

Self-Triggered Collision Avoidance Control for Multi-Vehicle Systems

Erick J. Rodríguez-Seda

Abstract—The implementation of collision detection mechanisms on-board of unmanned vehicles typically relies on digital platforms and shared communication networks. The discrete nature of these technologies does not only limit the frequency at which nearby obstacles and other vehicles are detected, but it also raises the demand for a better allocation of the vehicle's computational and communication resources. For instance, it is natural and convenient to adapt the frequency at which other vehicles are observed based on their distance and collision risk, a notion that relates to topics in minimum attention control and self-triggered control. In this paper, we report on a cooperative, decentralized self-triggered avoidance control law for an arbitrary large group of unmanned vehicles that reduces the attention and sampling frequency of other vehicles' position. The proposed control strategy guarantees the safe coordination of all vehicles at all times while reducing the sampling frequency of other vehicles' position based on their last available sample. We show that the sampling intervals are positive and lower bounded, which is critical for digital implementation. A simulation example is used to illustrate the performance of the avoidance control strategy.

I. INTRODUCTION

As autonomous vehicles immerse into urban areas [1] and gain ground in military and scientific missions [2], the need for rigorous safety control protocols becomes more evident. Guaranteeing collision avoidance at all times is not only critical for the safety of the vehicle but also for its surroundings. This immediate demand has spurred the proposal and study of diverse avoidance control policies for groups of autonomous vehicles (see [3]–[6] for examples).

The successful implementation of a collision avoidance strategy requires the autonomous vehicle to observe (or sense) the position of nearby obstacles including other vehicles. This is typically achieved by equipping the vehicle with advanced detection sensors (e.g., sonar radars and computer vision systems) or by sharing position information among agents via a common communication network. Unfortunately, these detection mechanisms may limit the frequency at which nearby vehicles are observed. For instance, image processing techniques employed by computer vision systems and other remote sensing technologies may take up to several seconds to update obstacle information depending on the vehicle's on-board computational capabilities [7]. Similarly, communications networks may restrict data transmission rates among vehicles due to bandwidth constraints and limited communication resources [8]. In these scenarios, the update rate of position information may not satisfy

the minimum sampling rates required by their avoidance control strategies. In addition, there are benefits on purposely operating the detection sensors and communication systems in a *sleep* and *wake-up* approach [9]. These include the saving of the vehicle's power source life and the better allocation and use of computational and communication resources. Moreover, this operating idea is also intuitive: It is natural to pay attention to obstacles and other vehicles according to their distance and collision risk.

The use of discrete obstacle observations for collision avoidance has been previously studied in [10]–[14] for pairs of dynamics systems and in [15] for multi-agent systems. The instants of observation can be predetermined (*i.e.*, open-loop) or updated using the last observation (*i.e.*, closed-loop feedback), which is also known as a self-triggered implementation [16]. Regardless of the update approach, the observation rate must be carefully selected: If the sampling frequency is not quick enough, the vehicle might not be able to prevent a collision.

In this paper, we propose a cooperative, decentralized avoidance control strategy for a group of autonomous vehicles with discrete obstacle observations based on the concepts of avoidance control [17], [18] and self-triggered control¹ [16], [21]–[23]. The goal of the control strategy is to guarantee collision avoidance at all times while maximizing the time intervals between observations based on the last available position sample. The observation intervals are updated in a closed-loop fashion, increase monotonically as the distance between two vehicles increases, and are shown to be positively lower bounded. In addition, the observation instants can be asynchronous among vehicles and the avoidance control law remains inactive whenever other vehicles are safely away as dictated by a bounded sensing region. Simulations results are presented to illustrate the performance of the proposed self-triggered avoidance control.

II. PRELIMINARIES

A. Notation

As standard notation, we denote the p -norm of a vector $\mathbf{x} = [x_1, \dots, x_n]^T \in \mathbb{R}^n$ as $\|\mathbf{x}\|_p := (|x_1|^p + \dots + |x_n|^p)^{1/p}$ for $1 \leq p < \infty$ and $\|\mathbf{x}\|_p = \max_i |x_i|$ for $p = \infty$, where $|x_i|$ is the absolute value of a real scalar x_i . We define the induced p -norm of a matrix $A \in \mathbb{R}^{n_1 \times n_2}$ as $\|A\|_p := \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|A\mathbf{x}\|_p}{\|\mathbf{x}\|_p}$. For the 2-norm of a vector or matrix we use the simpler

This research was supported in part by the Office of Naval Research Grant N001614WX30023.

