microstrip resonators employing the moment method in the Fourier transform domain [9].

Section I covers the basic equations, and Sections II–IV the differential equation formulations and Green's functions specific to the three coupled field sets.

## II. BASIC EQUATIONS

Time harmonic, plane wave solutions proportional to  $\exp(j\omega t)$  are assumed. Maxwell's equations in vector form are therefore

$$-\nabla \times \vec{E} = j\omega \vec{B} \quad (1)$$

$$\nabla \times \vec{H} = j\omega \vec{D} \quad (2)$$

where volume source terms have been dropped. Source terms are accounted for in the application of boundary conditions (BC). Scalar permeability  $\mu$  and tensor permittivity  $\hat{\epsilon}$  type mediums are considered implying

$$\vec{B} = \mu \vec{H} \quad (3)$$

$$\vec{D} = \hat{\epsilon} \vec{E}. \quad (4)$$

$\hat{\epsilon}$  is taken to be a diagonalized tensor in the coordinate system utilized, both for the biaxial case or its simplified specialization, the uniaxial case.

For planar structures with layering in the  $y$ -direction and having conductor (i.e., current source) discontinuities in the  $x$ - $z$  plane, the Fourier transformed problem is obtained by a double transform over all variables  $f(x, y, z)$ . Fourier transform pair is defined as

$$\tilde{f}(k_x, y, k_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) e^{-jk_x x} e^{-jk_z z} dx dz \quad (5a)$$

$$f(x, y, z) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k_x, k_y, z) e^{jk_x x} e^{jk_z z} dk_x dk_z. \quad (5b)$$

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Combining (1)-(4) and Fourier transforming by (5),

$$-\tilde{\nabla} \times \tilde{\vec{E}} = j\omega\mu\tilde{\vec{H}} \quad (6a)$$

$$\tilde{\nabla} \times \tilde{\vec{H}} = j\omega\epsilon\tilde{\vec{E}} \quad (6b)$$

where the operator  $\tilde{\nabla}$  is the Fourier transformed operator given by

$$\tilde{\nabla} = jk_x \hat{x} + \frac{d}{dy} \hat{y} + jk_z \hat{z}. \quad (7)$$

From (6) fundamental differential equations describing the field solution in terms of coupled field sets  $(E_i, H_i)$ ,  $i = x, y, z$  are presented in Sections III-V. The choice of these sets is shown to have particular value when the biaxial case is specialized to the uniaxial case. Each  $(E_i, H_i)$  set then corresponds to a specific uniaxial optic axis [10], that axis being just  $x_i$ . All of the three formulations based on  $(E_i, H_i)$  are equally valid and are reasonable to use in the biaxial case.

The structure to be analyzed in the following sections is shown in Fig. 1. A perfect ground plane is located at  $y = 0$ . Vector point source surface current is positioned at the inhomogeneous interface  $y = d$

$$\vec{J}_s = (\hat{x} + \hat{z})\delta(x - x_0)\delta(z - z_0). \quad (8)$$

Boundary conditions to be obeyed are

$$\hat{n} \times (\vec{E}^1 - \vec{E}^2) = 0 \quad (9a)$$

$$\hat{n} \times (\vec{H}^1 - \vec{H}^2) = \vec{J}_s \quad (9b)$$

at planar interfaces where  $\hat{n} = \hat{y}$  and superscripts indicate regions on either side of the interface,  $\hat{n}$  pointing into region 1.

### III. GREEN'S FUNCTION USING $(E_z, H_z)$

Solving the Fourier transform equations (6) after substantial manipulation results in two coupled differential equations.

$$\frac{d^2 \tilde{H}_z}{dy^2} + a\tilde{H}_z + b\frac{d\tilde{E}_z}{dy} = 0 \quad (10a)$$

$$\frac{d^2 \tilde{E}_z}{dy^2} + c\tilde{E}_z + d\frac{d\tilde{H}_z}{dy} = 0. \quad (10b)$$

Here

$$a = g_x \left( j\omega\mu - \frac{k_x^2}{g_y} \right) \quad (11a)$$

$$b = j \frac{k_x k_z}{\omega\mu} \left( \frac{g_x}{g_y} - 1 \right) \quad (11b)$$

$$c = q_y \left( j\omega\epsilon_z - \frac{k_z^2}{q_x} \right) \quad (11c)$$

$$d = j \frac{k_x k_z}{\omega} \left( \frac{1}{\epsilon_y} - \frac{q_y}{q_x \epsilon_x} \right) \quad (11d)$$

$$g_i = \frac{1}{j\omega\mu} (\omega^2 \mu \epsilon_i - k_z^2) \quad (12a)$$

$$q_i = \frac{1}{j\omega\epsilon_i} (\omega^2 \mu \epsilon_i - k_z^2), \quad (12b)$$

where  $i = x, y$ . Equations (10) are two coupled differential equa-

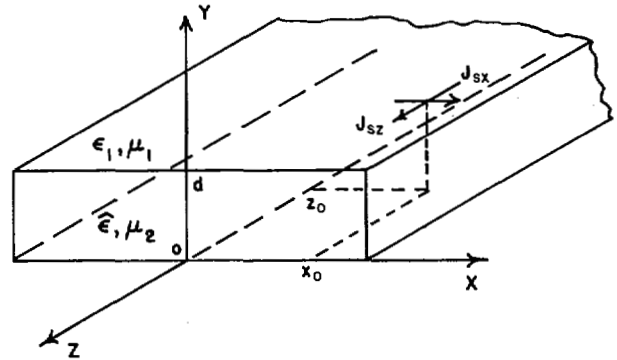


Fig. 1. The structure studied has infinite extent in the  $xz$ -plane, an interface at  $y = d$  between the isotropic region 1 and the anisotropic region 2, and a ground plane (perfect electric wall) at  $y = 0$ . Currents on radiators and resonators at the interface are represented by the surface current point source  $\vec{J}_s$  at  $(x_0, z_0, d)$ .

tions of second order in  $\tilde{E}_z$  and  $\tilde{H}_z$ . When the particular layer under consideration has unequal  $\epsilon_i$ ,  $b \neq 0$ , and  $d \neq 0$  causing the third terms in (10a) and (10b) to be nonzero. Thus  $TE_z$  or  $TM_z$  modes do not exist for the biaxial layer.

