

# Inverse Scattering With Non-over-determined Data

Alexander Ramm  
 Kansas State University  
 Manhattan, KS, USA  
 E-mail: ramm@math.ksu.edu

**Abstract**—It is proved that the scattering amplitude  $A(\beta, \alpha_0, k_0)$ , known for all  $\beta \in S^2$ , where  $S^2$  is the unit sphere in  $\mathbb{R}^3$ , and fixed  $\alpha_0 \in S^2$  and  $k_0 > 0$ , determines uniquely the surface  $S$  of the obstacle  $D$  and the boundary condition on  $S$ . The boundary condition on  $S$  is assumed to be the Dirichlet, or Neumann, or the impedance one. The uniqueness theorems for the solution of multidimensional inverse scattering problems with non-over-determined data was not known for many decades. Such theorems are presented in this paper.

**Index Terms**—scattering theory; obstacle scattering; uniqueness theorem; non-over-determined scattering data.

## I. INTRODUCTION

The uniqueness theorems for the solution of multidimensional inverse scattering problems with non-over-determined scattering data were not known since the origin of the inverse scattering theory, which goes, roughly speaking, to the middle of the last century. Such theorems are presented in this paper for 3D inverse scattering. In [9]–[11] such theorems are proved for the first time for inverse scattering by potentials. This Conference talk is based on papers [1], [2] from which parts are used verbatim. The data is called non-over-determined if it is a function of the same number of variables as the function to be determined from these data. In the case of the inverse scattering by an obstacle the unknown function describes the surface of this obstacle in  $\mathbb{R}^3$ , so it is a function of two variables. The non-over-determined scattering data is the scattering amplitude depending on a two-dimensional vector. The exact formulation of this inverse problem is given below.

Let us formulate the inverse obstacle problem discussed in this paper. Let  $D \subset \mathbb{R}^3$  be a bounded domain with a connected  $C^2$ -smooth boundary  $S$ ,  $D' := \mathbb{R}^3 \setminus D$  be the unbounded exterior domain and  $S^2$  be the unit sphere in  $\mathbb{R}^3$ . The smoothness assumption on  $S$  can be weakened.

Consider the scattering problem:

$$(\nabla^2 + k^2)u = 0 \quad \text{in } D', \quad \Gamma_j u|_S = 0, \quad u = e^{ik\alpha \cdot x} + v, \quad (1)$$

where the scattered field  $v$  satisfies the radiation condition:

$$v_r - ikv = o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty. \quad (2)$$

Here  $k > 0$  is a fixed constant called the wave number and  $\alpha \in S^2$  is a unit vector in the direction of the propagation of the incident plane wave  $e^{ik\alpha \cdot x}$ . The boundary conditions are assumed to be either the Dirichlet ( $\Gamma_1$ ), or Neumann ( $\Gamma_2$ ), or impedance ( $\Gamma_3$ ) type:

$$\Gamma_1 u := u, \quad \Gamma_2 u := u_N, \quad \Gamma_3 u := u_N + hu, \quad (3)$$

where  $N$  is the unit normal to  $S$  pointing out of  $D$ ,  $u_N$  is the normal derivative of  $u$  on  $S$ ,  $h = \text{const}$ ,  $\text{Im}h \geq 0$ ,  $h$  is the boundary impedance, and the condition  $\text{Im}h \geq 0$  guarantees the uniqueness of the solution to the scattering problem (1)–(2).

The scattering amplitude  $A(\beta, \alpha, k)$  is defined by the following formula:

$$v = A(\beta, \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty, \quad \frac{x}{r} = \beta, \quad (4)$$

where  $\alpha, \beta \in S^2$ ,  $\beta$  is the direction of the scattered wave,  $\alpha$  is the direction of the incident wave.

For a bounded domain  $D$  one has  $o(\frac{1}{r}) = O(\frac{1}{r^2})$  in formula (4). The scattering amplitude  $A(\beta, \alpha, k)$  can be measured experimentally. Let us call it *the scattering data*. It is known (see [3], p.25) that the solution to the scattering problem (1)–(2) does exist and is unique.