E. J. Rodríguez-Seda is with the Department of Weapons and Systems Engineering, United States Naval Academy, Annapolis, MD 21402, USA. rodrigue@usna.edu

¹To the knowledge of the author, event- and self-triggered control ideas [19], [20] have not been applied in the design of collision avoidance strategies for vehicles with dynamic models.

notation $\|\cdot\|$. As a shorthand for a matrix $A \in \mathbb{R}^{n_1 \times n_2}$ we use $[a_{lm}]_{n_1 \times n_2}$, where a_{lm} is the lm th entry of A . For simplicity, we will omit time dependence of signals except when considered necessary as in the case of zero-order-hold sample signals.

B. Potential and Avoidance Functions

Let \mathbf{v} and \mathbf{w} be two vectors in \mathbb{R}^n . We define the following *attractive potential function* $T : \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, \infty)$ as

$$T(\mathbf{v}, \mathbf{w}) = \frac{1}{2} \|\mathbf{v} - \mathbf{w}\|^2. \quad (1)$$

Similarly, let x be a positive real number. We define the following *repulsive potential function* (also called *avoidance function*) $A : \mathbb{R}^n \times \mathbb{R}^n \times (r, \infty) \rightarrow [0, \infty)$ as

$$A(\mathbf{v}, \mathbf{w}, x) = \left(\min \left\{ 0, \frac{\|\mathbf{v} - \mathbf{w}\|^2 - x^2}{\|\mathbf{v} - \mathbf{w}\|^2 - r^2} \right\} \right)^2 \quad (2)$$

where $r > 0$ is called the avoidance radius. The reader can easily verify that A is almost everywhere continuously differentiable and that

$$\begin{aligned} \frac{\partial A(\mathbf{v}, \mathbf{w}, x)^T}{\partial \mathbf{v}} &= - \frac{\partial A(\mathbf{v}, \mathbf{w}, x)^T}{\partial \mathbf{w}} \\ &= \begin{cases} \mathbf{0} & \text{if } \|\mathbf{v} - \mathbf{w}\| \geq x \\ \frac{4(x^2 - r^2)(\|\mathbf{v} - \mathbf{w}\|^2 - x^2)(\mathbf{v} - \mathbf{w})}{(\|\mathbf{v} - \mathbf{w}\|^2 - r^2)^3} & \text{if } r < \|\mathbf{v} - \mathbf{w}\| < x \\ \text{undefined} & \text{if } \|\mathbf{v} - \mathbf{w}\| = r \\ \mathbf{0} & \text{otherwise} \end{cases}. \end{aligned} \quad (3)$$

Furthermore, (3) has the property of being locally Lipschitz continuous in \mathbf{w} on a subset of $\mathbb{R}^n \times \mathbb{R}^n \times (r, \infty)$.

Lemma 2.1: Let $\mathcal{W} = \{\mathbf{w} \in \mathbb{R}^n, \|\mathbf{v} - \mathbf{w}\| \geq c\}$, where $c > r$. Then, $\partial A(\mathbf{v}, \mathbf{w}, x)/\partial \mathbf{v}$ is locally Lipschitz in \mathbf{w} on $\mathbb{R}^n \times \mathcal{W} \times (r, \infty)$ with Lipschitz constant given by

$$L(c, x) = \frac{4(x^2 - r^2)(3x^2 - r^2)^2 \left(1 + \frac{\sqrt{n-1}}{2}\right)}{(c^2 - r^2)^4} < \infty \quad (4)$$

Proof: First note that $\partial A(\mathbf{v}, \mathbf{w}, x)/\partial \mathbf{v}$ and $\partial^2 A(\mathbf{v}, \mathbf{w}, x)/\partial \mathbf{v} \partial \mathbf{w}$ are continuous on $\mathbb{R}^n \times \mathcal{W} \times (r, \infty)$. Therefore, $\partial A(\mathbf{v}, \mathbf{w}, x)/\partial \mathbf{v}$ is locally Lipschitz in \mathbf{w} on $\mathbb{R}^n \times \mathcal{W} \times (r, \infty)$ [24]. To prove (4), it is sufficient to compute an upper bound on $\|\partial^2 A(\mathbf{v}, \mathbf{w}, x)/\partial \mathbf{v} \partial \mathbf{w}\|$ on $\mathbb{R}^n \times \mathcal{W} \times (r, \infty)$. Accordingly, let us define $[a_{lm}]_{n \times n} = \partial^2 A(\mathbf{v}, \mathbf{w}, x)/\partial \mathbf{v} \partial \mathbf{w}$ and let a_{lm} be the lm th entry of $[a_{lm}]_{n \times n}$. Note that $a_{lm} = 0 \forall \mathbf{v}, \mathbf{w}, \|\mathbf{v} - \mathbf{w}\| > x$ and

