

Within the framework outlined in this note, it seems not possible to generalize the result of Theorem 2 to include all the eigenvalues of K of (2). Further, Theorem 1 or even the special case of it, viz. (10), cannot be obtained from Theorem 2.

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Eigenvalue Bounds in the Lyapunov and Riccati Matrix Equations

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Abstract—Two related theorems of Strang [1] are extended to provide upper and lower bounds on the eigenvalues of the Lyapunov and Riccati matrices, given by $Q=AB^H+BA$ and $R=AB^H+BA+2AHA$, where A and H are Hermitian, positive definite, complex matrices. We discuss inversion to obtain eigenvalue bounds on the matrix A for the usual case in which Q , R , and H are known.

In [1], Strang proved two theorems giving the best possible upper and lower bounds on the eigenvalues of certain functions of Hermitian matrices. Here, the theorems are given a simple extension in order to provide corresponding bounds on the solutions of the algebraic Lyapunov and Riccati equations [2].

There have been several previous investigations leading to similar eigenvalue bounds [3]–[5]. But none appears to have used Strang's best possible results.

In particular, if upper case Latin letters denote n by n complex matrices, the theorems of [1] concern Z_1 and Z_2 defined by

$$Z_1 = XY + YX \quad (1.1)$$

$$Z_2 = i(XY - YX) \quad (1.2)$$

where $i = \sqrt{-1}$. The Lyapunov and Riccati matrix functions are Q and R given by

$$Q = AB^H + BA \quad (2.1)$$

$$R = AB^H + BA + 2AHA. \quad (2.2)$$

Here B^H is the Hermitian transpose of B , all other matrices in (1.1), (1.2), (2.1), and (2.2) are Hermitian, and Q , R , and H are positive definite. The matrix A is unknown, hereafter referred to as the solution, while B , Q , R , and H are usually considered known. The eigenvalues of A are $a_1 \geq a_2 \geq \dots \geq a_n$, and similarly for the other Hermitian matrices.

Our method is first to extend Strang's theorem to find bounds on the eigenvalues of Q and R . We then discuss the inversion leading to bounds on the eigenvalues of the matrix A .

To extend Strang's theorems, introduce the Hermitian matrices

$$D = \frac{B+B^H}{2} \quad E = i\left(\frac{B-B^H}{2}\right)$$

$$F = AD + DA \quad G = i(AE - EA).$$

Note that F and G have the same form as Z_1 and Z_2 .

After elementary manipulations we find that

$$Q = F + G.$$

The minimum eigenvalue q_n of Q is now

$$q_n = \min_{x^H x \neq 0} \left[\frac{x^H F x + x^H G x}{x^H x} \right]$$

$$\geq \min_{x^H x \neq 0} \left[\frac{x^H F x}{x^H x} \right] + \min_{x^H x \neq 0} \left[\frac{x^H G x}{x^H x} \right]$$

$$\geq f_n + g_n \quad (3.1)$$

where x is an n by one complex vector with Hermitian transpose x^H . By parallel argument, the maximum eigenvalue q_1 of Q is subject to

$$q_1 \leq f_1 + g_1. \quad (3.2)$$

The theorems of [1] give the best possible upper and lower bounds on f_1 and f_n , and on g_1 and g_n . In particular,

$$\phi_1 \equiv \max[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5] \geq f_1 \geq f_n$$

$$\geq \min[\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5] \equiv \phi_n$$

where

$$\alpha_1 = 2a_1 d_1 \quad (4.1)$$

$$\alpha_2 = 2a_1 d_n \quad (4.2)$$

$$\alpha_3 = 2a_n d_1 \quad (4.3)$$

$$\alpha_4 = 2a_n d_n. \quad (4.4)$$

Also,

$$\alpha_5 = \begin{cases} \alpha_1, & \text{if } \max[\Gamma_1, \Gamma_2] \geq 1 \\ \frac{16a_1 a_n d_1 d_n - (a_1 - a_n)^2 (d_1 - d_n)^2}{4(a_1 + a_n)(d_1 + d_n)}, & \text{otherwise} \end{cases} \quad (4.5)$$

where

$$\Gamma_1 = \left| \frac{1}{2} \left[\frac{a_1 + a_n}{a_1 - a_n} \frac{(d_1 - d_n)^2}{(d_1 + d_n)^2} - \frac{a_1 + a_n}{a_1 - a_n} - \frac{a_1 - a_n}{a_1 + a_n} \right] \right|$$

$$\Gamma_2 = \left| \frac{1}{2} \left[\frac{d_1 + d_n}{d_1 - d_n} \frac{(a_1 - a_n)^2}{(a_1 + a_n)^2} - \frac{d_1 + d_n}{d_1 - d_n} - \frac{d_1 - d_n}{d_1 + d_n} \right] \right|.$$

Furthermore,

$$\beta \geq g_1 \geq g_n \geq -\beta \quad (4.6)$$

in which

$$\beta = \frac{1}{2}(a_1 - a_n)(e_1 - e_n).$$

Combining (4.1)–(4.6) as suggested by (3.1) and (3.2), our major result is obtained as

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$$\phi_1 + \beta \geq q_1 \geq q_n \geq \phi_n - \beta. \quad (5)$$

The bounds given by (5) apparently cannot be described as the best possible.