*The inverse scattering problem (IP) consists of finding  $S$  and the boundary condition on  $S$  from the scattering data.*

M.Schiffer was the first to prove in the sixties of the last century that if the boundary condition is the Dirichlet one then the surface  $S$  is uniquely determined by the scattering data  $A(\beta, \alpha_0, k)$  known for a fixed  $\alpha = \alpha_0$  all  $\beta \in S^2$  and all  $k \in (a, b)$ ,  $0 \leq a < b$ .

M. Schiffer did not publish his proof. This proof can be found, for example, in [3], p.85, and the acknowledgement of M.Schiffer's contribution is on p.399 in [3].

A. G. Ramm was the first to prove that the scattering data  $A(\beta, \alpha, k_0)$ , known for all  $\beta$  in a solid angle, all  $\alpha$  in a solid angle and a fixed  $k = k_0 > 0$  determine uniquely the boundary  $S$  and the boundary condition on  $S$ . This condition was assumed of one of the three types  $\Gamma_j$ ,  $j = 1, 2$  or  $3$ , (see [3], Chapter 2, for the proof of these results). By subindex zero fixed values of the parameters are denoted, for example,  $k_0, \alpha_0$ . By a solid angle in this paper an open subset of  $S^2$  is understood.

In [3], p.62, it is proved that for smooth bounded obstacles the scattering amplitude  $A(\beta, \alpha, k)$  is an analytic function of  $\beta$  and  $\alpha$  on the non-compact analytic variety  $M := \{z | z \in \mathbb{C}^3, z \cdot z = 1\}$ , where  $z \cdot z := \sum_{m=1}^3 z_m^2$ . The unit sphere  $S^2$  is a subset of  $M$ . If  $A(\beta, \alpha, k)$  as a function of  $\beta$  is known on an open subset of  $S^2$ , it is uniquely extended to all of  $S^2$  (and to all of  $M$ ) by analyticity. The same is true if  $A(\beta, \alpha, k)$  as a function of  $\alpha$  is known on an open subset of  $S^2$ . By this reason one may assume that the scattering

amplitude is known on all of  $S^2$  if it is known in a solid angle, that is, on open subsets of  $S^2$  as a function of  $\alpha$  and  $\beta$ .

In papers [6] and [7] a new approach to a proof of the uniqueness theorems for inverse obstacle scattering problem (IP) was given. This approach is used here. In paper [5] the uniqueness theorem for IP with non-over-determined data was proved for strictly convex smooth obstacles. The proof in [5] was based on the location of resonances for a pair of disjoint obstacles.

Here the uniqueness theorem for IP with non-over-determined scattering data for arbitrary  $S$  is formulated. The boundary is assumed  $C^2$ -smooth. This assumption can be weakened considerably (see, for example, [8], pp. 228-230). By the boundary condition any of the three conditions  $\Gamma_j$  are understood below. For definiteness one may assume that the Dirichlet boundary condition is used. Let  $A(\beta) := A(\beta, \alpha_0, k_0)$ .

**Theorem 1.** *The surface  $S$  and the boundary condition on  $S$  are uniquely determined by the data  $A(\beta)$  known in a solid angle.*

Let us explain the logic of the proof of Theorem 1. Assuming that the surface  $S$  is not uniquely determined by the non-over-determined scattering data, that is, that there exist at least two different surfaces  $S_1$  and  $S_2$ , I prove that equation (10) holds and derive from this a contradiction. This contradiction proves that the assumption  $S_1 \neq S_2$  is wrong. Thus, the desired conclusion of Theorem 1 follows. If it is proved that  $S_1 = S_2$ , then the type of the boundary condition one can uniquely determine by calculating  $u$  or  $\frac{u_N}{u}$  on  $S$ .

In the proof of Theorem 1 it is assumed that the surfaces  $S_1$  and  $S_2$  intersect. The cases when  $S_1$  and  $S_2$  do not intersect are discussed in Theorems 2 and 3.