Elimination of either  $\tilde{E}_z$  or  $\tilde{H}_z$  from the coupled set (10) makes either field satisfy

$$\frac{d^4 F}{dy^4} + (a + c - bd) \frac{d^2 F}{dy^2} + acF = 0. \quad (13)$$

Sinusoidal field solution of (13) occurs when

$$k_y^4 + (a + c - bd)k_y^2 + ac = 0, \quad (14)$$

$k_y$  being the argument. Two  $k_y^2$  solutions are suggested by (14).

$$k_{y(a,b)}^2 = \frac{a + c - bd}{2} \pm \frac{1}{2} \sqrt{(a + c - bd)^2 - 4ac}. \quad (15)$$

The plus sign in (15) is associated with solution  $k_{ya}^2$ . These solutions correspond to the two phase velocity magnitudes of the biaxial indicatrix [10]-[12] for a particular propagation direction. Suppose we choose  $F = \tilde{H}_z$ . Then by (10a)

$$\tilde{E}_z = -\frac{1}{b} \int \left[ \frac{d^2 \tilde{H}_z}{dy^2} + a\tilde{H}_z \right] dy. \quad (16)$$

From (13) and (16) with  $F = H_z$ , the other field components can be obtained to yield the total field solution in the biaxial layer.

$$\tilde{E}_x = \frac{1}{g_x} \left[ -\frac{d\tilde{H}_z}{dy} + \frac{jk_x k_z}{\omega\mu} \tilde{E}_z \right] \quad (17a)$$

$$\tilde{E}_y = \frac{1}{g_y} \left[ jk_x \tilde{H}_z + \frac{k_z}{\omega\mu} \frac{d\tilde{E}_z}{dy} \right] \quad (17b)$$

$$\tilde{H}_x = \frac{1}{q_y} \left[ \frac{d\tilde{E}_z}{dy} + \frac{jk_x k_z}{\omega\epsilon_y} \tilde{H}_z \right] \quad (17c)$$

$$\tilde{H}_y = \frac{1}{q_x} \left[ -jk_x \tilde{E}_z + \frac{k_z}{\omega\epsilon_x} \frac{d\tilde{H}_z}{dy} \right]. \quad (17d)$$

For isotropic layers (10) are the basic differential equations. They are decoupled,  $a = c = k_z^2$ , and the argument  $k_y$  must satisfy the separation equation

$$-k_y^2 = k_x^2 + k_z^2 - \omega^2 \mu_1 \epsilon_1. \quad (18)$$

Fields are still given by (17) with  $\mu = \mu_1$  and  $\epsilon_i = \epsilon_1$  and (10) represents the standard Helmholtz equation.

Field solutions for  $\tilde{E}_z$  and  $\tilde{H}_z$  are selected as

$$\tilde{H}_z = A_a \sin(k_{ya}y) + A_b \sin(k_{yb}y) + B_a \cos(k_{ya}y) + B_b \cos(k_{yb}y), \quad y < d \quad (19)$$

$$\tilde{H}_z = C_1 e^{-jk_{y1}(y-d)} \quad y > d. \quad (20a)$$

$$\tilde{E}_z = C_2 e^{-jk_{y1}(y-d)} \quad y > d. \quad (20b)$$

$C_1$  and  $C_2$  are independent because  $\text{TE}_z$  and  $\text{TM}_z$  are allowed for the isotropic half-space. We desire  $\text{Im}(k_{y1}) < 0$  so that the radiated field is bounded as  $y \rightarrow \infty$ . Boundary conditions (9a) combined with (16), (17a), (19), and (20) at  $y = 0$  make  $A_a = A_b = 0$ . The remaining BC (9) combined with (16), (17a), (17b), (19) and (20) lead to a matrix equation for the unknown coefficients  $B_a, B_b, C_1, C_2$  in (19) and (20):

$$\begin{bmatrix} \frac{g_1}{g_x} \left( k_{ya} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{ya}^2 - a}{bk_{ya}} \right) \sin(k_{ya}d) & \frac{g_1}{g_x} \left( k_{yb} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{yb}^2 - a}{bk_{yb}} \right) \sin(k_{yb}d) & -jk_{y1} & -\frac{jk_x y_z}{\omega \mu_1} & B_a & 0 \\ \frac{k_{ya}^2 - a}{bk_{ya}} \sin(k_{ya}d) & \frac{k_{yb}^2 - a}{bk_{yb}} \sin(k_{yb}d) & 0 & -1 & B_b & 0 \\ \left( \frac{k_{ya}^2 - a}{b} + \frac{jk_x k_z}{\omega \epsilon_y} \right) \frac{\cos(k_{ya}d)}{q_y} & \left( \frac{k_{yb}^2 - a}{b} + \frac{jk_x k_z}{\omega \epsilon_y} \right) \frac{\cos(k_{yb}d)}{q_y} & -\frac{jk_x k_z}{\omega q_1 \epsilon_1} & \frac{jk_{y1}}{q_1} & C_1 & \tilde{J}_{sz} \\ \cos(k_{ya}d) & \cos(k_{yb}d) & -1 & 0 & C_2 & -\tilde{J}_{sx} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{J}_{sz} \\ -\tilde{J}_{sx} \end{bmatrix} \quad (21)$$

We define tensor Green's function  $\hat{G}$  by

$$\begin{bmatrix} \tilde{E}_x \\ \tilde{E}_z \end{bmatrix} = \hat{G} \begin{bmatrix} \tilde{J}_{sx} \\ \tilde{J}_{sz} \end{bmatrix}. \quad (22)$$

Let the determinantal solution to (21) be

$$B_j = \Gamma_{ji} \tilde{J}_{si} \quad (23a)$$

$$C_j = \Lambda_{ji} \tilde{J}_{si} \quad (23b)$$

where  $j = a, b$  (or 1, 2) and  $i = x, z$  and repeated indices denote summation. Equations (23) combined with (16), (17), (19), (21) and (22) give the following tensor elements  $\tilde{G}_{ij}$  for  $y < d$

$$\tilde{G}_{11} = \frac{1}{g_x} \Gamma_{ax} \left[ k_{ya} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{ya}^2 - a}{bk_{ya}} \right] \sin(k_{ya}y) + \frac{1}{g_x} \Gamma_{bx} \left[ k_{yb} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{yb}^2 - a}{bk_{yb}} \right] \sin(k_{yb}y) \quad (24a)$$

$$\tilde{G}_{12} = \frac{1}{g_x} \Gamma_{az} \left[ k_{ya} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{ya}^2 - a}{bk_{ya}} \right] \sin(k_{ya}y) + \frac{1}{g_x} \Gamma_{bz} \left[ k_{yb} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{yb}^2 - a}{bk_{yb}} \right] \sin(k_{yb}y) \quad (24b)$$

$$\tilde{G}_{21} = \Gamma_{ax} \frac{k_{ya}^2 - a}{bk_{ya}} \sin(k_{ya}y) + \Gamma_{bx} \frac{k_{yb}^2 - a}{bk_{yb}} \sin(k_{yb}y) \quad (24c)$$

$$\tilde{G}_{22} = \Gamma_{az} \frac{k_{ya}^2 - a}{bk_{ya}} \sin(k_{ya}y) + \Gamma_{bz} \frac{k_{yb}^2 - a}{bk_{yb}} \sin(k_{yb}y). \quad (24d)$$

For  $y > d$ , using (17) and (20)–(23) produces

$$e^{jk_{y1}(y-d)} \hat{G} = \begin{bmatrix} \left( \frac{jk_{y1}}{g_1} \Lambda_{1x} + \frac{jk_x k_z}{\omega \mu_1 g_1} \Lambda_{2x} \right) & \left( \frac{jk_{y1}}{g_1} \Lambda_{1z} + \frac{jk_x k_z}{\omega \mu_1 g_1} \Lambda_{2z} \right) \\ \Lambda_{2x} & \Lambda_{2z} \end{bmatrix} \quad (25)$$

The spectral Green's function  $\hat{G}(y)$  given in (24) and (25) for, respectively,  $y < d$  and  $y > d$  can be evaluated at  $y = d$ ,  $\hat{G}(d)$ .