$$a_{lm} = \frac{-8(x^2 - r^2)(v_l - w_l)(v_m - w_m)(3x^2 - r^2 - 2\|\mathbf{v} - \mathbf{w}\|^2)}{(\|\mathbf{v} - \mathbf{w}\|^2 - r^2)^4}$$

if $l \neq m$ and

$$\begin{aligned} a_{mm} &= \frac{-8(x^2 - r^2)(v_m - w_m)^2(3x^2 - r^2 - 2\|\mathbf{v} - \mathbf{w}\|^2)}{(\|\mathbf{v} - \mathbf{w}\|^2 - r^2)^4} \\ &\quad - \frac{4(x^2 - r^2)(\|\mathbf{v} - \mathbf{w}\|^2 - x^2)(\|\mathbf{v} - \mathbf{w}\|^2 - r^2)}{(\|\mathbf{v} - \mathbf{w}\|^2 - r^2)^4} \end{aligned}$$

if $l = m$, $\forall \mathbf{v}, \mathbf{w}, \|\mathbf{v} - \mathbf{w}\| \in [c, x)$. Now, since $[a_{lm}]_{n \times n}$ is symmetric, we also have that $\|[a_{lm}]_{n \times n}\|_1 = \|[a_{lm}]_{n \times n}\|_\infty$ and, therefore,

$$\begin{aligned} \|[a_{lm}]_{n \times n}\| &\leq \sqrt{\|[a_{lm}]_{n \times n}\|_1 \|[a_{lm}]_{n \times n}\|_\infty} \\ &= \|[a_{lm}]_{n \times n}\|_1 = \max_m \sum_{l=1}^n |a_{lm}| \\ &= |a_{mm}| + \sum_{l=1, l \neq m}^n |a_{lm}| \end{aligned}$$

After some manipulation, we can show that the numerator of a_{lm} is maximized when $\|\mathbf{v} - \mathbf{w}\| = \frac{1}{2}\sqrt{3x^2 - r^2}$, whereas the denominator is minimized when $\|\mathbf{v} - \mathbf{w}\| = c$. Therefore, we can obtain the following bound

$$\begin{aligned} \|[a_{lm}]_{n \times n}\| &\leq \frac{4(x^2 - r^2)(3x^2 - r^2)^2 \left(1 + \frac{\sqrt{n-1}}{2}\right)}{(c^2 - r^2)^4} \\ &= L(c, x) \end{aligned}$$

where we also used the fact that $\sum_{l=1, l \neq m}^n (v_l - w_l)(v_l - w_l) \leq \frac{1}{2}\sqrt{n-1}\|\mathbf{v} - \mathbf{w}\|^2$. Since $\|[a_{lm}]_{n \times n}\| \leq L(c, x) < \infty$, we can conclude that $\partial A(\mathbf{v}, \mathbf{w}, x)/\partial \mathbf{v}$ is locally Lipschitz in \mathbf{w} on $\mathbb{R}^n \times \mathcal{W} \times (r, \infty)$ with Lipschitz constant $L(c, x)$. ■

III. PROBLEM FORMULATION

In this paper, we consider the task of controlling a group of N n -degree-of-freedom (DOF) vehicles with double-integrator dynamics given by

$$\ddot{\mathbf{q}}_i(t) = \mathbf{u}_i(t), \quad \text{for } i \in \mathcal{N} = \{1, \dots, N\} \quad (5)$$

where $\mathbf{q}_i \in \mathbb{R}^n$ and $\mathbf{u}_i \in \mathbb{R}^n$ represent, respectively, the position and control input for the i th vehicle and \mathcal{N} is the set of N vehicles. We would like to drive each vehicle in \mathcal{N} to its desired position $\mathbf{q}_i^d \in \mathbb{R}^n$ while keeping a distance greater than $r > 0$ (namely the avoidance radius) among all vehicles at all times, i.e., $\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| > r \forall t \geq 0, i, j \in \mathcal{N}, i \neq j$. To this end, we assume that each vehicle can discretely observe, either by the use of on-board sensors or a communication network, the position of other vehicles that are located within a bounded sensing range $R > r$. The observation of nearby vehicles by the i th agent is assumed to take place at discrete time-varying sampling intervals $\tau_{ij,k} > 0$ to be determined, where k is the cardinal sequence of observations. We define the observed position of the j th vehicle by the i th vehicle as

$$\mathbf{q}_j^i(t) = \mathbf{q}_j(t_{ij,k}), \quad \forall t \in [t_{ij,k}, t_{ij,k+1}) \quad (6)$$

where $t_{ij,k+1} = t_{ij,k} + \tau_{ij,k}$. For simplicity, we let $t_{ij,0} = 0$. Then, we can summarize our control objective as follows: Given R and r , design a decentralized control law \mathbf{u}_i and an update rule for $\tau_{ij,k}$ such that the sampling intervals (also known as inter-execution times) are as large as possible while still guaranteeing collision avoidance at all times.