**Theorem 2.** *If  $A_1(\beta) = A_2(\beta)$  for all  $\beta$  in a solid angle, then it is not possible that  $D_1 \neq D_2$  and  $D_1 \cap D_2 = \emptyset$ .*

**Theorem 3.** *If  $A_1(\beta) = A_2(\beta)$  for all  $\beta$  in a solid angle, then it is not possible that  $D_1 \neq D_2$  and  $D_1 \subset D_2$ .*

## II. OUTLINE OF PROOFS

Denote by  $G(x, y, k)$  the Green's function corresponding to the scattering problem (1)-(2). The parameter  $k > 0$  is assumed fixed in what follows. For definiteness we assume below the Dirichlet boundary condition, but our proof is valid for the Neumann and impedance boundary conditions as well. If there are two surfaces  $S_m$ ,  $m = 1, 2$ , we denote by  $G_m$  the corresponding Green's functions of the Dirichlet Helmholtz operator in  $D'_m$ .

**Lemma 1.** ([3], p. 46) *One has:*

$$G(x, y, k) = g(|y|)u(x, \alpha, k) + O\left(\frac{1}{|y|^2}\right), \quad (5)$$

$$|y| \rightarrow \infty, \quad \frac{y}{|y|} = -\alpha.$$

Here  $g(|y|) := \frac{e^{ik|y|}}{4\pi|y|}$ ,  $u(x, \alpha, k)$  is the scattering solution, that is, the solution to problem (1)-(2), and the notation  $\gamma(r) := 4\pi g(r) = \frac{e^{ik|r|}}{|r|}$  is used below.

The solutions to equation (1) have the unique continuation property:

*If  $u$  solves equation (1) and vanishes on a set  $\tilde{D} \subset D'$  of positive Lebesgue measure, then  $u$  vanishes everywhere in  $D'$ .*

Denote by  $D_{12} := D_1 \cup D_2$ ,  $D'_{12} := \mathbb{R}^3 \setminus D_{12}$ ,  $S_{12} := \partial D_{12}$ ,  $\tilde{S}_1 := S_{12} \setminus S_2$ , that is,  $\tilde{S}_1$  does not belong to  $D_2$ ,  $B'_R := \mathbb{R}^3 \setminus B_R$ ,  $B_R := \{x : |x| \leq R\}$ . The number  $R$  is sufficiently large, so that  $D_{12} \subset B_R$ .

An important part of our proof is based on the global perturbation lemma, Lemma 2 below, which is proved in [7], see there formula (4), and a similar lemma is proved for potential scattering in [8], see there formula (5.1.30). For convenience of the readers a short proof of Lemma 2 is given below.

**Lemma 2.** *One has:*

$$4\pi[A_1(\beta, \alpha, k) - A_2(\beta, \alpha, k)] = \int_{S_{12}} [u_1(s, \alpha, k)u_{2N}(s, -\beta, k) - u_{1N}(s, \alpha, k)u_2(s, -\beta, k)] ds, \quad (6)$$

where the scattering amplitude  $A_m(\beta, \alpha, k)$  corresponds to obstacle  $S_m$ ,  $m = 1, 2$ .

**Lemma 3.** ([3], p. 25) *If  $\lim_{r \rightarrow \infty} \int_{|x|=r} |v|^2 ds = 0$  and  $v$  satisfies the Helmholtz equation (1), then  $v = 0$  in  $B'_R$ .*

**Lemma 4.** (lifting lemma) *If  $A_1(\beta, \alpha, k) = A_2(\beta, \alpha, k)$  for all  $\beta, \alpha \in S^2$ , then  $G_1(x, y, k) = G_2(x, y, k)$  for all  $x, y \in D'_{12}$ . If  $A_1(\beta, \alpha_0, k) = A_2(\beta, \alpha_0, k)$  for all  $\beta \in S^2$  and a fixed  $\alpha = \alpha_0$ , then  $G_1(x, y_0, k) = G_2(x, y_0, k)$  for all  $x \in D'_{12}$  and  $y_0 = -\alpha_0\tau + \eta$ , where  $\tau > 0$  is a number and  $\eta$  is an arbitrary fixed vector orthogonal to  $\alpha_0$ ,  $\alpha_0 \cdot \eta = 0$ .*