This particular Green's function relates the surface currents found on the radiator patch or resonator conductor to the tangential surface electric field components.  $\hat{G}(d)$  can be used in method of moment calculations enlisting Galerkin's technique to numerically find the actual current distributions. Knowledge of  $\tilde{J}_s$  leads also to the determination of the other field components over varying  $y$  by the use of (17) and (22). Elements of  $\hat{G}(d)$  are provided in the Appendix.

Uniaxial limiting case of the biaxial layer has particular simplicity if  $\epsilon_s = \epsilon_x = \epsilon_y \neq \epsilon_z$  because  $b = d = 0$  by (11) and  $k_{y(a,b)}^2 = a, c$ , by (15).

$$k_{ya}^2 = a = \omega^2 \epsilon_s \mu_2 - k_x^2 - k_z^2 \quad (26a)$$

$$k_{yb}^2 = c = \omega^2 \epsilon_z \mu_2 - k_x^2 - \frac{\epsilon_z}{\epsilon_s} k_z^2. \quad (26b)$$

Equations (26) are the uniaxial separation equations, (26a) being for the ordinary wave and (26b) for the extraordinary wave. Instead of (13), we use (10) to characterize the uniaxial layer. It is seen that  $\tilde{E}_z$  and  $\tilde{H}_z$  are uncoupled,  $\text{TE}_z$  and  $\text{TM}_z$  modes exist with the  $z$ -axis being the optic axis, and  $\tilde{E}_z$  is associated with the extraordinary wave and  $\tilde{H}_z$  with the ordinary wave. The uniaxial  $\tilde{E}_z, \tilde{H}_z$  field forms differ for  $y < d$  from (19):

$$\tilde{E}_z = B_b \sin(k_{yb}y) + A_b \cos(k_{yb}y) \quad (27a)$$

$$\tilde{H}_z = A_a \sin(k_{ya}y) + B_a \cos(k_{ya}y). \quad (27b)$$

(24c) Application of BC (9a) at  $y = 0$  imposes  $A_a = A_b = 0$ . The other

BC generate a matrix solution for  $B_a$ ,  $B_b$ ,  $C_1$ , and  $C_2$ :

$$\begin{bmatrix} \frac{g_1}{g_s} k_{ya} \sin(k_{ya}d) & \frac{g_1}{g_s} \frac{jk_x k_z}{\omega \mu_2} \sin(k_{yb}d) & -jk_{y1} & -\frac{jk_x k_z}{\omega \mu_1} \\ 0 & \sin(k_{yb}d) & 0 & -1 \\ \frac{jk_x k_z}{\omega \epsilon_s} \frac{\cos(k_{ya}d)}{q_s} & k_{yb} \frac{\cos(k_{yb}d)}{q_s} & -\frac{jk_x k_z}{\omega \epsilon_1 q_1} & \frac{jk_{y1}}{q_1} \\ \cos(k_{ya}d) & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} B_a \\ B_b \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{J}_{sz} \\ -\tilde{J}_{sx} \end{bmatrix} \quad (28)$$

$\hat{G}$  using (17), (22), (23), (27), and (28) is expressed as

$$\tilde{G}_{11} = \frac{k_{ya}}{g_s} \Gamma_{ax} \sin(k_{ya}y) + \frac{jk_x k_z}{\omega \mu_2 g_s} \Gamma_{bx} \sin(k_{yb}y) \quad (29a)$$

$$\tilde{G}_{12} = \frac{k_{ya}}{g_s} \Gamma_{az} \sin(k_{ya}y) + \frac{jk_x k_z}{\omega \mu_2 g_s} \Gamma_{bz} \sin(k_{yb}y) \quad (29b)$$

$$\tilde{G}_{21} = \Gamma_{bx} \sin(k_{yb}y) \quad (29c)$$

$$\tilde{G}_{22} = \Gamma_{bz} \sin(k_{yb}y) \quad (29d)$$

for  $y < d$ . Using (22), (23), and (28) we find the form (25) holds for  $y > d$ .

#### IV. GREEN'S FUNCTION USING $(E_y, H_y)$

Again solving (6) as in Section III,

$$\frac{d^2 \tilde{H}_y}{dy^2} + a \tilde{H}_y + b \frac{d \tilde{E}_y}{dy} = 0 \quad (30a)$$

$$\frac{d^2 \tilde{E}_y}{dy^2} + c \tilde{E}_y + d \frac{d \tilde{H}_y}{dy} = 0. \quad (30b)$$

Here

$$a = -(k_x^2 + k_z^2) \left[ 1 - \frac{\omega^2 \mu \epsilon_x \epsilon_z}{k_x^2 \epsilon_x + k_z^2 \epsilon_z} \right] \quad (31a)$$

$$b = \frac{j \omega k_x k_z \epsilon_y (\epsilon_x - \epsilon_z)}{k_x^2 \epsilon_x + k_z^2 \epsilon_z} \quad (31b)$$

$$c = -(k_x^2 \epsilon_x + k_z^2 \epsilon_z) \left[ \frac{1}{\epsilon_y} - \frac{\omega^2 \mu}{k_x^2 + k_z^2} \right] \quad (31c)$$

$$d = \frac{\epsilon_x - \epsilon_z}{\epsilon_y} \frac{j \omega \mu k_x k_z}{k_x^2 + k_z^2} \quad (31d)$$