In general, the observation of other vehicles' position does not need to be synchronous, i.e., $t_{ij,k}$ does not have to be

equal to $t_{ji,k}$ or $t_{il,k}$ for $i \neq j, j \neq l$. Without loss of generality, we assume that the avoidance and detection radii, i.e., r and R , are the same for all vehicles.

IV. SELF-TRIGGERED AVOIDANCE RESPONSE

To achieve our control objective of collision avoidance and set-point tracking, we propose the following decentralized control law

$$\begin{aligned} \mathbf{u}_i = & -k_p \frac{\partial \mathcal{T}^T(\mathbf{q}_i, \mathbf{q}_i^d)}{\partial \mathbf{q}_i} - \sum_{j \in \mathcal{N}_i} \frac{\partial \mathcal{A}^T(\mathbf{q}_i, \mathbf{q}_j^i, \bar{R})}{\partial \mathbf{q}_i} \\ & - \frac{1}{\nu} \left\| k_p \frac{\partial \mathcal{T}^T(\mathbf{q}_i, \mathbf{q}_i^d)}{\partial \mathbf{q}_i} + \sum_{j \in \mathcal{N}_i} \frac{\partial \mathcal{A}^T(\mathbf{q}_i, \mathbf{q}_j^i, \bar{R})}{\partial \mathbf{q}_i} \right\| \dot{\mathbf{q}}_i \\ & - k_v \dot{\mathbf{q}}_i + \text{sgn}(\dot{\mathbf{q}}_i) \sum_{j \in \mathcal{N}_i} \gamma_{ij,k} \end{aligned} \quad (7)$$

where $\mathcal{T}(\cdot)$ and $\mathcal{A}(\cdot)$ are defined as in Section II-B, ν , k_p , and k_v are positive scalar parameters, $\mathcal{N}_i \subset \mathcal{N}/i$ is the set of vehicles within the i th vehicle's sensing range R , and $\text{sgn}(\cdot)$ is the sign function computed as

$$\text{sgn}(\dot{\mathbf{q}}_i) = \begin{cases} \frac{\dot{\mathbf{q}}_i}{\|\dot{\mathbf{q}}_i\|} & \text{if } \dot{\mathbf{q}}_i \neq \mathbf{0} \\ \mathbf{0} & \text{otherwise} \end{cases}.$$

The new range variable \bar{R} is defined as $\bar{R} = R - \tau_{\max}\nu$, where τ_{\max} represents the maximum admissible observation interval. For simplicity, we will define it as

$$\tau_{\max} = \frac{R - r}{2\nu}.$$

The sets of gains $\gamma_{ij,k}$ are binary functions equal to some design parameter $\gamma > 0$ if $\mathcal{A}(\mathbf{q}_i, \mathbf{q}_j^i, R) > 0$ and 0 otherwise. Overall, the proposed strategy (7) is a piece-wise continuous control law comprised of five terms. The first term to the right of the equality is used for set-point regulation, the second term guarantees collision avoidance, while the third, fourth, and fifth terms regulate the velocity of the i th vehicle and provide robustness to the discrete sampling process.

Now that we have defined a cooperative control law for the multi-vehicle system, we will proceed to establish a self-triggering rule for the observation rate. Ideally, we would like to maximize the inter-execution times to reduce the use of the vehicles' detection resources, while still guaranteeing collision avoidance. To this end, let us define

$$\mathcal{S}_{ij,k} = \left\{ \tau : \tau \in \left(0, \frac{d_{ij,k} - r}{2\nu} \right], 0 < \tau L(d_{ij,k} - \tau\nu, \bar{R}) \leq \frac{\gamma}{\nu} \right\} \quad (8)$$

as the set of admissible inter-execution times, where $d_{ij,k} = \|\mathbf{q}_i(t_{ij,k}) - \mathbf{q}_j(t_{ij,k})\|$ is the Euclidean distance between the two vehicles at $t_{ij,k}$. Then, we propose to compute the observation intervals as

$$\tau_{ij,k} = \begin{cases} \sup_{\tau \in \mathcal{S}_{ij,k}} \tau & \text{if } \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j^i, R) > 0 \\ \tau_{\max} & \text{otherwise.} \end{cases} \quad (9)$$

Note that $\tau_{ij,k}$ is upper bounded by τ_{\max} and monotonically decreases as the distance between the two vehicles

decreases (i.e., the observation rate of other vehicles is a function of their distance). Furthermore, in Section V we will show that a positive solution to (9) exists as long as $\|\mathbf{q}_i(t_{ij,k}) - \mathbf{q}_j(t_{ij,k})\| > r \forall k \geq 0$.

