Navier–Stokes Equations

Mathematics of Fluids in 2- and 3-Space: Dirichlet Boundary Conditions plus Asymptotic Analysis

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Abstract. Synopsis of Navier–Stokes equations (incompressible version) in 2- and 3-dimensional space with reference to Dirichlet boundary conditions: homogeneous and non-homogeneous case. Asymptotic analysis of a Navier–Stokes flow.

Keywords: Dirichlet boundary conditions (homogeneous and non-homogeneous case), incompressible flow, Navier–Stokes equations.

Contents

1. Incompressible Navier–Stokes Equations in 2- and 3-Space of Homogeneous Fluids

The Navier–Stokes equations [13] [24] [25] [26] [27] are analyzed according to Lipschitz domain [12] and Dirichlet boundary conditions [5], both when homogeneity is present (Sec. 1.1.1) and when it is absent (Sec. 1.1.2).

1.1. Lipschitz Domain and No-slip Dirichlet Boundary Conditions

The *no-slip* condition is for a condition which states that a viscous fluid have zero (average flow or bulk) velocity at a solid boundary.

The subject-matter addressed here is an ideal fluid with a pressure, or mathematically a scalar function $\mathsf{P}(x,t)$, so that, for a certain unit normal $\hat{N}$, a force of stress $\mathsf{P}(x,t)\hat{N}$ exerted on the surface $\partial\Omega$ is imagined per unit area at $x \in \partial\Omega$ at time t. Surface force is $\int_{\partial\Omega_t} \tau_{\mathrm{s}}(\hat{N})dy$, where τ_{s} is the stress in the fluid.

1.1.1. The Homogeneity Case

Theorem 1.1 *(Part I). Let*

$$\varphi_{\mathrm{L}} \in \mathfrak{H}^{-1}(\Omega), \ \ \Omega \in \mathbb{R}^n, \ \ n = 2, 3, \tag{1}$$

be a Lipschitz-type function, where $\mathfrak{H}$ is the Hilbert space [14] [15]. Then, for the flow velocity vector $\upsilon(x,t)$ in $\mathbb{R}^n$ and the fluid pressure P, there is a pair

$$(\upsilon, \mathsf{P}) \in \mathfrak{H}^{-1}(\Omega) \times L^2_0(\Omega) \tag{2}$$

that is the solution of the Navier–Stokes problem

$$\left.\begin{array}{l} -\frac{\operatorname{div}(\nabla\upsilon)}{\mathrm{Re}} + (\upsilon\nabla)\upsilon + \nabla\mathsf{P} = \varphi_{\mathrm{L}} \\ \operatorname{div}\upsilon = 0 \end{array}\right\} \text{in } \Omega \in \mathbb{R}^n, \tag{3a}$$

$$\upsilon_{|\partial\Omega} = 0, \ \ \partial\Omega \times (0,T) =]0,T[, \tag{3b}$$

satisfying

$$\begin{cases} \|\nabla\upsilon\|_{L^2} \leqslant \mathrm{Re}\|\varphi_{\mathrm{L}}\|_{\mathfrak{H}^{-1}}, & (4a) \\ \|\mathsf{P}\|_{L^2} \leqslant K_\Omega \left(\|\varphi_{\mathrm{L}}\|_{\mathfrak{H}^{-1}} + (\mathrm{Re}\|\varphi_{\mathrm{L}}\|)^2_{\mathfrak{H}^{-1}} \right), & (4b) \end{cases}$$

where

$\mathrm{Re} = \frac{\upsilon\ell_{\mathrm{c}}}{\nu}$ *is the Reynolds number (a dimensionless quantity), in which ℓ_{c} is the characteristic length of the device and ν is the kinematic viscosity,*

$\operatorname{div}(\nabla\upsilon) = \triangle\,\upsilon$, *with the Laplacian $\triangle$,*

$\nabla\mathsf{P}$ *is the pressure gradient,*

L^2 *is the Hilbert space with respect to the Lebesgue measure [8], and*

K_Ω *is a convenient value.*

Proof. The demonstration will be divided into various parts.

(α) Let $\varkappa\colon \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, such that there exist

(i) a function ρ representing the density (field) of the fluid,

(ii) an equality $\varkappa(\zeta) = \xi_j$, with

$$\varkappa(\zeta) = \frac{\|\nabla\upsilon_{n\geqslant 1}\|^2_{L^2}}{\mathrm{Re}} + \int_{\Omega\in\mathbb{R}^n} (\upsilon_n\nabla)\,\upsilon_n \times \upsilon_n dx - \langle\varphi_{\mathrm{L}}, \upsilon_n\rangle_{\mathfrak{H}^{-1},\ \mathfrak{H}^1_0} \geqslant \frac{\|\nabla\upsilon_n\|^2_{L^2}}{\mathrm{Re}} - \|\varphi_{\mathrm{L}}\|_{\mathfrak{H}^{-1}}\|\upsilon_n\|_{\mathfrak{H}^1_0}, \tag{5}$$

and

$$\xi_j = \left(\frac{1}{\mathrm{Re}} \int_{\Omega\in\mathbb{R}^n} \nabla\upsilon_n \,\bigg|\, \nabla\mathfrak{v}_j dx + \int_{\Omega\in\mathbb{R}^n} (\upsilon_n\nabla)\,\upsilon_n \times \mathfrak{v}_j dx - \langle\varphi_{\mathrm{L}}, \mathfrak{v}_j\rangle_{\mathfrak{H}^{-1},\ \mathfrak{H}^1_0} \right)_{1\leqslant j\leqslant n}. \tag{6}$$

The function

$$\upsilon_n \stackrel{\text{eqv}}{=\!=} \left(\sum_{j=1}^{n} \zeta_j \mathfrak{v}_j \right) \in \mathfrak{S}_n, \tag{7}$$

where $\mathfrak{S}_n$ is a vector space of finite dimension resulted from $\mathfrak{z}_j$, is the solution of

$$\frac{\nabla \upsilon_n}{\mathrm{Re}} \left| \left(\nabla \mathfrak{s}_n dx + \int_{\Omega \in \mathbb{R}^n} (\upsilon_n \nabla)\, \upsilon_n \times \mathfrak{s}_n dx = \langle \varphi_{\mathrm{L}}, \mathfrak{s}_n \rangle_{\mathfrak{H}^{-1},\ \mathfrak{H}_0^1} \right)_{\mathfrak{s}_n \in \mathfrak{S}_n} \right. . \tag{8}$$