Equations (30) demonstrate that  $TE_y$  and  $TM_y$  modes cannot exist. Elimination of  $\tilde{E}_y$  or  $\tilde{H}_y$  in (30) produces a single equation of the form (13) with separation equation (14) and solution (15).  $k_{y(a,b)}$  solutions determined using (31) are the same as found in Section III since the  $F$  derivative coefficients in (13) for the  $(E_y, H_y)$  formulation approach here are identical to those found for

the  $(E_z, H_z)$  formulation. Once  $\tilde{E}_y$  and  $\tilde{H}_y$  are determined, the remaining four components are given by

$$\tilde{E}_x = \frac{1}{\omega k_z (\epsilon_x - \epsilon_z)} \left[ \frac{d^2 \tilde{H}_y}{dy^2} + (\omega^2 \mu \epsilon_z - k_x^2 - k_z^2) \tilde{H}_y \right] \quad (32a)$$

$$\tilde{E}_z = \frac{1}{\omega k_x (\epsilon_x - \epsilon_z)} \left[ \frac{d^2 \tilde{H}_y}{dy^2} + (\omega^2 \mu \epsilon_x - k_x^2 - k_z^2) \tilde{H}_y \right] \quad (32b)$$

$$\tilde{H}_x = \frac{1}{\omega \mu k_z (\epsilon_x - \epsilon_z)} \left[ \epsilon_y \frac{d^2 \tilde{E}_y}{dy^2} + (\omega^2 \mu \epsilon_x \epsilon_y - \epsilon_x k_x^2 - \epsilon_z k_z^2) \tilde{E}_y \right] \quad (32c)$$

$$\tilde{H}_z = \frac{1}{\omega \mu k_x (\epsilon_x - \epsilon_z)} \left[ \epsilon_y \frac{d^2 \tilde{E}_y}{dy^2} + (\omega^2 \mu \epsilon_y \epsilon_z - \epsilon_x k_x^2 - \epsilon_z k_z^2) \tilde{E}_y \right] \quad (32d)$$

Field solutions for  $\tilde{E}_y$  and  $\tilde{H}_y$  are given by (19) and (20), replacing  $\tilde{E}_z$ ,  $\tilde{H}_z$  by  $\tilde{E}_y$ ,  $\tilde{H}_y$  to give the biaxial result for  $y < d$  and isotropic result for  $y > d$ . For  $y > d$ , (18) is used for the isotropic case with  $k_1^2 = \omega^2 \mu_1 \epsilon_1$ , and since (32) no longer holds, a vector potential construction [13] gives the remaining fields as a  $TE_y$  and  $TM_y$  superposition:

$$\tilde{E}_x = \frac{jk_x}{k_1^2 - k_{y1}^2} \frac{d \tilde{E}_y}{dy} - \frac{\omega \mu_1 k_z}{k_1^2 - k_{y1}^2} \tilde{H}_y \quad (33a)$$

$$\tilde{E}_z = \frac{jk_z}{k_1^2 - k_{y1}^2} \frac{d \tilde{E}_y}{dy} + \frac{\omega \mu_1 k_x}{k_1^2 - k_{y1}^2} \tilde{H}_y \quad (33b)$$

$$\tilde{H}_x = \frac{\omega \epsilon_1 k_z}{k_1^2 - k_{y1}^2} \tilde{E}_y + \frac{jk_x}{k_1^2 - k_{y1}^2} \frac{d \tilde{H}_y}{dy} \quad (33c)$$

$$\tilde{H}_z = \frac{\omega \epsilon_1 k_x}{k_1^2 - k_{y1}^2} \tilde{E}_y + \frac{jk_z}{k_1^2 - k_{y1}^2} \frac{d \tilde{H}_y}{dy} \quad (33d)$$

Boundary condition (9a) at  $y = 0$  using (19), (20), (32a), (32b), and (33a), (33b) makes  $B_a = B_b = 0$ . Remaining BC (9) combined with (19), (20), (32), and (33) produces a matrix equa-

tion for the unknown coefficients  $A_a, A_b, C_1$ , and  $C_2$ :

$$\begin{bmatrix} \theta_{1a} \sin(k_{ya}d) & \theta_{1b} \sin(k_{yb}d) & -\frac{j\omega\mu_1 k_z}{k_1^2 - k_{y1}^2} & \frac{jk_x k_{y1}}{k_1^2 - k_{y1}^2} \\ \theta_{2a} \sin(k_{ya}d) & \theta_{2b} \sin(k_{yb}d) & \frac{\omega\mu_1 k_x}{k_1^2 - k_{y1}^2} & \frac{k_{y1} k_z}{k_1^2 - k_{y1}^2} \\ \theta_{3a} \frac{k_{ya}^2 - a}{bk_{ya}} \cos(k_{ya}d) & \theta_{3b} \frac{k_{yb}^2 - a}{bk_{yb}} \cos(k_{yb}d) & \frac{k_x k_{y1}}{k_1^2 - k_{y1}^2} & \frac{\omega\epsilon_1 k_z}{k_1^2 - k_{y1}^2} \\ \theta_{4a} \frac{k_{yb}^2 - a}{bk_{ya}} \cos(k_{ya}d) & \theta_{4b} \frac{k_{yb}^2 - a}{bk_{yb}} \cos(k_{yb}d) & \frac{k_{y1} k_z}{k_1^2 - k_{y1}^2} & -\frac{\omega\epsilon_1 k_x}{k_1^2 - k_{y1}^2} \end{bmatrix} \begin{bmatrix} A_a \\ A_b \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\tilde{J}_{sz} \\ \tilde{J}_{sx} \end{bmatrix} \quad (34)$$

Here

$$\theta_{1(a,b)} = \frac{j}{\omega(\epsilon_z - \epsilon_x)k_z} [\omega^2 \mu_2 \epsilon_z - k_x^2 - k_z^2 - k_{y(a,b)}^2] \quad (35a)$$

$$\theta_{2(a,b)} = \frac{1}{\omega(\epsilon_z - \epsilon_x)k_x} [\omega^2 \mu_2 \epsilon_x - k_x^2 - k_z^2 - k_{y(a,b)}^2] \quad (35b)$$

$$\theta_{3(a,b)} = \frac{1}{\omega\mu(\epsilon_x - \epsilon_z)k_z} [\omega^2 \mu_2 \epsilon_x \epsilon_y - \epsilon_x k_x^2 - \epsilon_z k_z^2 - \epsilon_y k_{y(a,b)}^2] \quad (35c)$$

$$\theta_{4(a,b)} = \frac{1}{\omega\mu(\epsilon_x - \epsilon_z)k_x} [\omega^2 \mu_2 \epsilon_y \epsilon_z - \epsilon_x k_x^2 - \epsilon_z k_z^2 - \epsilon_y k_{y(a,b)}^2] \quad (35d)$$