The second part of Lemma 4 is a particular case of its first part for  $\alpha = \alpha_0$ . In Lemma 6 (see below) it is proved that if

$$A_1(\beta) := A_1(\beta, \alpha_0, k) = A_2(\beta, \alpha_0, k) := A_2(\beta) \quad \forall \beta \in S^2, \quad (7)$$

then  $w(x, y_0) = 0$ , where  $x \in D'_{12}$  is arbitrary,  $y_0 = -\tau\alpha_0 + \eta$ ,  $\tau > 0$  is a number and  $\eta$  is an arbitrary fixed vector orthogonal to  $\alpha_0$ ,  $\eta \cdot \alpha_0 = 0$ .

**Lemma 5.** *One has*

$$\lim_{x \rightarrow t} G_{2N}(x, s, k) = \delta(s - t), \quad t \in S_2, \quad (8)$$

where  $\delta(s - t)$  denotes the delta-function on  $S_2$  and  $x \rightarrow t$  denotes a limit along any straight line non-tangential to  $S_2$ .

Let us point out the following implications:

$$G(x, y, k) \rightarrow u(x, \alpha, k) \rightarrow A(\beta, \alpha, k),$$

which hold by Lemma 1. The first arrow means that the knowledge of  $G(x, y, k)$  determines uniquely the scattering solution  $u(x, \alpha, k)$  for all  $\alpha \in S^2$ , and the second arrow means that the scattering solution  $u(x, \alpha, k)$  determines uniquely the scattering amplitude  $A(\beta, \alpha, k)$ .

The reversed implications also hold:

$$A(\beta, \alpha, k) \rightarrow u(x, \alpha, k) \rightarrow G(x, y, k).$$

The reversed implications also hold for  $\alpha = \alpha_0$ :

$$A(\beta, \alpha_0, k) \rightarrow u(x, \alpha_0, k) \rightarrow G(x, y_0, k).$$

Let us assume that  $A_1(\beta) = A_2(\beta)$  for all  $\beta$  but  $S_1 \neq S_2$ . We want to derive from this assumption a contradiction. This contradiction will prove that the assumption  $S_1 \neq S_2$  is false, so  $S_1 = S_2$ .

If  $A_1(\beta) = A_2(\beta)$ , then Lemma 2 yields the following conclusion:

$$0 = \int_{S_{12}} [u_1(s, \alpha_0, k)u_{2N}(s, -\beta, k) - u_{1N}(s, \alpha_0, k)u_2(s, -\beta, k)] ds, \quad \forall \beta \in S^2, \quad (9)$$

where  $k > 0$  and  $\alpha_0 \in S^2$  are fixed.

Lemma 6 (see below) allows one to claim that equation (9) implies the following equation:

$$0 = \int_{S_{12}} [G_1(s, y_0, k)G_{2N}(s, x, k) - G_{1N}(s, y_0, k)G_2(s, x, k)] ds, \quad \forall x \in D'_{12}, \quad (10)$$

where  $y_0 = -\alpha_0\tau + \eta$ ,  $\tau > 0$ ,  $\eta \cdot \alpha_0 = 0$ ,  $\eta$  is an arbitrary fixed vector orthogonal to  $\alpha_0$ . Thus, vector  $-\alpha$  is a free vector in the sense that its origin can be placed at any point  $\eta$  such that  $\eta \cdot \alpha_0 = 0$  in the chosen coordinate system. Indeed, the incident plane wave  $e^{ik\alpha_0 \cdot x}$  is not changed when  $x$  is replaced by  $x + \eta$ , provided that  $\eta \cdot \alpha_0 = 0$ . The scattered field  $v$ , satisfying the radiation condition, will satisfy the radiation condition when the origin of the coordinate system is moved to a point  $\eta$ , provided that  $\eta \cdot \alpha_0 = 0$ , and there is always a coordinate origin such that the vector  $-\alpha_0\tau$  intersects the point  $t \in S_2$ .

