The Euler Product is not equal to the Riemann Zeta Function

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Abstract: The Euler product is equal to the Riemann zeta function: it was proved by the earlier mathematician. The author of this article shows that the Euler product is not equal to the Riemann zeta function.

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1. Introduction

The Riemann Zeta function is defined by

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + 4^{-s} + 5^{-s} + \cdots = \sum n^{-s}, \quad \forall \, \mathbb{R}e(s) > 1. \tag{1}$$

Where $\mathbb{R}e(s)$ is the real part of all complex numbers.

2. Euler Product not equal to Riemann Zeta Function

The Euler product for the Riemann zeta function [1-4] is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{i=1}^{\infty} \frac{1}{1 - \frac{1}{p_i^s}}, \tag{2}$$

where $\prod_{i=1}^{\infty} \frac{1}{1 - \frac{1}{p_i^s}}$ is the Euler product and p_i is the i^{th} prime.

Let us show that the Euler product is not equal to the Riemann zeta function.

$$\prod_{i=1}^{\infty} \frac{1}{1 - \frac{1}{p_i^s}} = \frac{1}{1 - \frac{1}{2^s}} \times \frac{1}{1 - \frac{1}{3^s}} \times \frac{1}{1 - \frac{1}{5^s}} \times \frac{1}{1 - \frac{1}{7^s}} \times \frac{1}{1 - \frac{1}{11^s}} \times \frac{1}{1 - \frac{1}{13^s}} \times \cdots \tag{3}$$

By substituting $s = 1$in Equation (3), we get

$$\prod_{i=1}^{\infty} \frac{1}{1 - \frac{1}{p_i}} = \frac{1}{1 - \frac{1}{2}} \times \frac{1}{1 - \frac{1}{3}} \times \frac{1}{1 - \frac{1}{5}} \times \frac{1}{1 - \frac{1}{7}} \times \frac{1}{1 - \frac{1}{11}} \times \frac{1}{1 - \frac{1}{13}} \times \cdots \tag{4}$$

By simplifying Equation (4), we obtain

$$\prod_{i=1}^{\infty} \frac{1}{1-\frac{1}{p_i}} = \frac{2}{1} \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6} \times \frac{11}{10} \times \frac{13}{12} \times \cdots > 3 \Rightarrow \prod_{i=1}^{\infty} \frac{1}{1-\frac{1}{p_i^s}} > \prod_{i=1}^{\infty} \frac{1}{1-\frac{1}{p_i}} > 3. \quad (5)$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} < 3 \Rightarrow \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} < \sum_{n=1}^{\infty} \frac{1}{n} < 3. \quad (6)$$

From the above expressions (5) and (6), we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} < \prod_{i=1}^{\infty} \frac{1}{1-\frac{1}{p_i^s}} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^s} \neq \prod_{i=1}^{\infty} \frac{1}{1-\frac{1}{p_i^s}}$$

Therefore, the Euler product is not equal to the Riemann zeta function.

3. Conclusion

In the article, the author shows that the Euler product is not equal to the Riemann zeta function.

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