Green's function  $\hat{G}$  for the biaxial layer for  $y < d$  is given by (22) replacing the field vector by  $(\tilde{E}_y, \tilde{H}_y)^T$ , using the form of (19), enlisting (32) and (34) and employing (23a) to define  $A_j$  determinantal coefficients  $\Gamma_{ji}$ . Tensor elements  $\tilde{G}_{ij}$  for  $y < d$  are

$$\tilde{G}_{11} = \frac{1}{\omega k_z(\epsilon_x - \epsilon_z)} [(\omega^2 \mu_2 \epsilon_z - k_x^2 - k_{ya}^2 - k_z^2) \Gamma_{ax} \cdot \sin(k_{ya}y) + (\omega^2 \mu_2 \epsilon_z - k_x^2 - k_{yb}^2 - k_z^2) \Gamma_{bx} \cdot \sin(k_{yb}y)] \quad (36a)$$

$$\tilde{G}_{12} = \frac{1}{\omega k_z(\epsilon_x - \epsilon_z)} [(\omega^2 \mu_2 \epsilon_z - k_x^2 - k_{ya}^2 - k_z^2) \Gamma_{az} \cdot \sin(k_{ya}y) + (\omega^2 \mu_2 \epsilon_z - k_x^2 - k_{yb}^2 - k_z^2) \Gamma_{bz} \cdot \sin(k_{yb}y)] \quad (36b)$$

$$\tilde{G}_{21} = \frac{1}{\omega k_x(\epsilon_x - \epsilon_z)} [(\omega^2 \mu_2 \epsilon_x - k_x^2 - k_{ya}^2 - k_z^2) \Gamma_{ax} \cdot \sin(k_{ya}y) + (\omega^2 \mu_2 \epsilon_x - k_x^2 - k_{yb}^2 - k_z^2) \Gamma_{bx} \cdot \sin(k_{yb}y)] \quad (36c)$$

$$\tilde{G}_{22} = \frac{1}{\omega k_x(\epsilon_x - \epsilon_z)} [(\omega^2 \mu_2 \epsilon_x - k_x^2 - k_{ya}^2 - k_z^2) \Gamma_{az} \cdot \sin(k_{ya}y) + (\omega^2 \mu_2 \epsilon_x - k_x^2 - k_{yb}^2 - k_z^2) \Gamma_{bz} \cdot \sin(k_{yb}y)] \quad (36d)$$

For  $y > d$ , using the form of (20), and (23), (33), and (34), the spectral Green's function  $\hat{G}$  is

$$\begin{aligned} e^{jk_{y1}(y-d)} \hat{G} &= \frac{1}{k_1^2 - k_{y1}^2} \cdot \begin{bmatrix} (k_x k_{y1} \Lambda_{2x} - \omega\mu_1 k_z \Lambda_{1x}) (k_x k_{y1} \Lambda_{2z} - \omega\mu_1 k_z \Lambda_{1z}) \\ (k_z k_{y1} \Lambda_{2x} + \omega\mu_1 k_x \Lambda_{1x}) (k_z k_{y1} \Lambda_{2z} + \omega\mu_1 k_x \Lambda_{1z}) \end{bmatrix} \end{aligned} \quad (37)$$

Choosing the uniaxial limiting case with optic axis in the  $y$ -direction causes (30) to decouple and

$$k_{ya}^2 = a = \omega^2 \mu_2 \epsilon_s - k_x^2 - k_z^2 \quad (38a)$$

$$k_{yb}^2 = c = \omega^2 \mu_2 \epsilon_s - \frac{\epsilon_s}{\epsilon_y} (k_x^2 + k_z^2). \quad (38b)$$

Here  $\epsilon_s = \epsilon_x = \epsilon_z \neq \epsilon_y$ . Equation (38a) corresponds to the ordinary wave ( $\tilde{H}_y$ ) and (38b) to the extraordinary wave ( $\tilde{E}_y$ ). TE<sub>y</sub> and TM<sub>y</sub> modes do exist. Going back to (6) since (32) cannot be used, choosing  $\tilde{E}_y$  and  $\tilde{H}_y$  according to (27), and applying the BC (9) gives

$$\tilde{E}_x = \frac{jk_x}{k_x^2 + k_z^2} \frac{\epsilon_y}{\epsilon_s} \frac{d\tilde{E}_y}{dy} - \frac{\omega\mu_2 k_z}{k_x^2 + k_z^2} \tilde{H}_y \quad (39a)$$

$$\tilde{E}_z = \frac{jk_z}{k_x^2 + k_z^2} \frac{\epsilon_y}{\epsilon_s} \frac{d\tilde{E}_y}{dy} + \frac{\omega\mu_2 k_x}{k_x^2 + k_z^2} \tilde{H}_y \quad (39b)$$

$$\tilde{H}_x = \frac{\omega\epsilon_y k_z}{k_x^2 + k_z^2} \tilde{E}_y + \frac{jk_x}{k_x^2 + k_z^2} \frac{d\tilde{H}_y}{dy} \quad (39c)$$

$$\tilde{H}_z = \frac{-\omega\epsilon_y k_x}{k_x^2 + k_z^2} \tilde{E}_y + \frac{jk_z}{k_x^2 + k_z^2} \frac{d\tilde{H}_y}{dy} \quad (39d)$$

for the remaining field components (note form is the same as (33) using (38)) and

$$\begin{bmatrix} -j\omega\mu_2 k_z \sin(k_{ya}d) & k_x \frac{\epsilon_y}{\epsilon_s} k_{yb} \sin(k_{yb}d) & j\omega\mu_1 k_z & -jk_x k_{y1} \\ \omega\mu k_x \sin(k_{ya}d) & -jk_z \frac{\epsilon_y}{\epsilon_s} k_{yb} \sin(k_{yb}d) & -\omega\mu_1 k_x & -k_z k_{y1} \\ jk_x k_{ya} \cos(k_{ya}d) & \omega\epsilon_y k_z \cos(k_{yb}d) & -k_x k_{y1} & -\omega\epsilon_1 k_z \\ jk_z k_{ya} \cos(k_{ya}d) & -\omega\epsilon_y k_x \cos(k_{yb}d) & -k_z k_{y1} & \omega\epsilon_1 k_x \end{bmatrix} \begin{bmatrix} A_a \\ A_b \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{J}_{sz}(k_x^2 + k_z^2) \\ -\tilde{J}_{sx}(k_x^2 + k_z^2) \end{bmatrix} \quad (40)$$

$B_a = B_b = 0$  by the BC constraints.