If we consider the fact that

$$\|\upsilon_n\|_{\mathfrak{H}_0^1} = \|\nabla \upsilon_n\|_{L^2}, \tag{9}$$

it is easy to understand that $\kappa(\zeta) \geqslant 0$ as long as

$$\|\upsilon_n\|_{\mathfrak{H}_0^1} \geqslant \mathrm{Re}\|\varphi_{\mathrm{L}}\|_{\mathfrak{H}^{-1}}. \tag{10}$$

Then, the focal equality, which meets the requirement of (8),

$$\kappa(\zeta) = 0 \tag{11}$$

is given by

$$|\zeta|_{\mathbb{R}^n} = \rho > 0. \tag{12}$$

From Eqq. (5) plus (11) we can extract an inequality estimate:

$$\|\nabla \upsilon_n\|_{L^2} \leqslant \mathrm{Re}\|\varphi_{\mathrm{L}}\|_{\mathfrak{H}^{-1}}. \tag{13}$$

The latter is connected to the next step.

(β) Take a Banach space [1], say $(\mathfrak{B}, \|\cdot\|)$, and a sequence, say $\{x\}_n$, of elements of $\mathfrak{B}$ and of $\dot{\mathfrak{B}}$, with two types of convergence:

(i) weak process of converging for $\{x\}_n \in \mathfrak{B}$,

(ii) weak-⋆ process of converging for $\{x\}_n \in \dot{\mathfrak{B}}$.

It turns out that

$$\begin{cases} \|x\|_{\mathfrak{B}} \leqslant \liminf\limits_{n\to\infty} \|x_n\|_{\mathfrak{B}}, & (14a) \\ \|x\|_{\dot{\mathfrak{B}}} \leqslant \liminf\limits_{n\to\infty} \|x_n\|_{\dot{\mathfrak{B}}}. & (14b) \end{cases}$$

The evidence of (14) is easily achievable. Let ϵ be a positive value. If $\{x\}_n$ did not converge weakly, we would have

$$\left| \langle \varphi_{\mathrm{L}}, x_{f_n} - x \rangle_{\dot{\mathfrak{B}},\ \mathfrak{B}} \right| \geqslant \epsilon, \tag{15}$$

so

$$n \to \infty \begin{cases} \left\langle \varphi_{\mathrm{L}}, x_{f(\mathfrak{s}_n)} - x \right\rangle_{\mathfrak{B}} \to 0, & (16a) \\ \left\langle \varphi_{\mathrm{L}}, x_{f(\mathfrak{s}_n)} - x \right\rangle_{\dot{\mathfrak{B}}} \to 0, & (16b) \end{cases}$$

which is in contradiction with the main assumption.

(γ) Inequalities (13) and (14) lead directly to

$$\|\nabla \upsilon\|_{L^2} \leqslant \liminf_{n\to\infty} \|\nabla \upsilon_r\|_{L^2} \leqslant \mathrm{Re}\|\varphi_{\mathrm{L}}\|_{\mathfrak{H}^{-1}}. \tag{17}$$

(δ) Moving from Eq. (8), and using a Hilbert space $\mathfrak{H}^* \subset L^p(\Omega)$, $p = 6$, we can fix that $\upsilon \in \mathfrak{H}_0^1(\Omega)$ is the ad hoc solution for

$$\frac{1}{\mathrm{Re}} \int_{\Omega \in \mathbb{R}^n} \nabla \upsilon \left| \left(\nabla \mathfrak{s} dx + \int_{\Omega \in \mathbb{R}^n} (\upsilon \nabla)\, \upsilon \times \mathfrak{s} dx = \langle \varphi_{\mathrm{L}}, \mathfrak{s} \rangle_{\mathfrak{H}^{-1},\ \mathfrak{H}_0^1} \right)_{\mathfrak{s}_n \in \mathfrak{H}^*} \right. , \tag{18}$$

from which

$$\left\langle [\![: (\upsilon\nabla)\upsilon - \frac{\mathrm{Re}\varphi_{\mathrm{L}} + \triangle \upsilon}{\mathrm{Re}} :]\!], \mathfrak{s} \right\rangle_{\mathfrak{H}^{-1},\ \mathfrak{H}_0^1} = 0. \tag{19}$$

(ε) Suppose a function Z meeting the requirement

$$\operatorname{div} Z_{|\Omega} = 0,$$

for which

$$\langle \varphi_{\mathrm{L}}, Z \rangle_{\mathfrak{H}^{-1},\ \mathfrak{H}_0^1} = 0. \tag{20}$$

Then there is a function $\mathsf{P} \in L_0^2(\Omega)$ such that

$$\varphi_{\mathrm{L}} = \nabla \mathsf{P}. \tag{21}$$

The demonstration is buried in the book of G. de Rham [19] = [20], so we refer to the Swiss.

From this proposition we conclude that here too there exists a $\mathsf{P} \in L^2_0(\Omega)$ such that

$$[\![: \cdots :]\!] = \nabla \mathsf{P}, \tag{22}$$

and we are done.

□

Scholium 1.1 (Two clarifications).

(1) Why the Hilbertian set $\mathfrak{H}^*$ in (δ) is embedded into a L^p space having $p = 6$? Because, in the dimensionality under discussion, it has been established that $n \leqslant 3$.

(2) In Eq. (19) I make use of the repeat sign $[\![: \cdots :]\!]$ derived from musical notation for the umpteenth time.