**Lemma 6.** *If (9) holds then (10) holds.*

*Proof.* Denote by  $w(x, y_0)$  the integral in (10),  $y_0 := -\tau\alpha_0 + \eta$ , where  $\eta \cdot \alpha_0 = 0$ ,  $\eta$  is fixed. Recall that  $\tilde{S}_1$  is the part of  $S_{12}$  that does not belong to  $D_2$ , and  $\tilde{S}_2$  is defined similarly. Rewrite equations (9) and (10) as

$$\int_{\tilde{S}_2} u_1(s, \alpha_0)u_{2N}(s, \beta) ds - \int_{\tilde{S}_1} u_{1N}(s, \alpha_0)u_2(s, \beta) ds = 0, \quad \forall \beta \in S^2, \quad (11)$$

and

$$w(x, y_0) = \int_{\tilde{S}_2} G_1(s, y_0)G_{2N}(s, x) ds - \int_{\tilde{S}_1} G_{1N}(s, y_0)G_2(s, x) ds, \quad \forall x \in D'_{12}. \quad (12)$$

We assume that  $w(x, y_0) \neq 0$  while (11) holds, and derive a contradiction. This contradiction shows that  $w(x, y_0) = 0$  and this implies, according to the proof of Theorem 1 below, that  $S_1 = S_2$ .

Let  $x \rightarrow t \in \tilde{S}_2$  in (12). By Lemma 5 one gets

$$w(t, y_0) = G_1(t, y_0) - \int_{\tilde{S}_1} G_{1N}(s, y_0)G_2(s, t) ds = G_1(t, y_0), \quad (13)$$

where  $t \in \tilde{S}_2$ . This is true because  $G_2(s, t) = 0$  if  $t \in \tilde{S}_2$ .

Let  $y_0 = -\tau\alpha_0 + \eta$ ,  $\eta \cdot \alpha_0 = 0$ , and let  $\tau \rightarrow \infty$  in (13). By Lemma 1 one gets

$$w(t, y_0) = \frac{e^{ik\tau}}{4\pi\tau} u_1(t, \alpha_0, k) + O\left(\frac{1}{\tau^2}\right). \quad (14)$$

It follows from formula (11) and Lemma 3 that

$$\int_{\tilde{S}_2} u_1(s, \alpha_0, k)G_{2N}(s, x) ds - \int_{\tilde{S}_1} u_{1N}(s, \alpha_0, k)G_2(s, x) ds = 0 \quad \forall x \in D'_{12}. \quad (15)$$

Let  $\tau \rightarrow \infty$  in (12), use (14) and (15) and conclude that

$$w(x, y_0) = O\left(\frac{1}{\tau^2}\right), \quad \tau \rightarrow \infty. \quad (16)$$

From (16) and (14) one concludes that

$$u_1(t, \alpha_0, k) = 0 \quad \forall t \in \tilde{S}_2. \quad (17)$$

Therefore formulas (15) and (17) imply

$$\varphi(x) := \int_{\tilde{S}_1} u_{1N}(s, \alpha_0, k)G_2(s, x) ds = 0 \quad \forall x \in D'_{12}. \quad (18)$$

Because the set  $\{G_2(s, x)\}_{\forall x \in D'_{12}}$  is dense in  $L^2(\tilde{S}_1)$ , it follows from (18) that

$$u_{1N}(s, \alpha_0, k) = 0 \quad \forall s \in \tilde{S}_1. \quad (19)$$

To check that the set  $\{G_2(s, x)\}_{\forall x \in D'_{12}}$  is dense in  $L^2(\tilde{S}_1)$  it is sufficient to check that it is dense in a dense subset of  $L^2(\tilde{S}_1)$ . As such a subset, let us choose  $C_0^\infty(\tilde{S}_1)$ . Let  $\phi \in C_0^\infty(\tilde{S}_1)$  and assume that  $\phi$  is orthogonal to  $G_2(s, x)$  for all  $x \in D'_{12}$ . Then the simple layer potential  $\psi(x) := \int_{\tilde{S}_1} G_2(s, x)\phi(s) ds = 0$  for all  $x \in D'_{12}$ . Since  $\phi$  has compact support in  $\tilde{S}_1$ , one may apply the jump relation for the normal derivatives of the potential of single layer  $\psi(x)$  across  $\tilde{S}_1$  and conclude that  $\phi(s) = 0$ . This proves the statement about density of the set  $\{G_2(s, x)\}_{\forall x \in D'_{12}}$  and one concludes that (19) holds.