Spectral Green's function, using the forms of (23) and (27), and (22), (39) and (40), for the uniaxial case is given by

with

$$\tilde{G}_{11} = -\frac{1}{k_x^2 + k_z^2} \left[ \omega\mu_2 k_z \Gamma_{ax} \sin(k_{ya}y) + jk_x k_{yb} \frac{\epsilon_y}{\epsilon_s} \Gamma_{bx} \sin(k_{yb}y) \right] \quad (41a)$$

$$a = -\frac{p_z}{p_y} k_z^2 - p_z \quad (43)$$

$$b = -\frac{jk_x k_z}{\omega\mu} \left( \frac{p_z}{p_y} - 1 \right) \quad (44a)$$

$$\tilde{G}_{12} = -\frac{1}{k_x^2 + k_z^2} \left[ \omega\mu_2 k_z \Gamma_{az} \sin(k_{ya}y) + jk_x k_{yb} \frac{\epsilon_y}{\epsilon_s} \Gamma_{bz} \sin(k_{yb}y) \right] \quad (41b)$$

$$c = -\frac{p_y}{p_z} \frac{\epsilon_z}{\epsilon_y} k_z^2 - \frac{\epsilon_x}{\epsilon_y} p_y \quad (44b)$$

$$d = -\frac{jk_x k_z}{\omega\epsilon_y} \left( 1 - \frac{p_y}{p_z} \right) \quad (44c)$$

$$\tilde{G}_{21} = -\frac{1}{k_x^2 + k_z^2} \left[ -\omega\mu_2 k_x \Gamma_{ax} \sin(k_{ya}y) + jk_z k_{yb} \frac{\epsilon_y}{\epsilon_s} \Gamma_{bx} \sin(k_{yb}y) \right] \quad (41c)$$

$$p_i = k_x^2 - \omega^2 \mu \epsilon_i, \quad i = y, z. \quad (44d)$$

Coupled equations (42) are reducible to a single fourth order differential equation of the form (13). Knowledge of  $\tilde{E}_x$  and  $\tilde{H}_x$  gives

$$\tilde{E}_y = \frac{\omega\mu}{jp_y} \left[ -jk_z \tilde{H}_x + \frac{k_x}{\omega\mu} \frac{d\tilde{E}_x}{dy} \right] \quad (45a)$$

$$\tilde{H}_y = \frac{k_x}{jp_z} \left[ \frac{d\tilde{H}_x}{dy} + \frac{j\omega\epsilon_z k_z}{k_x} \tilde{E}_x \right] \quad (45b)$$

$$\tilde{E}_z = \frac{\omega\mu}{jp_z} \left[ \frac{d\tilde{H}_x}{dy} + \frac{jk_x k_z}{\omega\mu} \tilde{E}_x \right] \quad (45c)$$

$$\tilde{H}_z = \frac{jk_x}{p_y} \left[ -jk_z \tilde{H}_x + \frac{\omega\epsilon_y}{k_x} \frac{d\tilde{E}_x}{dy} \right] \quad (45d)$$

when  $y < d$ . For  $y > d$ , we find that the form of (37) is valid for  $\tilde{G}$ .

## V. GREEN'S FUNCTION USING $(E_x, H_x)$

The second-order differential equations based on (6) are

$$\frac{d^2 \tilde{H}_x}{dy^2} + a \tilde{H}_x + b \frac{d\tilde{E}_x}{dy} = 0 \quad (42a)$$

$$\frac{d^2 \tilde{E}_x}{dy^2} + c \tilde{E}_x + d \frac{d\tilde{H}_x}{dy} = 0 \quad (42b)$$

Field solutions for  $\tilde{E}_x$  and  $\tilde{H}_x$  are chosen as the forms in (19) and (20). Equations (45) give the other field components for both the biaxial and isotropic layer. Boundary condition (9a) at  $y = 0$  makes  $A_a = A_b = 0$ . Boundary condition at  $y = d$  generates a matrix equation for the unknown coefficients  $B_a, B_b, C_1$ , and

$C_2$ :

$$\begin{aligned}
 & \begin{bmatrix} \frac{k_{ya}^2 - a}{bk_{ya}} \sin(k_{ya}d) & \frac{k_{yb}^2 - a}{bk_{yb}} \sin(k_{yb}d) & 0 & -1 \\ \frac{\mu_2 p_1}{\mu_1 p_z} \left( k_{ya} - \frac{jk_x k_z}{\omega \mu_2} \frac{k_{ya}^2 - a}{bk_{ya}} \right) \sin(k_{ya}d) & \frac{\mu_2 p_1}{\mu_1 p_z} \left( k_{yb} - \frac{jk_x k_z}{\omega \mu_2} \frac{k_{yb}^2 - a}{bk_{yb}} \right) \sin(k_{yb}d) & -jk_{y1} & \frac{jk_x k_z}{\omega \mu_1} \\ \cos(k_{ya}d) & \cos(k_{yb}d) & -1 & 0 \\ \frac{jk_x}{p_y} \left( -jk_z + \frac{\omega \epsilon_y}{k_x} \frac{k_{ya}^2 - a}{b} \right) \cos(k_{ya}d) & \frac{jk_x}{p_y} \left( -jk_z + \frac{\omega \epsilon_y}{k_x} \frac{k_{yb}^2 - a}{b} \right) \cos(k_{yb}d) & -\frac{k_x k_z}{p_1} & -\frac{\omega \epsilon_1 k_{y1}}{p_1} \end{bmatrix} \begin{bmatrix} B_a \\ B_b \\ C_1 \\ C_2 \end{bmatrix} \\
 & = \begin{bmatrix} 0 \\ 0 \\ \tilde{J}_{sz} \\ -\tilde{J}_{sx} \end{bmatrix}. \tag{46}
 \end{aligned}$$

Green's function  $\hat{G}$  for  $y < d$  is found employing a form like (16) based on (42a), (19), (22), (23), (45), and (46). For  $y < d$

$$\tilde{G}_{11} = \Gamma_{ax} \frac{k_{ya}^2 - a}{bk_{ya}} \sin(k_{ya}y) + \Gamma_{bx} \frac{k_{yb}^2 - a}{bk_{yb}} \sin(k_{yb}y) \tag{47a}$$

$$\tilde{G}_{12} = \Gamma_{az} \frac{k_{ya}^2 - a}{bk_{ya}} \sin(k_{ya}y) + \Gamma_{bz} \frac{k_{yb}^2 - a}{bk_{yb}} \sin(k_{yb}y) \tag{47b}$$

$$\begin{aligned}
 \tilde{G}_{22} = & \frac{\omega \mu_2}{jp_z} \Gamma_{az} \left[ -k_{ya} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{ya}^2 - a}{bk_{ya}} \right] \sin(k_{ya}y) \\
 & + \frac{\omega \mu_2}{jp_z} \Gamma_{bz} \left[ -k_{yb} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{yb}^2 - a}{bk_{yb}} \right] \sin(k_{yb}y). \tag{47d}
 \end{aligned}$$

When  $y > d$ , availing oneself of the form of (20) and using (22), (23), (45), and (46), the spectral Green's function is

$$e^{jk_{y1}(y-d)} \hat{G} = \begin{bmatrix} \Lambda_{2x} & \Lambda_{2z} \\ \frac{\omega \mu_1}{jp_1} \left( -jk_{y1} \Lambda_{1x} + \frac{jk_x k_z}{\omega \mu_1} \Lambda_{2x} \right) & \frac{\omega \mu_1}{jp_1} \left( -jk_{y1} \Lambda_{1z} + \frac{jk_x k_z}{\omega \mu_1} \Lambda_{2z} \right) \end{bmatrix}. \tag{48}$$

$$\begin{aligned}
 \tilde{G}_{21} = & \frac{\omega \mu_2}{jp_z} \Gamma_{ax} \left[ -k_{ya} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{ya}^2 - a}{bk_{ya}} \right] \sin(k_{ya}y) \\
 & + \frac{\omega \mu_2}{jp_z} \Gamma_{bx} \left[ -k_{yb} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{yb}^2 - a}{bk_{yb}} \right] \sin(k_{yb}y)
 \end{aligned}$$