Remark 4.1: The proposed avoidance strategy requires each vehicle to evaluate an avoidance function per neighbor. It can be shown that the size of the set of neighbors, \mathcal{N}_i , is always bounded by a finite number \bar{N} [25] and, therefore, the number of computations required for the execution of (7) is also bounded. For instance, if $n = 2$ we have that

$$\bar{N} \leq \min \left\{ N - 1, \left\lceil \frac{\pi}{\sin^{-1}\left(\frac{r}{2R-r}\right)} \left(\sqrt{\frac{R-r}{3r^2}} + \frac{1}{2} \right) \right\rceil \right\}.$$

V. STABILITY AND COLLISION AVOIDANCE ANALYSIS

Herein we analyze the effectiveness of (7) in guaranteeing the safe convergence of the vehicles to their desired position despite the use of time-varying discrete observations. To this end, let us first introduce the following results.

Lemma 5.1 (Velocity Regulation): Consider a vehicle with dynamics and control law given by (5) and (7). Assume that $\|\dot{\mathbf{q}}_i(0)\| \leq \nu$. Then, $\|\dot{\mathbf{q}}_i(t)\| < \nu \forall t > 0$.

Proof: Assume the following Lyapunov-candidate function $\mathcal{V}_i(\dot{\mathbf{q}}_i) = \frac{1}{2} \dot{\mathbf{q}}_i^T \dot{\mathbf{q}}_i$. Taking its time-derivative yields

$$\begin{aligned} \dot{\mathcal{V}}_i = & -\dot{\mathbf{q}}_i^T \left(k_p \frac{\partial \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d)}{\partial \mathbf{q}_i} + \sum_{j \in \mathcal{N}_i} \frac{\partial \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j^i, \bar{R})}{\partial \mathbf{q}_i} \right) \\ & - \left\| k_p \frac{\partial \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d)}{\partial \mathbf{q}_i} + \sum_{j \in \mathcal{N}_i} \frac{\partial \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j^i, \bar{R})}{\partial \mathbf{q}_i} \right\| \frac{\|\dot{\mathbf{q}}_i\|^2}{\nu} \\ & - k_v \|\dot{\mathbf{q}}_i\|^2 - \|\dot{\mathbf{q}}_i\| \sum_{j \in \mathcal{N}_i} \gamma_{ij,k} \end{aligned} \quad (10)$$

$$\begin{aligned} \leq & \left\| k_p \frac{\partial \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d)}{\partial \mathbf{q}_i} + \sum_{j \in \mathcal{N}_i} \frac{\partial \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j^i, \bar{R})}{\partial \mathbf{q}_i} \right\| \times \\ & \left(\|\dot{\mathbf{q}}_i\| - \frac{\|\dot{\mathbf{q}}_i\|^2}{\nu} \right) - k_v \|\dot{\mathbf{q}}_i\|^2 \end{aligned} \quad (11)$$

which is strictly negative $\forall \|\dot{\mathbf{q}}_i\| \geq \nu - \epsilon$, where ϵ is an arbitrarily small positive constant. Therefore, we can conclude that $\|\dot{\mathbf{q}}_i(t)\| \leq \nu - \epsilon < \nu \forall t > 0$. ■

Lemma 5.2: Let the inter-execution intervals be given by (9) and assume that $\|\dot{\mathbf{q}}_i(0)\| \leq \nu \forall i \in \mathcal{N}$. If $d_{ij,k} > r$ for some $k \geq 0$, then

- i) $\exists \tau_{ij,k} > 0$ that satisfies (9) and
- ii) $d_{ij,k+1} > r$.

Proof: First, assume that $d_{ij,k} \geq R$. Then, $\tau_{ij,k} = \tau_{\max} > 0$ and

$$\begin{aligned} d_{ij,k+1} & \geq d_{ij,k} - \int_{t_{ij,k}}^{t_{ij,k+1}} \|\dot{\mathbf{q}}_i(\theta) - \dot{\mathbf{q}}_j(\theta)\| d\theta \\ & > R + 2\nu\tau_{\max} > r \end{aligned}$$

where we applied Lemma 5.1. Now, let us assume that $d_{ij,k} = r_o \in (r, R)$. We want to show that $\mathcal{S}_{ij,k} \neq \emptyset$.