For the Riccati matrix function R ,

$$\begin{aligned} r_n &\geq \phi_n - \beta + 2 \min_{x^H x \neq 0} \left[\frac{x^H A H A x}{x^H x} \right] \\ &\geq \phi_n - \beta + 2 \left\{ \min_{x^H x \neq 0} \left[\frac{x^H A H A x}{x^H x} \right] \right\} \left\{ \min_{x^H x \neq 0} \left[\frac{x^H A A x}{x^H x} \right] \right\} \\ &\geq \phi_n - \beta + 2 h_n a_n^2. \end{aligned}$$

Using parallel arguments for r_1 , the second major result is

$$\phi_1 + \beta + 2 h_1 a_1^2 \geq r_1 \geq r_n \geq \phi_n - \beta + 2 h_n a_n^2. \quad (6)$$

The inequalities (5), (6) at first glance appear to be artificial, since Q and R are usually considered known while A is viewed as the as-yet-unknown solution of (2.1), (2.2). However, they can be inverted to furnish bounds on a_n/a_1 .

For example, consider the case in which

$$d_1 > 0 \quad (7.1)$$

$$d_n < 0 \quad (7.2)$$

$$d_n^2/d_1^2 < 1. \quad (7.3)$$

For simplicity we also assume that

$$a_n^2/a_1^2 < 1 \quad (7.4)$$

$$\max[\Gamma_1, \Gamma_2] > 1. \quad (7.5)$$

Regarding (7.1), if $d_1 < 0$, then $a_1 < 0$ and A is negative definite. Further, if $d_n > 0$, then Q may be set equal to D , with the result that $A=I$. So $d_1 < 0$ and $d_n > 0$ are not cases of any interest. Finally, under (7.1) it follows that $a_1 > 0$.

Omitting the algebra, only two cases need be considered:

- i) $a_1 d_1 > a_n d_n$ $a_n d_1 > a_1 d_n$
- ii) $a_1 d_1 > a_n d_n$ $a_1 d_n > a_n d_1$.

The quantity a_n may only be positive in the first case.

For case i)

$$q_1 \leq 2 a_1 d_1 + \frac{1}{2} (e_1 - e_n) (a_1 - a_n)$$

$$q_n \geq 2 a_1 d_n - \frac{1}{2} (e_1 - e_n) (a_1 - a_n)$$

and upon inversion

$$\frac{a_n}{a_1} \leq \frac{\frac{1}{2} (e_1 - e_n) + 2 d_1 - q_1/a_1}{\frac{1}{2} (e_1 - e_n)} \quad (8.1)$$

$$\frac{a_n}{a_1} \leq \frac{\frac{1}{2} (e_1 - e_n) - 2 d_n + q_n/a_1}{\frac{1}{2} (e_1 - e_n)}. \quad (8.2)$$

Similarly, for case ii)

$$q_1 \leq 2 a_1 d_1 + \frac{1}{2} (e_1 - e_n) (a_1 - a_n)$$

$$q_n \geq 2 a_n d_1 - \frac{1}{2} (e_1 - e_n) (a_1 - a_n)$$

from which

$$\frac{a_n}{a_1} \leq \frac{\frac{1}{2} (e_1 - e_n) + 2 d_1 - q_1/a_1}{\frac{1}{2} (e_1 - e_n)} \quad (9.1)$$

$$\frac{a_n}{a_1} \leq \frac{\frac{1}{2} (e_1 - e_n) + q_n/a_1}{\frac{1}{2} (e_1 - e_n) + 2 d_1}. \quad (9.2)$$

For R , the counterpart of (8.1), (8.2), (9.1), and (9.2) may be derived in a parallel fashion. Finally, it is just a matter of tedious algebra to derive the inverse bounds on a_n when $\max[\Gamma_1, \Gamma_2] < 1$ and when $a_n^2/a_1^2 > 1$.

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Explicit Solutions of the Discrete-Time Lyapunov Matrix Equation and Kalman-Yakubovich Equations

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Abstract—A general solution for the nonsquare nonsymmetric Lyapunov matrix equation in a canonical form is presented. The solution is shown to be a Toeplitz matrix which may be calculated using the backwards Levinson algorithm. This solution is then applied to the Kalman-Yakubovich equations to derive a method for generating strictly positive-real functions via the positive-real lemma. This latter result has an application in system identification.

I. INTRODUCTION

Linear matrix equations of the form

$$X - A X B = C \quad (1)$$

arise in many applications. In the control theory area this equation and its specialization the Lyapunov matrix equation

$$X - A^T X A = L L^T \quad (2)$$

occur naturally in the study of discrete-time system stability [1], in covariance calculation [2], and in the iterative solution of the matrix Riccati equation [3]¹ and of the optimal constant output feedback problem [4], [5].¹ As a consequence there has been considerable effort devoted to its solution [6]-[10], [31] as well as to the solution of its continuous-time analogue (see, e.g., [13], [22], [31]).

As shown in [11], [12], provided A and B have no mutually reciprocal eigenvalues, the general solution to (1) may be found by solving an equation of the form $Dy=e$ for vector y consisting of elements of X , with D being a Kronecker product. Dimensionality problems occur. Alternatively [12], it is easily seen that, provided A and B have eigenvalues of magnitude less than one,

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¹While these references deal with the continuous-time Lyapunov equation and its occurrence with continuous-time systems, their results carry over *mutatis mutandis* to discrete-time.