Theorem 1.2 *(Part II). Let*

$$\varphi_{\text{L}} \in L^2(\Omega),\ \ \mathscr{C}^{1,1}(\Omega) \in \mathbb{R}^n,\ \ n = 2, 3. \tag{23}$$

Then

$$(\upsilon, \mathsf{P}) \in \mathfrak{H}^2(\Omega) \times \mathfrak{H}^1(\Omega), \tag{24}$$

meets

$$\left.\begin{aligned} &-\tfrac{\operatorname{div}(\nabla \upsilon)}{\text{Re}} + \nabla \mathsf{P} = \varphi_{\text{L}} - (\upsilon \nabla)\, \upsilon \\ &\operatorname{div} \upsilon = 0 \end{aligned}\right\}\ in\ \Omega \in \mathbb{R}^n, \tag{25a}$$

$$\upsilon_{|\partial\Omega} = 0,\ \ \partial\Omega \times (0, T) =]0, T[. \tag{25b}$$

Proof. It is about proving that

$$\upsilon \in L^{k<\infty}(\Omega),\ \ k < +\infty. \tag{26}$$

(1) If $n = 2$, the demonstration is a consequence of an instance of Sobolev compact embedding; see the Sobolevian spatial structures [22] [23]. Let

$$W^{1,p}(\Omega) \subset L^k(\Omega) \tag{27}$$

be a W-embedding; take for granted that

$$1 \leqslant k < s, \tag{28}$$

on the basis of

$$\begin{cases} 1 \leqslant p < +\infty, & (29a) \\ 1 \leqslant k \leqslant s, & (29b) \end{cases}$$

where s is so defined:

(i) $1 \leqslant s < +\infty$, for $p = n$,
(ii) $s = +\infty$, for $p > n$,
(iii) $s(p - n) = -np$, for $p < n$.

It is effortlessly see that

$$\mathfrak{H}^1(\Omega) \subset L^{k<\infty}(\Omega). \tag{30}$$

(2) If $n = 3$, the above gives

$$\mathfrak{H}^1(\Omega) \subset L^k(\Omega),\ \ k = 6, \tag{31}$$

from which

$$\begin{cases} \nabla \upsilon \in L^2(\Omega),^{\text{a}} & (32a) \\ (\upsilon \nabla)\, \upsilon \in L^{\frac{3}{2}}(\Omega). & (32b) \end{cases}$$

Regarding the regularity problem (25), for an increasing deduction procedure, it can be inferred that

$$\upsilon \in \begin{cases} W^{2,\frac{3}{2}}(\Omega), & (33a) \\ W^{1,3}(\Omega), & (33b) \\ L^{k<\infty}(\Omega), & (33c) \end{cases}$$

[a] It is this (for the sake of completeness): $\left[L^2(\Omega)\right]^{3\times 3}$.

and so

$$(\upsilon\nabla)\,\upsilon \in L^{p<2}(\Omega) \tag{34}$$

plus

$$\upsilon \in \begin{cases} W^{2,p<\infty}(\Omega), & (35a)\\ L^{\infty}(\Omega). & (35b)\end{cases}$$

We are done. Which allows us to state that the $\in$-relation (24) is true (i.e. verifiable); and, if we generalize,[a] that

$$\varphi_{\mathrm{L}} \in L^{(2<p<\infty)}(\Omega), \tag{36}$$

therefore

$$(\upsilon,\, \mathsf{P}) \in W^{2,p}(\Omega) \times W^{1,p}(\Omega) \subset \begin{cases} L^{k}(\Omega), & (37a)\\ \mathscr{C}^{0,\alpha}(\hat{\Omega}), & (37b)\end{cases}$$

once it is made clear that

$\alpha \in]0,1]$ is the Hölder exponent, and

$\hat{\Omega}$ denotes a non-empty (measurable) set.

□

1.1.2. The Non-homogeneity Case

The Navier–Stokes problem with non-homogeneous Dirichlet condition is the same as that in Eq. (3) except for sub-Eq. (3b), which should be replaced by

$$\upsilon_{|\partial\Omega} = \jmath,$$

presuming that $\jmath$ is some function, i.e.

$$\left.\begin{aligned} -\tfrac{\operatorname{div}(\nabla\upsilon)}{\mathrm{Re}} + (\upsilon\nabla)\upsilon + \nabla\mathsf{P} = \varphi_{\mathrm{L}}\\ \operatorname{div}\upsilon = 0\end{aligned}\right\} \text{ in } \Omega \in \mathbb{R}^n, \tag{38a}$$

$$\upsilon_{|\partial\Omega} = \jmath. \tag{38b}$$

Theorem 1.3. *Let w be the outward unit normal vector (a vector of unit length which pointis outward, away from the surface $\partial\Omega$ of Ω), and $\tau \in \partial\Omega$ some smooth tensor field. And we impose $\gamma_1^{(c)} \ldots \gamma_k^{(c)}$ as the connected components of the boundary $\partial\Omega$. Here too Lipschitz domain Ω is replicated. Then one has*

$$K_{(1,2)\Omega\in\mathbb{R}^n} > 0, \quad n = 2, 3,$$

so the pair

$$\varphi_{\mathrm{L}} \in \mathfrak{H}^{-1}(\Omega), \tag{39}$$

$$\tau \in \mathfrak{H}^{\frac{1}{2}}(\partial\Omega), \tag{40}$$

satisfy

$$\int_{\partial\Omega} (\jmath) w d\tau = 0, \tag{41a}$$

$$\sum_{j=1}^{k} \int_{\gamma^{(c)}_{1\leqslant j\leqslant k}} (\jmath) w d\tau \leqslant \frac{K_{1,\Omega}}{\mathrm{Re}}. \tag{41b}$$

[a] Cf. the membership relation (23).

The solution to the non-linear problem is given by

$$\begin{aligned} 0 &= \int_{\Omega\in\mathbb{R}^n} \operatorname{tr}(\nabla\upsilon)\,dx \\ &= \int_{\Omega\in\mathbb{R}^n} \operatorname{div}\upsilon dx \\ &= \int_{\partial\Omega} (\text{л}) w d\tau. \end{aligned} \tag{42}$$

The solution will be

$$(\upsilon, \mathsf{P}) \in \mathfrak{H}_0^1(\Omega) \times L_0^2(\Omega), \tag{43}$$

which goes to solve

$$\begin{aligned} \|\nabla\upsilon\|_{L^2} &\leqslant K_{2,\Omega}\Big(\mathrm{Re}\|\varphi_{\mathrm{L}}\|_{\mathfrak{H}^{-1}}\Big) \\ &\quad + K_{2,\Omega}\|\text{л}\|_{\mathfrak{H}^{\frac{1}{2}}} e^{K_{2,\Omega}\left(\mathrm{Re}\|\text{л}\|_{\mathfrak{H}^{\frac{1}{2}}}\right)}. \end{aligned} \tag{44}$$

Proof. Let ζ be a non-negative $\mathbb{R}$-number,

$$\vec{X} \in \mathfrak{H}^1(\Omega)$$

a vector field, and

$$\text{ъ} \in \mathfrak{H}_0^1(\Omega)$$

a certain quantity, such that

$$\begin{cases} \|\vec{X}\otimes \text{ъ}\|_{L^2} \leqslant \zeta\|\text{ъ}\|_{\mathfrak{H}_0^1}, & (45a) \\ \|\nabla\vec{X}\|_{L^2} \leqslant K_\Omega\|\text{л}\|_{\mathfrak{H}^{\frac{1}{2}}} e^{\left((K_\Omega)/\zeta\|\text{л}\|_{\mathfrak{H}^{\frac{1}{2}}}\right)}. & (45b) \end{cases}$$

Here the fine eye can see a resurrecting of Hopf's [6] ruminations; thence it will be allowed to gloss over the demonstrative passages about the inequalities in (45).