Therefore, it suffices to prove that $\exists \tau \in (0, \frac{r_o - r}{2\nu}]$ such that $\tau L(d_{ij,k} - \tau\nu, \bar{R})\nu \leq \gamma$. Accordingly, define $\tau_1 \in (0, \frac{r_o - r}{2\nu}]$. Then, we have that

$$L(r_o - \tau\nu, \bar{R})|_{\tau \leq \tau_1} \leq \frac{4(3\bar{R}^2 - r^2)^2 \left(1 + \frac{\sqrt{n-1}}{2}\right)}{(\bar{R}^2 - r^2)^{-1} ((r_o - \tau_1\nu)^2 - r^2)^4} = M_\varepsilon < \infty$$

since $r_o - \tau_1\nu \geq r_o - \frac{r_o - r}{2} = \frac{r_o + r}{2} > r$. Now, choosing $\tau_{ij,k} = \tau = \frac{\gamma}{M_\varepsilon\nu} > 0$ we obtain that $\tau L(d_{ij,k} - \tau\nu, \bar{R})\nu \leq \frac{\gamma}{M_\varepsilon\nu} M_\varepsilon\nu = \gamma$ which proves the first statement. For the second statement, we have that $d_{ij,k+1} > d_{ij,k} - 2\tau_1\nu > r_o - (r_o - r) > r$, and the proof is complete. ■

Lemmas 5.1 and 5.2 provide sufficient conditions to guarantee 1) that the velocities of the vehicles remain bounded for all time and 2) that the vehicles do not collide within one observation interval and that there is a positive solution to (9) as long as the distance between the two vehicles is greater than r . Note that Lemma 5.2 is not used to establish collision avoidance for all times, as it does not show if $\sum_{k=0}^{\infty} \tau_{ij,k}$ diverges to infinity. Instead, collision avoidance for all times will be established via the next statement.

Theorem 5.1 (Collision Avoidance): Consider the multi-vehicle system in (5) with control law and self-triggering rule given by (7) and (9), respectively. Assume that $\|\dot{\mathbf{q}}_i(0)\| \leq \nu$ and $\|\mathbf{q}_i(0) - \mathbf{q}_j(0)\| > r \ \forall i, j \in \mathcal{N}, i \neq j$. Then, $\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| > r \ \forall t \geq 0$ and $\forall i, j \in \mathcal{N}, i \neq j$.

Proof: Assume that $\|\dot{\mathbf{q}}_i(0)\| \leq \nu$ and $\|\mathbf{q}_i(0) - \mathbf{q}_j(0)\| > r \ \forall i, j \in \mathcal{N}, i \neq j$ and consider the following Lyapunov-candidate function

$$\mathcal{V}_C = \sum_{i \in \mathcal{N}} \left(k_p \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d) + \mathcal{V}_i(\dot{\mathbf{q}}_i) + \frac{1}{2} \sum_{j \in \mathcal{N}_i} \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j, \bar{R}) \right) \quad (12)$$

which is positive definite. Taking its time derivative yields

$$\dot{\mathcal{V}}_C = \sum_{i \in \mathcal{N}} \left(k_p \frac{\partial \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d)}{\partial \mathbf{q}_i} \dot{\mathbf{q}}_i + \dot{\mathcal{V}}_i(\dot{\mathbf{q}}_i) \right) + \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} \left(\frac{\partial \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j, \bar{R})}{\partial \mathbf{q}_i} \dot{\mathbf{q}}_i + \frac{\partial \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j, \bar{R})}{\partial \mathbf{q}_j} \dot{\mathbf{q}}_j \right).$$

Then, using (3) and (10) we obtain that

$$\begin{aligned} \dot{\mathcal{V}}_C &= \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} \left(\frac{\partial \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j, \bar{R})}{\partial \mathbf{q}_i} - \frac{\partial \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j^i, \bar{R})}{\partial \mathbf{q}_i} \right) \dot{\mathbf{q}}_i \\ &\quad - \sum_{i \in \mathcal{N}} \left(k_v \|\dot{\mathbf{q}}_i\|^2 + \|\dot{\mathbf{q}}_i\| \sum_{j \in \mathcal{N}_i} \gamma_{ij,k} \right) \\ &\quad - \left\| k_p \frac{\partial \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d)}{\partial \mathbf{q}_i} + \sum_{j \in \mathcal{N}_i} \frac{\partial \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j^i, \bar{R})}{\partial \mathbf{q}_i} \right\| \frac{\|\dot{\mathbf{q}}_i\|^2}{\nu} \\ &\leq \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} \left\| \frac{\partial \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j, \bar{R})}{\partial \mathbf{q}_i} - \frac{\partial \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j^i, \bar{R})}{\partial \mathbf{q}_i} \right\| \|\dot{\mathbf{q}}_i\| \\ &\quad - \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} \|\dot{\mathbf{q}}_i\| \gamma_{ij,k}. \end{aligned} \quad (13)$$