Therefore

$$u_{1N}(s, \alpha_0, k) = 0, \quad u_1(s, \alpha_0, k) = 0 \quad \forall s \in \tilde{S}_1. \quad (20)$$

By the uniqueness of the solution to the Cauchy problem for the Helmholtz equation, relations (20) imply that  $u_1 = 0$  in  $D'_1$ . This is a contradiction since  $|u_1(x, \alpha_0, k)| \rightarrow 1$  as  $|x| \rightarrow \infty$ . This contradiction proves that  $w(x, y_0) = 0$ . The consequence of this, by Theorem 1 (see below), is the relation  $S_1 = S_2$ .

Lemma 6 is proved.  $\square$

*Proof of Theorem 1.* Since  $G_m = 0$  on  $S_m$ ,  $m = 1, 2$ , one can write (10) as

$$\int_{\tilde{S}_2} G_1(s, y_0, k)G_{2N}(s, x, k) ds - \int_{\tilde{S}_1} G_{1N}(s, y_0, k)G_2(s, x, k) ds, \quad \forall x \in D'_{12}, \quad (21)$$

where  $\tilde{S}_1$  is the part of  $S_{12}$  which does not belong to  $S_2$ , and  $\tilde{S}_2$  is defined similarly.

Let the point  $x$  in (21) tend to  $t \in \tilde{S}_2$ . Use Lemma 5 and the boundary condition  $G_2(s, t, k) = 0$  for  $t \in \tilde{S}_2$  to get

$$G_1(t, y_0, k) = 0, \quad \forall t \in \tilde{S}_2, \quad y_0 = -\alpha_0\tau + \eta, \quad \eta \cdot \alpha_0 = 0. \quad (22)$$

Let  $y_0 \rightarrow t$  along vector  $-\alpha_0$ . This is possible if  $\eta$  is properly chosen, namely, if it is equal to the projection of  $t$  onto the direction orthogonal to  $-\alpha_0$ . Then, on the one hand,  $G_1(t, y_0, k) \rightarrow \infty$ , because  $G_1(t, y_0, k)$  has a singularity as  $y_0 \rightarrow t$ :

$$|G_1(t, y_0, k)| = O\left(\frac{1}{|t - y_0|}\right), \quad t \in \tilde{S}_2. \quad (23)$$

On the other hand,  $G_1(t, y_0, k) = 0$  for  $y_0 = -\alpha_0\tau + \eta$ ,  $\tau > 0$ ,  $\eta \cdot \alpha_0 = 0$ , by formula (22) since  $t \in S_2$ .

This contradiction, which is due to the assumption  $S_1 \neq S_2$ , proves that  $S_1 = S_2$ .

If  $S_1 = S_2 := S$  then  $D_1 = D_2 := D$  and  $u_1(x, \alpha_0, k) = u_2(x, \alpha_0, k) := u(x, \alpha_0, k)$  for  $x \in D'$ , and, consequently, the boundary condition on  $S$  is uniquely determined: if  $u|_S = 0$ , then one has the Dirichlet boundary condition  $\Gamma_1$ , otherwise calculate  $\frac{u_N}{u}$  on  $S$ . If this ration vanishes, then one has the Neumann boundary condition  $\Gamma_2$ , otherwise one has the impedance (Robin) boundary condition  $\Gamma_3$ , and the boundary impedance  $h = -\frac{u_N}{u}$  on  $S$ .

Theorem 1 is proved.  $\square$

The above proof is given under the assumption that the boundary condition on  $S_2$  is the Dirichlet one, but it remains valid for other boundary conditions  $\Gamma_j$ ,  $j = 2, 3$ .

**Corollary.** *It follows from Theorems 2 and 3 that the solution to problem (1)– (2) (the scattering solution) cannot have a closed surface of zeros except the surface  $S$ , the boundary of the obstacle.*

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