For notational reasons, let us add this substitution-equivalence: $\vec{X} \overset{\text{eqv}}{=\!=} \chi$, with which we go to write

$$\left.\begin{aligned} &-\tfrac{\triangle \text{ъ}}{\mathrm{Re}} + (\tilde{\chi}\nabla)\text{ъ} + (\text{ъ}\nabla)\tilde{\chi} + (\text{ъ}\nabla)\text{ъ} + \nabla\mathsf{P} = \varphi_{\mathrm{L}} + \tfrac{\triangle\tilde{\chi}}{\mathrm{Re}} - (\tilde{\chi}\nabla)\tilde{\chi} \\ &\operatorname{div}\text{ъ} = 0 \\ &\text{ъ} = 0 \end{aligned}\right\}\text{ in } \Omega\in\mathbb{R}^n. \tag{46a}$$

What is needed now is to set up an energy estimate for the problem in question, which is equivalent to finding an estimate for

$$E(\tilde{\chi}, \text{ъ}) = \int_{\Omega\in\mathbb{R}^n} \Big((\text{ъ}\nabla)\tilde{\chi}\Big)\text{ъ}\ dx + \int_{\Omega\in\mathbb{R}^n} \Big((\tilde{\chi}\nabla)\text{ъ}\Big)\text{ъ}\ dx, \tag{47}$$

$$= \int_{\Omega\in\mathbb{R}^n} (\tilde{\chi}\otimes\text{ъ}) \Big|\ \nabla\text{ъ}\ dx - \int_{\Omega\in\mathbb{R}^n} (\text{ъ}\otimes\tilde{\chi}) \Big|\ \nabla\text{ъ}\ dx, \tag{48}$$

given an E-function.

We define $\vec{Y} \in \mathfrak{H}^1(\Omega)$ to be a vector field in order that

$$\left.\begin{aligned} -\triangle\vec{Y} + \nabla \digamma = 0 \\ \operatorname{div}\vec{Y} = 0 \end{aligned}\right\}\text{ in } \Omega\in\mathbb{R}^n, \tag{49a}$$

$$\vec{Y} = \left(\frac{1}{|\gamma_j^{(\mathrm{c})}|}\int_{\gamma^{(\mathrm{c})}_{1\leqslant j\leqslant k}} (\text{л}) w d\tau\right) w \text{ on } \gamma^{(\mathrm{c})}_{1\leqslant j\leqslant k}, \tag{49b}$$

choosing $\digamma \in L^2(\Omega)$ as a scalar field.

Via (45) one gets $|E(\vec{X},\text{ъ})| \leqslant 2\zeta\|\nabla\text{ъ}\|^2_{L^2}$, and thru (41) one achieves

$$\Big|E(\vec{Y},\text{ъ})\Big| \leqslant 2\|\text{ъ}\|_{L^6}\|\vec{Y}\|_{L^3}\|\nabla\text{ъ}\|_{L^2} \leqslant K_\Omega\|\vec{Y}\|_{L^3}\|\nabla\text{ъ}\|^2_{L^2} \leqslant K_\Omega\|\vec{Y}\|_{\mathfrak{H}^1}\|\nabla\text{ъ}\|^2_{L^2} \leqslant \frac{\|\nabla\text{ъ}\|^2_{L^2}}{4\mathrm{Re}}. \tag{50}$$

The multi-Eq. (46) is nothing other than the Navier–Stokes problem transcribed in such a way as to produce a homogeneous specificity about the Dirichlet boundary condition. An estimate on such an equational set takes this form of inequalities,

$$\frac{\|\nabla \mathfrak{v}\|_{L^2}^2}{\mathrm{Re}} \leqslant \left(\frac{8\zeta\,(\mathrm{Re}+1)}{4\mathrm{Re}}\right)\|\nabla \mathfrak{v}\|_{L^2}^2 + [\![:\left(\|\varphi_{\mathrm{L}}\|_{\mathfrak{H}^{-1}} + [\![:\left\|\tfrac{\triangle\tilde{\chi}}{\mathrm{Re}} - (\tilde{\chi}\nabla)\tilde{\chi}\right\|_{\mathfrak{H}^{-1}} :]\!]\right) :]\!]\,\|\nabla \mathfrak{v}\|_{L^2} \leqslant 2\mathrm{Re}\,[\![:\cdots:]\!], \tag{51}$$

which provide a solution to the problem of this Section. From the formalism combination (equation plus inequality) of (41) and from the Hopfian inequalities (45), it can be concluded that the non-equal relation (44) is (mathematically) true by

$$[\![:\cdots:]\!] \leqslant \|\tilde{\chi}\otimes\tilde{\chi}\|_{L^2} + \frac{\|\nabla\tilde{\chi}\|}{\mathrm{Re}} \leqslant K_\Omega\left(\|\mathfrak{a}\|_{\mathfrak{H}^{\frac{1}{2}}}\Big/\mathrm{Re}\right)e^{K_\Omega\|\mathfrak{a}\|_{\mathfrak{H}^{\frac{1}{2}}}\mathrm{Re}}. \tag{52}$$

□

1.2. Asymptotic Analysis (of a Navier–Stokes Flow)

Let us set the problem with this set of equations,

$$\left.\begin{aligned}&\tfrac{\partial \upsilon(t)_{t\to+\infty}}{\partial t} - \tfrac{\mathrm{div}(\nabla\upsilon)}{\mathrm{Re}} + (\upsilon\nabla)\upsilon + \nabla\mathsf{P} = \varphi_{\mathrm{L}}\\ &\mathrm{div}\,\upsilon = 0,\end{aligned}\right\}\text{in } \Omega\in\mathbb{R}^n, \tag{53a}$$

$$\upsilon_{|\partial\Omega}, \tag{53b}$$

$$\upsilon_{|t=0} = \upsilon(0) = \upsilon_0, \tag{53c}$$

dictating that there exists a solution

$$(\upsilon,\mathsf{P})_\infty \in \mathfrak{H}_0^1(\Omega)\times L_0^2(\Omega). \tag{54}$$