Placing the optic axis for the uniaxial limiting case in the x-direction makes (42) decouple and

$$k_{ya}^2 = a = \omega^2 \mu_2 \epsilon_s - k_x^2 - k_z^2 \tag{49a}$$

$$k_{yb}^2 = c = \omega^2 \mu_2 \epsilon_x - \frac{\epsilon_x}{\epsilon_s} k_x^2 - k_z^2. \tag{49b}$$

(47c) Here  $\epsilon_s = \epsilon_y = \epsilon_z \neq \epsilon_x$ . Equation (49a) relates to the ordinary

wave ( $\tilde{H}_x$ ) and (49b) to the extraordinary wave ( $\tilde{E}_x$ ).  $TE_x$  and  $TM_x$  modes exist. Enforcing the BC (9) using  $\tilde{E}_x$ ,  $\tilde{H}_x$  (from (27)), and (45) leads to a matrix equation in the unknown coefficients after noting that  $A'_a = A_b = 0$ :

$$\begin{bmatrix} 0 & \sin(k_{yb}d) & 0 & -1 \\ \frac{\mu_2}{\mu_1} \frac{p_1}{p_s} k_{ya} \sin(k_{ya}d) & -\frac{j}{\omega\mu_1} \frac{p_1}{p_s} k_x k_z \sin(k_{yb}d) & -jk_{y1} & \frac{jk_x k_z}{\omega\mu_1} \\ \cos(k_{ya}d) & 0 & -1 & 0 \\ \frac{k_x k_z}{p_s} \cos(k_{ya}d) & \frac{j\omega\epsilon_y k_{yb}}{p_s} \cos(k_{yb}d) & -\frac{k_x k_z}{p_1} & -\frac{\omega\epsilon_1 k_{y1}}{p_1} \end{bmatrix} \begin{bmatrix} B_a \\ B_b \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \tilde{J}_{sz} \\ -\tilde{J}_{sx} \end{bmatrix} \quad (50)$$

Green's function for the uniaxial limiting case with the help of (22), (23), (45), (50) and the form of (27), for  $y < d$  is now given by

$$\tilde{G}_{11} = \Gamma_{bx} \sin(k_{yb}y) \quad (51a)$$

$$\tilde{G}_{12} = \Gamma_{bz} \sin(k_{yb}y) \quad (51b)$$

$$\tilde{G}_{21} = \frac{\omega\mu_2}{jp_s} \left[ -k_{ya} \Gamma_{ax} \sin(k_{ya}y) + \frac{jk_x k_z}{\omega\mu_2} \Gamma_{bx} \sin(k_{yb}y) \right] \quad (51c)$$

$$\tilde{G}_{22} = \frac{\omega\mu_2}{jp_s} \left[ -k_{ya} \Gamma_{az} \sin(k_{ya}y) + \frac{jk_x k_z}{\omega\mu_2} \Gamma_{bz} \sin(k_{yb}y) \right] \quad (51d)$$

$\hat{G}$  for  $y > d$  is given by expression (48) using  $\Lambda_{ji}$  as determined by (50).

## VI. CONCLUDING REMARKS

A process for obtaining the Green's function in the double Fourier transformed domain has been demonstrated here for planar problems with conductive planar discontinuities where the layers are biaxial anisotropic dielectrics. It has been proven that problem solutions based on  $TE_i$  and  $TM_i$  modal constructions where  $i$  = axis (or direction) are not possible, but nevertheless manageable methods can still be applied to obtain the Green's function.

One realizes that procedures used to treat examples based on a two-layered structure (a finite layer and a semi-infinite half-space) in this paper are extendable to any number of layers. However, the algebraic ease or transparency with which some problems can be treated because of  $TE_i$  and  $TM_i$  mode existence and their modal decoupled nature used in finding the hybrid mode construction can not be utilized for the biaxial layer. Although, layers which are uniaxial [14] or isotropic [15] in a sandwich structure can still be approached in that manner.

The approach in this paper is amenable toward solving radiation and resonator structures having biaxial (or uniaxial) layers in conjunction with the moment method using Galerkin's technique. Use of biaxial (or uniaxial) materials controllable by the electro-

optic or piezo-electric effects, for example, could allow control of radiation patterns from patch antennas or resonant frequency change of patch resonators. Such structures are compatible with both hybrid and monolithic circuit technology operating in the microwave and millimeter frequency regions.

## APPENDIX DYADIC SPECTRAL GREEN'S FUNCTION AT THE INTERFACE

The dyadic or tensor spectral Green's function evaluated at the  $y = d$  interface is used in method of moment approaches to first find surface currents on the conductors and then other quantities of interest, including fields, radiated power, and resonant frequencies.  $\tilde{G}(d)$  defined by (22) is a  $2 \times 2$  immittance tensor, with units of ohms, relating surface electric field and surface current. Because tangential electric field is continuous across the  $y = d$  interface,  $\tilde{G}(y)$  with  $y \rightarrow d^-$  from (24) and  $\tilde{G}(y)$  with  $y \rightarrow d^+$  from (25) have a unique limit at  $y = d$ ,  $\tilde{G}(d)$ . Using (24) and (25) in the  $(E_z, H_z)$  formulation enables  $\tilde{G}(d)$  to be expressed in terms of  $g_i$ ,  $q_i$ ,  $a$ ,  $b$ ,  $c$ , and  $d$ . Elements of  $\tilde{G}(d)$ ,  $\tilde{G}_{ij}(d)$ , are written down after substantial algebraic manipulation as

$$\tilde{G}_{ij} = \tilde{G}'_{ij}/D_m \quad (52)$$

$$\tilde{G}'_{11}(d)$$

$$\begin{aligned} &= \frac{k_z}{q_1 g_x} \frac{a}{b} \left[ k_{y1} - \frac{k_x^2 k_z}{\omega^2 \mu_1 \epsilon_1} \right] \left[ \frac{k_{ya}}{k_{yb}} - \frac{k_{yb}}{k_{ya}} \right] \sin(k_{ya}d) \\ &\quad \cdot \sin(k_{yb}d) - \frac{k_z}{q_y} \left[ \frac{k_{yb}^2 - a}{b} + \frac{jk_x k_z}{\omega \epsilon_y} \right] \left[ \frac{jk_{ya}}{g_x} \right. \\ &\quad \left. + \frac{k_x}{\omega} \left\{ \frac{k_{y1} + k_z}{\mu_1 g_1} - \frac{k_z}{\mu_2 g_x} \right\} - \frac{k_{ya}^2 - a}{bk_{ya}} \right] \sin(k_{ya}d) \\ &\quad \cdot \cos(k_{yb}d) + \frac{k_z}{q_y} \left[ \frac{k_{ya}^2 - a}{b} + \frac{jk_x k_z}{\omega \epsilon_y} \right] \left[ \frac{jk_{yb}}{g_x} + \frac{k_x}{\omega} \right. \\ &\quad \left. \cdot \left\{ \frac{k_{y1} + k_z}{\mu_1 g_1} - \frac{k_z}{\mu_2 g_x} \right\} - \frac{k_{yb}^2 - a}{bk_{yb}} \right] \sin(k_{yb}d) \cos(k_{ya}d) \end{aligned} \quad (53a)$$