Applying Lemmas 2.1, 5.1, and 5.2, we can upper bound (13) by

$$\dot{\mathcal{V}}_C \leq \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}_i} (\Lambda_{ij,k} \tau_{ij,k} \nu - \gamma_{ij,k}) \|\dot{\mathbf{q}}_i\|.$$

where $\Lambda_{ij,k} = L(d_{ij,k} - \tau_{ij,k}\nu, \bar{R})$ if $d_{ij,k} < R$ and $\Lambda_{ij,k} = 0$ otherwise. For the latter case, we simply have that $\mathcal{V}_C \leq 0$. For the first case, we have that if $d_{ij,k} < R$ for some i, j , then, $\gamma_{ij,k} = \gamma$ and $\tau_{ij,k} \in (0, \frac{\gamma}{L(d_{ij,k} - \tau_{ij,k}\nu, \bar{R})\nu}]$, which also implies that $\dot{\mathcal{V}}_C \leq 0$. Since $\dot{\mathcal{V}}_C \leq 0$, we have that $\mathcal{V}_C(t) \leq \mathcal{V}_C(0) < \infty \ \forall t \geq 0$. Now suppose that for some pair $i, j, i \neq j$, we have that $\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| \rightarrow r$. This would imply that $\mathcal{V}_C(t) \geq \mathcal{A}(\mathbf{q}_i, \mathbf{q}_j, \bar{R}) \rightarrow \infty$, which is a contradiction. Consequently, we can conclude that $\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| \not\rightarrow r$ and that $\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| > r \ \forall t \geq 0$ and $\forall i, j$. ■

The above theorem establishes the main result of this paper by providing sufficient conditions that guarantee collision avoidance at all times when implementing the proposed control strategy. For completeness, the next two results provide a bound on the observation intervals and showcase the effectiveness of the control law (7) in guaranteeing position convergence once the conflicts among agents have been resolved.

Corollary 5.1 (Lower Bound of Observation Intervals):

Assume that Theorem 5.1 holds. Then, $\exists \tau_0 > 0$ such that $\tau_{ij,k} \geq \tau_0 \ \forall i \in \mathcal{N}, j \in \mathcal{N}_i, k \geq 0$.

Proof: Assume that Theorem 5.1 holds and consider (12). Since $\dot{\mathcal{V}}_C \leq 0$, we have that $\mathcal{V}_C(t) \leq \mathcal{V}_C(0) = M_C^2 < \infty \ \forall t \geq 0$, where $M_C > 0$ is a finite constant. This implies that $\mathcal{A}(\mathbf{q}_i, \mathbf{q}_j, \bar{R}) \leq M_C^2 \Rightarrow \|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| \geq \sqrt{\frac{\bar{R}^2 - r^2}{M_C^2 + 1} + r^2} = r_0 > r \ \forall t$. Since $\tau_{ij,k}$ is a monotonically decreasing function of $d_{ij,k}$ (which is lower bounded by r_0), we have that $\tau_{ij,k}$ is also lower bounded by τ_0 for all k , where $\tau_0 > 0$ is the solution to (9) when $d_{ij,k} = r_0$. Finding an analytical solution to (9) can be tedious. Instead, we will compute a more conservative yet simpler lower bound. Let

$$\tau' = \min \left\{ \frac{r_0 - r}{2\nu}, \frac{\gamma(r_0 - r)^8}{c_0\nu} \right\} > 0$$

where $c_0 = 4(\bar{R}^2 - r^2)(3\bar{R}^2 - r^2)^2 \left(1 + \frac{\sqrt{n-1}}{2}\right)$. Note that if a solution $\tau^* \in (0, \tau']$ satisfies $\tau' L(r_0 - \tau^*\nu, \bar{R}) = \frac{\gamma}{\nu}$, then it will also satisfy (9). Therefore, solving $L(r_0 - \tau^*\nu, \bar{R}) = \frac{\gamma}{\tau^*\nu}$ for τ^* yields

$$\tau^* = \frac{1}{\nu} \left(r_0 - \sqrt[4]{c_0 \tau' \nu / \gamma + r^2} \right) > 0$$

from which we conclude that $\tau_{ij,k} \geq \tau_0 = \min\{\tau', \tau^*\} > 0 \ \forall i \in \mathcal{N}, j \in \mathcal{N}_i, k \geq 0$. ■

Remark 5.1: Corollary 5.1 states the existence of a positive lower bound on $\tau_{ij,k}$ that is satisfied $\forall k$. This is of special relevance as it implies that the proposed self-triggering rule does not exhibit the infamous Zeno behavior and, therefore, it can be successfully implemented via the use of digital computers. Note, however, that Corollary 5.1 cannot provide a universal lower bound as the value depends on the system's initial conditions (i.e., M_C).