We can thereby better specify the details for the asymptotic aspect. Putting

$$\begin{cases}\upsilon + \upsilon_\infty \overset{\mathrm{eqv}}{=\!=} \mathfrak{v}, & (55\mathrm{a})\\ \mathsf{P} \overset{\mathrm{eqv}}{=\!=} \mathsf{P}_\infty + \mathsf{p}, & (55\mathrm{b})\end{cases}$$

the equational set (53) looks like this:

$$(85)\begin{cases}\dfrac{\partial\mathfrak{v}(t)}{\partial t} - \dfrac{\mathrm{div}(\nabla\mathfrak{v})}{\mathrm{Re}} + (\mathfrak{v}\nabla)\mathfrak{v} + (\mathfrak{v}\nabla)\upsilon_\infty + (\upsilon_\infty\nabla)\mathfrak{v} + \nabla\mathsf{p} = 0,\ \ t\to+\infty, & (56\mathrm{a})\\ \mathrm{div}\,\mathfrak{v} = 0, & (56\mathrm{b})\\ \mathfrak{v} = 0, & (56\mathrm{c})\\ \mathfrak{v}(0) \overset{\mathrm{eqv}}{=\!=} (\upsilon_0 - \upsilon_\infty),\ \ t = 0. & (56\mathrm{d})\end{cases}$$

Dimensionality represents a crossroads.
If $n = 2$, then we have a unique global weak solution regardless of the initial data.
If $n = 3$, there is more than one global weak solution. (Strong solutions may not be global).

1.2.1. Stability of a 2-Space Stationary Solution

Let us take a closer look at the 2-dimensionality.

Theorem 1.4. *Let*

$$\varphi_{\mathrm{L}} \in \mathfrak{H}^{-1}(\Omega),\ \ \Omega\in\mathbb{R}^2, \tag{57}$$

be a Lipschitz-type function, and the $\in$*-relation* (54) *be the steady solution of* (38)*. The unique global weak solution* (υ,P) *of* (53) *meets*

$$\|\nabla\upsilon_\infty\|_{L^2} \leqslant K_\Omega\Big/\mathrm{Re}\begin{cases}\|\nabla\upsilon(t) - \nabla\upsilon_\infty\|_{L^2} \leqslant K_\Omega\|\upsilon_0 - \upsilon_\infty\|_{L^2}, & (58\mathrm{a})\\ \|\upsilon(t) - \upsilon_\infty\|_{L^2} \leqslant \|\upsilon_0 - \upsilon_\infty\|_{L^2}, & (58\mathrm{b})\end{cases}$$

with $\exp(-\zeta t)$, $t \geqslant 1$, *in* (58a), *and* $\exp(-\zeta t)$, $t\geqslant 0$, *in* (58b), *for each initial data* $\upsilon_0 \in \mathfrak{H}^{(\cdot)}$.

Proof.

(α) Let $m(\varpi, \upsilon, \mathfrak{v})$, $\varpi, \upsilon, \mathfrak{v} \in \mathfrak{H}_0^1(\Omega)$, be a trilinear form on $\mathfrak{H}_0^1(\Omega) \times \mathfrak{H}_0^1(\Omega) \times \mathfrak{H}_0^1(\Omega)$, so as to

$$|m(\varpi, \upsilon, \mathfrak{v})| \leqslant K_\Omega \|\varpi\|_{L^2}^{\frac{4-n}{4}} \|\varpi\|_{\mathfrak{H}^1}^{\frac{n}{4}} \|\upsilon\|_{L^2}^{\frac{4-n}{4}} \|\upsilon\|_{\mathfrak{H}^1}^{\frac{n}{4}} \|\mathfrak{v}\|_{\mathfrak{H}^1}. \tag{59}$$

From (56), with the help of Young's inequality [32], one finds

$$\frac{1}{2} \cdot \frac{d\left(\|\mathfrak{v}\|_{L^2}^2\right)}{dt} + \frac{\|\nabla \mathfrak{v}\|_{L^2}^2}{\mathrm{Re}} \leqslant K_\Omega \|\nabla \upsilon_\infty\|_{L^2} \|\mathfrak{v}\|_{L^2} \|\nabla \mathfrak{v}\|_{L^2}, \tag{60}$$

$$\frac{d}{dt}\|\mathfrak{v}\|_{L^2}^2 + \frac{\|\nabla \mathfrak{v}\|_{L^2}^2}{\mathrm{Re}} \leqslant K_\Omega \Big(\mathrm{Re}\|\nabla \upsilon_\infty\|_{L^2}^2 \|\mathfrak{v}\|_{L^2}^2\Big). \tag{61}$$

To be more precise, the non-equality (60) is gained by multiplying the equational group (56) by $\mathfrak{v}$, whereas the non-equality (61) is reached by Young's inequality.

(β) Let λ_1 be the first eigenvalue of the Stokes operator. The Poincaré inequality [16]

$$\begin{cases} \|\upsilon\|_{\mathfrak{H}}^2 \leqslant \dfrac{\|\nabla \upsilon\|_{L^2}^2}{\lambda_1}, & (62a) \\ \|\nabla \upsilon\|_{L^2}^2 \leqslant \dfrac{\|\sum_{j \geqslant 1} (\lambda \upsilon \mathfrak{v})_j\|_{L^2}^2}{\lambda_1}, & (62b) \end{cases}$$

comes in handy to spell out that

$$\begin{cases} \dfrac{d}{dt}\|\mathfrak{v}\|_{L^2}^2 + \dfrac{1}{\mathrm{Re}} K_{1,\Omega} - \underbrace{K_\Omega \cdot \mathrm{Re}\|\nabla \upsilon_\infty\|_{L^2}^2}_{K_\Omega (\mathrm{Re}\|\nabla \upsilon_\infty\|_\infty)^2 < K_{1,\Omega}} \|\mathfrak{v}\|_{L^2}^2 \leqslant 0, & (63a) \\ \dfrac{d}{dt}\|\mathfrak{v}\|_{L^2}^2 + 2\zeta \|\mathfrak{v}\|_{L^2}^2 \leqslant 0. & (63b) \end{cases}$$

Ergo $\|\mathfrak{v}(t)\|_{L^2}^2 \leqslant \|\mathfrak{v}_0\|_{L^2}^2$ with $\exp(-2\zeta t)$.

