

THE PRACTICALITY OF PROCESSING UNDERSAMPLED WAVEFORMS

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Abstract

The effects of processing digital signals that were obtained by sampling continuous-time waveforms at less than the Nyquist rate are reviewed. The reasoning for considering the process is discussed and several applications are presented.

Introduction

Most measurement systems which implement digital signal processing techniques contain anti-aliasing filters which insure that the continuous-time signal (waveform) is being sampled at a rate equal to or greater than the Nyquist rate. This guarantees that the filtered continuous-time signal can be recovered exactly from its samples, i.e., all the information contained in the filtered continuous-time signal is contained in the samples. This approach parallels the treatment of most introductory text books on the subject which typically introduces the basics of sampling theory in an early chapter. Then, in the later chapters, it is assumed that the digital signal being processed has been obtained by sampling at a rate equal to or greater than the Nyquist rate. This allows the text to parallel the development of digital signal processing theory to the reader's previous exposure to continuous-time signal processing theory. The processing of undersampled signals (sampled at less than the Nyquist rate) is very seldom considered.

Since sampling at or above the Nyquist rate preserves all of the information contained in the continuous-time waveforms, why consider processing samples that do not? The answer to this question is two-fold. First, not all of the information contained in the continuous-time signal is distorted by undersampling. This is especially true if the general characteristics of the signal being sampled are known. Secondly, off-the-shelf instruments (high speed digital voltmeters) exist which have a larger bandwidth than the folding frequency allowed by the throughput rate of the digital system (computer) collecting and processing the samples. This allows the Nyquist frequency to be greater than the folding frequency, i.e., a continuous-time signal can be undersampled by such a system.

Review and Applications

The principles of sampling theory and the effects of the sampling interval on the frequency content of the digital signals are covered in detail in most text books on the subject (1,2) and, for the sake of brevity, are not repeated here. In general, however, the portion of a sampled continuous-time signal with spectral components above the folding frequency is indistinguishable from a different set of frequencies which are less than the folding frequency. These frequencies are called aliases of one another and the process of confounding frequencies is called aliasing. It can be shown that a sinusoid at a frequency, f , which is greater than the folding frequency, $f_s/2$, (f_s = sampling frequency) aliases a frequency, f_a , as shown below.

$$f_a = |f \pm kf_s|$$

where k is an integer such that

$$0 \leq f_a \leq f_s/2$$

the following sub-sections illustrate how important signal characteristics can be obtained by taking advantage of this relationship.

1. Harmonic Distortion

The percent harmonic distortion of a continuous-time signal can be calculated if the amplitudes of the fundamental frequency, f , and all of its integer multiples (harmonics) are known. In general, the spectrum of a signal being analyzed for percent harmonic distortion contains only these components and the fundamental frequency is known. If the signal contains a finite number of harmonics with significant amplitudes and the sampling frequency, f_s , is chosen to satisfy the following equation

$$f = \frac{f_s}{2N} + Mf_s$$

where N is the number of the largest multiple of f that has significant amplitude and M is an integer, then f will alias $f_s/2N$ when sampled. The k th harmonic of f will alias $kf_s/2N$. Therefore, each harmonic will uniquely fold to a frequency less than or equal the folding frequency. If $2N$ samples

of the signal are taken, calculating the discrete Fourier transform (DFT) of the samples will reveal the amplitude of the fundamental and its harmonics. (1,2,3)

2. Phase Difference

The phase difference between two sinusoids of the same frequency is not distorted by aliasing. Therefore, two discrete Fourier transforms can be performed on the two sinusoids to determine the phase of the signals that they alias. The phase difference can then be computed.

3. Modulation Characteristics

If the bandwidth of a modulated sinusoid is less than $f_s/2$ and f_s is chosen such that the sinusoid being modulated aliases $f_s/4$, then the characteristics of the modulated signal (AM, modulating frequency, etc.) can be determined by digital processing of the samples. This can be accomplished because the spectrum of the sampled waveform has the same shape as the continuous-time waveform. The only difference is that it is now located between zero and $f_s/2$.

4. Frequency

As shown in the beginning of this section, the frequency that a sinusoid will alias is determined by the sampling frequency. If the frequency of interest is known to be less than some maximum value, the frequency can be determined by taking multiple sample sequences of the sinusoid at different sampling intervals. This is true because the equation relating f , f_a , and f_s must be satisfied for each sample sequence.

Conclusion

Useful measurement information can be obtained by processing digital signals that were obtained by sampling continuous-time waveforms at less than the Nyquist rate. The applications outlined above have been successfully implemented. The reader, however, should be cautioned that the accuracy of these techniques are governed by many factors such as the signal-to-noise ratio of the signal being sampled, quantization noise, and timing errors in the sampling. A complete analysis of the acquisition and processing system is required in order to determine measurement accuracy.

Note: The terminology used in this paper is consistent with that recommended in reference 4.

References

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