$$\begin{aligned}
\tilde{G}'_{12}(d) &= -\frac{jk_x k_z}{\omega \mu_1 g_x} \frac{a}{b} \left[ \frac{k_{ya}}{k_{yb}} - \frac{k_{yb}}{k_{ya}} \right] \sin(k_{ya}d) \sin(k_{yb}d) \\
&\quad - k_z \left[ \frac{jk_{ya}}{g_x} + \frac{k_x}{\omega} \left\{ \frac{k_{y1} + k_z}{\mu_1 g_1} - \frac{k_z}{\mu_2 g_x} \right\} \frac{k_{ya}^2 - a}{bk_{ya}} \right] \\
&\quad \cdot \sin(k_{ya}d) \cos(k_{yb}d) + k_z \left[ \frac{jk_{yb}}{g_x} + \frac{k_x}{\omega} \left\{ \frac{k_{y1} + k_z}{\mu_1 g_1} \right. \right. \\
&\quad \left. \left. - \frac{k_z}{\mu_2 g_x} \right\} \frac{k_{yb}^2 - a}{bk_{yb}} \right] \sin(k_{ya}d) \cos(k_{yb}d) \quad (53b)
\end{aligned}$$

$$\begin{aligned}
\tilde{G}'_{21}(d) &= -\frac{jk_x k_z}{\omega \epsilon_1 q_1} \frac{g_1}{g_x} \frac{a}{b} \left[ \frac{k_{ya}}{k_{yb}} - \frac{k_{yb}}{k_{ya}} \right] \sin(k_{ya}d) \sin(k_{yb}d) \\
&\quad + \frac{jk_{y1}}{q_y} \frac{k_{ya}^2 - a}{kb_{ya}} \left[ \frac{k_{yb}^2 - a}{b} + \frac{jk_x k_z}{\omega \epsilon_y} \right] \sin(k_{ya}d) \cos(k_{yb}d) \\
&\quad - \frac{jk_{y1}}{q_y} \frac{k_{yb}^2 - a}{bk_{yb}} \left[ \frac{k_{ya}^2 - a}{b} + \frac{jk_x k_z}{\omega \epsilon_y} \right] \sin(k_{yb}d) \cos(k_{ya}d). \quad (53c)
\end{aligned}$$

$$\begin{aligned}
\tilde{G}'_{22}(d) &= \frac{g_1}{g_x} \frac{a}{b} \left[ \frac{k_{yb}}{k_{ya}} - \frac{k_{ya}}{k_{yb}} \right] \sin(k_{ya}d) \cos(k_{yb}d) \\
&\quad + jk_{y1} \frac{k_{ya}^2 - a}{bk_{ya}} \sin(k_{ya}d) \cos(k_{yb}d) \\
&\quad - jk_{y1} \frac{k_{yb}^2 - a}{bk_{yb}} \sin(k_{yb}d) \cos(k_{ya}d) \quad (53d) \\
D_m &= \frac{jk_{y1}}{q_1} \frac{g_1}{g_x} \frac{a}{b} \left[ \frac{k_{yb}}{k_{ya}} - \frac{k_{ya}}{k_{yb}} \right] \sin(k_{ya}d) \sin(k_{yb}d) \\
&\quad + \frac{jk_{y1}}{bq_y} (k_{ya}^2 - k_{yb}^2) \cos(k_{ya}d) \cos(k_{yb}d) \\
&\quad - \frac{k_{ya}^2 - a}{bk_{ya}} \left[ \frac{k_{yb}^2}{q_1} + \frac{(k_x k_z)^2}{\omega^2 \mu_1} \left( \frac{1}{\epsilon_1 q_1} - \frac{1}{\epsilon_y q_y} \right) \right] \\
&\quad + \frac{jk_x k_z}{\omega \mu_1 q_y} \frac{k_{yb}^2 - a}{b} \sin(k_{ya}d) \cos(k_{yb}d) \\
&\quad + \frac{g_1}{g_x} \left[ k_{ya} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{ya}^2 - a}{bk_{ya}} \right] \left[ \frac{jk_x k_z}{\omega} \left( \frac{1}{\epsilon_y q_y} \right. \right. \\
&\quad \left. \left. - \frac{1}{\epsilon_1 q_1} \right) + \frac{1}{q_y} \frac{k_{yb}^2 - a}{b} \right] \sin(k_{ya}d) \cos(k_{yb}d)
\end{aligned}$$

$$\begin{aligned}
&+ \frac{k_{yb}^2 - a}{bk_{yb}} \left[ \frac{k_{yb}^2}{q_1} + \frac{(k_x k_z)^2}{\omega^2 \mu_1} \left( \frac{1}{\epsilon_1 q_1} - \frac{1}{\epsilon_y q_y} \right) \right] \\
&+ \frac{jk_x k_z}{\omega \mu_1 q_y} \frac{k_{ya}^2 - a}{b} \sin(k_{yb}d) \cos(k_{ya}d) \\
&- \frac{g_1}{g_x} \left[ k_{yb} + \frac{jk_x k_z}{\omega \mu_2} \frac{k_{yb}^2 - a}{bk_{yb}} \right] \left[ \frac{jk_x k_z}{\omega} \left( \frac{1}{\epsilon_y q_y} \right. \right. \\
&\left. \left. - \frac{1}{\epsilon_1 q_1} \right) + \frac{1}{q_y} \frac{k_{ya}^2 - a}{b} \right] \sin(k_{yb}d) \cos(k_{ya}d). \quad (54)
\end{aligned}$$

The limiting uniaxial or isotropic behavior of  $\tilde{G}_{ij}(d)$  in (52) may be examined directly by dividing  $\tilde{G}'_{ij}$  and  $D_m$  in (53) and (54) by  $(k_{ya} - k_{yb}) \sin(k_{ya}d) \cos(k_{yb}d)/b$ , and studying, respectively, the cases  $\epsilon_x \rightarrow \epsilon_y \neq \epsilon_z$  and  $\epsilon_x \rightarrow \epsilon_y \rightarrow \epsilon_z$ .

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