□

1.2.2. Stability of a 3-Space Stationary Solution

Let us analyze the 3D issue.

***Theorem 1.5**. Let*

$$\varphi_{\mathrm{L}} \in \mathfrak{H}^{-1}(\Omega), \quad \Omega \in \mathbb{R}^3, \tag{64}$$

be a Lipschitz-type function, and

$$(\upsilon, \mathsf{P})_\infty \in \mathfrak{H}_0^1(\Omega) \times L_0^2(\Omega) \tag{65}$$

be a steady solution of (38)*. The unique global strong solution* (υ, P) *of* (56) *and hence of* (53) *meets*

$$\Big\{\|\nabla \upsilon(t) - \nabla \upsilon_\infty\| \leqslant \|\nabla \upsilon_0 - \nabla \upsilon_\infty\|\Big\}_{L^2} \exp(-\zeta t), \tag{66}$$

$t \geqslant 0$*, admitting that*

$$\left.\begin{matrix} \|\nabla \upsilon_\infty\|_{L^2} \\ \|\nabla \upsilon_0 - \nabla \upsilon_\infty\|_{L^2} \end{matrix}\right\} \leqslant K_\Omega \Big/ \mathrm{Re}. \tag{67}$$

Scholium 1.2. The unique local strong solution of (56) is easier to prove. It is set on an $\mathbb{R}$-domain of $\mathscr{C}^{1,1}(\Omega)$, with $\varphi_{\mathrm{L}} \in L^2_{\mathrm{loc}}(\cdot)$, where $\cdot$ is for $[0, +\infty[, L^2(\Omega)$ in $\mathbb{R}^3$. The result provides that

$$\text{for some } M > 0 \begin{cases} \upsilon \in \mathscr{C}^0(\cdot) \cap L^2_{\mathrm{loc}}, \text{ where } \cdot = [0, M[, \mathfrak{H}^*, & (68a) \\ \mathsf{P} \in L^2_{\mathrm{loc}}(\cdot), \text{ where } \cdot = [0, M[, \mathfrak{H}^1(\Omega), & (68b) \\ \frac{d\upsilon}{dt} \in L^2_{\mathrm{loc}}(\cdot), \text{ where } \cdot = [0, M[, \mathfrak{H}. & (68c) \end{cases}$$

Proof. This time too the demonstration will be divided into several steps.

(α) We designate $\triangle_{\mathrm{S}}$ to be the (positive self-adjoint) Stokes operator, which unbounded and it behaves like the Laplace operator $\triangle = \nabla^2 = \nabla \cdot \nabla$, when $\triangle_{\mathrm{S}}$ is investigated in relation to Dirichlet boundary conditions. If one sets out that

$$\mathfrak{v} \in \mathfrak{D}_{\mathrm{S}}(\triangle_{\mathrm{S}}) = \left\{ \mathfrak{v} \in \mathfrak{H}^2(\Omega, \mathbb{R}^3) \mid \operatorname{div} \mathfrak{v}_{|\Omega} = 0 \text{ and } \mathfrak{v}_{|\partial\Omega} = 0 \right\}, \tag{69}$$

where $\mathfrak{D}_{\mathrm{S}}$ is the domain of the Stokes operator, and

$$[\![: \left\{ \mathfrak{v} \in \mathfrak{H}^1(\Omega, \mathbb{R}^3) \mid \operatorname{div} \mathfrak{v}_{|\Omega} = 0 \text{ and } \mathfrak{v}_{|\partial\Omega} = 0 \right\} :]\!] \tag{70}$$

the Stokes operator can be summarized as

$$\begin{cases} \triangle_{\mathrm{S}}\, \mathfrak{v} = -\pi_{\mathrm{L}} \triangle \mathfrak{v} = - \triangle \mathfrak{v}, & (71a) \\ \mathfrak{D}_{\mathrm{S}} :\ \subset [\![: \cdots :]\!] \longrightarrow [\![: \cdots :]\!]. & (71b) \end{cases}$$

And here a linear operator was introduced ad hoc. This is the Leray projector π_{L} [9] [10], which is
(i) the orthogonal projection from $L^2(\Omega)$ onto $\mathfrak{H}$,

$$L^2(\Omega) \xrightarrow{\pi_{\mathrm{L}}} \mathfrak{H},$$

whose equational reference comes from (71a),
(ii) or (inasmuch as $\mathfrak{D}_{\mathrm{S}}$ possesses some smoothness properties) a projection in $\mathfrak{H}^1$ onto itself,

$$\|\pi_{\mathrm{L}}(\mathfrak{v})\|_{\mathfrak{H}^1} \leqslant K_\Omega \|\mathfrak{v}\|_{\mathfrak{H}^1},$$

as reflected in (71b).
From the foregoing, it can be deduced that

$$\langle \triangle_{\mathrm{S}}\, \mathfrak{v}, \varpi \rangle = -\int_{\Omega \in \mathbb{R}^3} \pi_{\mathrm{L}} \triangle \mathfrak{v}(\varpi) = -\int_{\Omega \in \mathbb{R}^3} \triangle \mathfrak{v}(\varpi) = -\int_{\Omega \in \mathbb{R}^3} \mathfrak{v} \cdot \triangle \varpi = \langle \mathfrak{v} \triangle_{\mathrm{S}}, \varpi \rangle. \tag{72}$$

(β) From (56) and (59), by making use of the operator (71), one builds

$$\begin{aligned} \left\| \frac{\triangle_{\mathrm{S}}\, \mathfrak{v}}{\mathrm{Re}} \right\|_{L^2}^2 + \frac{1}{2} \cdot \frac{d\left(\|\nabla \mathfrak{v}\|_{L^2}^2\right)}{dt} &\leqslant \|\upsilon_\infty\|_{L^6} \|\nabla \mathfrak{v}\|_{L^3} \| \triangle_{\mathrm{S}}\, \mathfrak{v}\|_{L^2} + \|\mathfrak{v}\|_{L^\infty} \|\nabla \upsilon\|_{L^2} \| \triangle_{\mathrm{S}}\, \mathfrak{v}\|_{L^2} \\ &+ \|\mathfrak{v}\|_{L^\infty} \|\nabla \upsilon_\infty\|_{L^2} \| \triangle_{\mathrm{S}}\, \mathfrak{v}\|_{L^2} \leqslant K_\Omega \cdot \|\nabla \upsilon_\infty\|_{L^2} \|\nabla \mathfrak{v}\|_{L^2}^{\frac{1}{2}} \| \triangle_{\mathrm{S}}\, \mathfrak{v}\|_{L^2}^{\frac{3}{2}} + \|\nabla \mathfrak{v}\|_{L^2}^{\frac{3}{2}} \| \triangle_{\mathrm{S}}\, \mathfrak{v}\|_{L^2}^{\frac{3}{2}}. \end{aligned} \tag{73}$$

According to Young's inequality [32] and Poincaré inequality (62), one has subsequently that

$$\begin{cases} \dfrac{\| \triangle_{\mathrm{S}}\, \mathfrak{v}\|_{L^2}^2}{\mathrm{Re}} + \dfrac{d}{dt}\|\nabla \mathfrak{v}\|_{L^2}^2 \leqslant [\![: K_\Omega \left(\mathrm{Re}^3 \|\nabla \mathfrak{v}\|_{L^2}^6\right) + K_\Omega \left(\mathrm{Re}^3 \|\nabla \upsilon_\infty\|_{L^2}^4 \|\nabla \mathfrak{v}\|_{L^2}^2\right) :]\!], & (74a) \\ \dfrac{K_{1,\Omega} \|\nabla \mathfrak{v}\|_{L^2}^2}{\mathrm{Re}} + \dfrac{d}{dt}\|\nabla \mathfrak{v}\|_{L^2}^2 \leqslant [\![: \cdots :]\!]. & (74b) \end{cases}$$

More accurately, Young's inequality acts on the first formulistic line (74a); Poincaré inequality acts on the second formulistic line (74b).

From the plexus of previous inequalities, in agreement with convenience values, one sketches that

$$\mathfrak{v}(0) \overset{\text{eqv}}{=\!=} (\upsilon_0 - \upsilon_\infty) \begin{cases} [\![: \dfrac{d\left(\|\nabla \mathfrak{v}\|_{L^2}^2\right)}{dt} :]\!] + \dfrac{K_{1,\Omega}}{2\mathrm{Re}} - K_\Omega \left(\mathrm{Re}^3 \|\nabla \mathfrak{v}\|_{L^2}^4\right) \|\nabla \mathfrak{v}\|_{L^2}^2 \leqslant 0, & (75a) \\ [\![: \cdots :]\!] + \underbrace{\dfrac{K_{1,\Omega}}{2\mathrm{Re}} - K_\Omega \left(\mathrm{Re}^3 \|\nabla \mathfrak{v}(0)\|_{L^2}^4\right)}_{\geqslant K_{1,\Omega} / 4\mathrm{Re}}. & (75b) \end{cases}$$

(γ) Let $t \in [0, T]$, with $T > 0$, and

$$T_{\mathrm{MAX}} \in \,]0, +\infty] = \{(\cdot) \in \mathbb{R}^3 \mid 0 < (\cdot) \leqslant +\infty\}. \tag{76}$$

The $\mathfrak{v}$-solution dictates that

$$\left.\begin{matrix} T \\ T_{\mathrm{MAX}} \end{matrix}\right\} = +\infty. \tag{77}$$

If that was not true, one would have that

$$T = T_{\text{MAX}} \neq +\infty, \tag{78}$$

namely, $T < +\infty$, and $T < T_{\text{MAX}}$, with

$$\int_0^{T_{\text{MAX}}} \|\mathfrak{v}\|_{L^2}^2 dt = +\infty. \tag{79}$$

From the denial of equality in (78) let us go write, in the first instance,

$$[\![: \cdots :]\!] + \frac{K_{1,\Omega}\|\nabla \mathfrak{v}\|_{L^2}^2}{8\text{Re}} \leqslant 0, \tag{80}$$

$$\|\nabla \mathfrak{v}(t)\|_{L^2} \leqslant \|\nabla \mathfrak{v}(0)\|_{L^2}, \tag{81}$$

and, secondly,

$$\begin{cases} \underbrace{\dfrac{K_{1,\Omega}}{2\text{Re}} - K_\Omega \left(\text{Re}^3 \|\nabla \mathfrak{v}(T)\|_{L^2}^4\right)}_{\geqslant K_{1,\Omega}/4\text{Re}}, & (82\text{a}) \\ \underbrace{\dfrac{K_{1,\Omega}}{2\text{Re}} - K_\Omega \left(\text{Re}^3 \|\nabla \mathfrak{v}(t)\|_{L^2}^4\right)}_{\geqslant K_{1,\Omega}/8\text{Re}}. & (82\text{b}) \end{cases}$$

In the inequality (82b) the value of t is assigned by this $\in$-relation,

$$t \in [0, T+\epsilon[\, = \{(\cdot) \in \mathbb{R}^3 \mid 0 \leqslant (\cdot) < T + \epsilon\}, \;\; \epsilon > 0. \tag{83}$$

Here we go. The relation (83) goes against both the fundamental $\in$-relation (76) and the equalities in (77), which epitomize nothing other than the effective $\mathfrak{v}$-solution for the proof of the Theorem 1.5.

□

1.3. The P-Force

We are talking about the pressure

$$\mathsf{P}\colon \Omega \times \mathbb{R}_+ \to \mathbb{R} \tag{84}$$

of the fluid, that is,

$$\begin{cases} \mathsf{P}\colon \Omega \times (0,T) \to \mathbb{R}, \;\; t \in (0,T), & (85\text{a}) \\ \mathsf{P}\colon \Omega \times (0,T) \times D_{\mathbb{R}} \subset \mathbb{R}^n \to \mathbb{R}, \;\; n = 2,3, & (85\text{b}) \end{cases}$$

in consonance with Euler's coordinates. Remember Eqq. contained in (85) and (56).

Please be aware that the pressure can be gained from an elliptic partial differential equation (including the Laplacian) of Poisson-type [17]:

$$-\triangle \mathsf{P} = \sum_{i,j} v^i_{x_j} v^j_{x_i}, \tag{86}$$

having a function $\varphi_{\text{L}} \in \mathfrak{H}^{-1}(\Omega)$, and $v \in \mathfrak{H}^1_0(\Omega)$, such that $-\triangle v = \varphi_{\text{L}}$ in $\Omega \in \mathbb{R}^n$.

How can we develop the asymptoticity of pressure? For this purpose, first of all, we determine

$$\begin{cases} 1/\mathfrak{w} > 0 \displaystyle\int_t^{t+\mathfrak{w}} \mathsf{P}(x)dx, \;\; t \to +\infty, & (87\text{a}) \\ \mathsf{P} \xrightarrow{\mathsf{P}(x) \in L^2(\Omega)} \mathsf{P}_{+\infty}. & (87\text{b}) \end{cases}$$

Thereafter we go on to describe the convergence behavior of $\int_t^{t+\mathfrak{w}} \mathsf{p}(x)dx$.[a]

[a] Cf. Eq. (55b).

Here are the steps to complete this action:

$$\underbrace{\int_t^{t+\omega} \mathsf{P}(x)dx}_{\nabla} = \frac{\partial}{\partial t}\Big(\mathfrak{v}(t) - \mathfrak{v}(t+\omega)\Big) + \operatorname{div} [\![: \int_t^{t+\omega} \frac{\nabla \mathfrak{v}}{\mathrm{Re}} - \mathfrak{v}\otimes\mathfrak{v} - \mathfrak{v}\otimes v_\infty - v_\infty \otimes \mathfrak{v}\; dx :]\!]. \quad (88)$$

Let us impose some inequalities, which act as a guide. The first is

$$\|\mathsf{P}\|_{L^k(\Omega)} = K_\Omega \|\nabla \mathsf{P}\| \overset{\text{eqv}}{=\!=} \|\mathsf{P}\|_{L^k(\Omega)} \leqslant K_\Omega\Big(\|\mathsf{P}\|_{W^{-1,k}(\Omega)} + \|\nabla \mathsf{P}\|_{W^{-1,k}(\Omega)}\Big),$$
$$\overset{\text{eqv}}{=\!=} \|\mathsf{P}\|_{L^k(\Omega)} \leqslant K_\Omega \|\nabla \mathsf{P}\|_{\mathfrak{H}^{-1}(\Omega)} = K_\Omega \sup \int_\Omega \mathsf{P}(x)\operatorname{div}\mathsf{F}(x)dx \;\Big|\; \|\mathsf{F}\|_{\mathfrak{H}^1_0(\Omega)} \leqslant 1, \quad (89)$$

$1 < k < +\infty$, for any function $\mathsf{P} \in L^k(\Omega)$, $\mathsf{P} \in \mathscr{C}^\infty_{\mathrm{c}}$, where $\mathscr{C}^\infty_{\mathrm{c}}$ is the class of all functions with compact supported in $\mathscr{C}^\infty_{\mathrm{c}}$-form, generating a vector $\mathbb{R}$-space, and for a certain function F.

The second is

$$\|\mathsf{P}\|_{\mathfrak{H}^{-1}} \leqslant K_\Omega \left(\frac{\left|\int_\Omega \mathsf{P}dx\right|}{|\Omega|} + \|\nabla \mathsf{P}\|_{\mathfrak{H}^{-1}}\right), \quad (90)$$

in which, at long last, $\|\mathsf{P}\|_{\mathfrak{H}^{-1}} = \lim_{r\to\infty} \|\mathsf{P}_{n_r} \in L^2(\Omega)\|_{\mathfrak{H}^{-1}}$ holds.

In both inequalities (89) and (90), the letter Ω, clearly, is the usual Lipschitz domain that stands out throughout the article.

And accordingly

$$\left\|\int_t^{t+\omega} \mathsf{P}(x)dx\right\| \leqslant K_\Omega \left\|\underbrace{\int_t^{t+\omega} \mathsf{P}(x)dx}_{\nabla}\right\|_{\mathfrak{H}^{-1}} \leqslant [\![: \|\mathfrak{v}(t) + \mathfrak{v}(t+\omega)\|_{\mathfrak{H}^{-1}} :]\!]$$
$$+ \|[\![: \cdots :]\!]\|_{L^2} \leqslant [\![: \cdots :]\!]$$
$$+ K_\Omega \left(\int_t^{t+\omega} \underbrace{\|\nabla \mathfrak{v}\|_{L^2}}_{\to 0} + \|\mathfrak{v}\cdot v_\infty\|_{L^4} + \|\mathfrak{v}\|^2_{L^4}\right) dx. \quad (91)$$

2. Caudal Section: Concluding Passage

Wide-ranging guidelines omitted in this didascalic article, which is characterized by a merely synoptic approach, can be found in [11] [30] [2], for instance. Insights on the Navier–Stokes equations for incompressible non-homogeneous fluids are in [21] [4] [3] [7].

However, the present paper is ironically in line with the hypertrophic production of current mathematics, viz. with the industrialization of scientific articles generated by the papier-mâché Empire of publishing companies.

What has mathematics become? A Queneau-type buffo exercise, a self-replicating onanism through the action of the apery, a pile of teratological excrescences grafted onto a chucklesome practice of bibliometrics [31], and consequently a blind trip towards nothingness outside of itself. Lastly, mathematics has become a game that entertain tedious souls (un giuoco che «le tediose alme trastulla»).

True mathematics, if it genuinely rates the name yet, should not be done in this way, which is so fashionable nowadays. The authentic meaning of the noun máthēma (μάθημα) has been lost. Well, it leads back to the path of humility… And reality—of which mathematics is only a descriptive illusion (like any other description: the poetic one, the artistic one, etc.)—certainly lies elsewhere.

*

* *

R. Queneau, in an unpublished essay dating between 1944 and 1948 [18], writes:

The ideal that scientists have built throughout this beginning of the century has been a presentation of science not as knowledge but as a rule and method. We give (indefinable) notions, axioms and instructions for use, in short, a system of conventions. But is this not a game that has nothing different from chess or bridge? [...] [A]nd what do we know in mathematics? Precisely: nothing. And there is nothing to know. We do not know the point, the number, the group, the set, the function any more than we know the electron, [the concept of] life [or the] human behavior. We do not know the world of functions and differential equations any more than we "know" the Concrete Terrestrial and Everyday Reality. All we know is a method accepted (permitted) as true by the scientific community, a method which also has the advantage of connecting to fabrication techniques. But this method is also a game, more precisely what is called a *jeu d'esprit*. Therefore the whole of science, in its complete form, presents itself both as a technique and as a game. It is neither more nor less than the other human activity: Art.

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