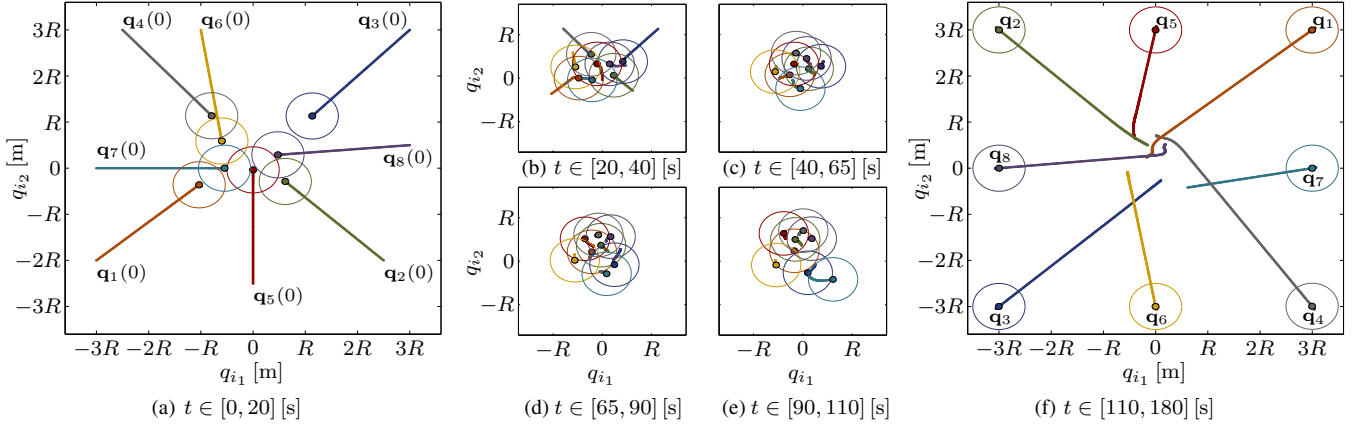


Fig. 1. Sequential motion of the multi-vehicle system for $t \in [0, 180]$ [s]. The small color-filled circles and the large open circles traced at the end of each simulation interval delimit areas of diameters r and R , respectively. Therefore, an overlapping of two small circles would imply a collision. Similarly, an overlapping of two large open circles implies that the two vehicles are within each other's sensing range.

Theorem 5.2 (Set-Point Tracking): Consider a vehicle with dynamics and control law given by (5) and (7). Assume that $\exists t_0 \geq 0$ such that $\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| \geq R \forall t \geq t_0, j \in \mathcal{N}_i$. Then, $\mathbf{q}_i(t) \rightarrow \mathbf{q}_i^d$ as $t \rightarrow \infty$.

Proof: Assume that $\exists t_0 \geq 0$ such that $\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| \geq R \forall t \geq t_0, j \in \mathcal{N}_i$. Then, $\mathcal{A}(\mathbf{q}_i, \mathbf{q}_j^d, \bar{R}) \equiv \gamma_{ij,k} \equiv 0 \forall j \in \mathcal{N}_i$.

Now, consider $\mathcal{V}_T = k_p \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d) + \mathcal{V}_i(\dot{\mathbf{q}}_i)$ as a Lyapunov function. Taking the time-derivative of \mathcal{V}_T we obtain that

$$\begin{aligned} \dot{\mathcal{V}}_T &= k_p \frac{\partial \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d)}{\partial \mathbf{q}_i} \dot{\mathbf{q}}_i - k_p \frac{\partial \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d)}{\partial \mathbf{q}_i} \dot{\mathbf{q}}_i \\ &\quad - \left(k_v + \frac{1}{\nu} \left\| k_p \frac{\partial \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d)}{\partial \mathbf{q}_i} \right\| \right) \dot{\mathbf{q}}_i^T \dot{\mathbf{q}}_i \leq -k_v \|\dot{\mathbf{q}}_i\|^2. \end{aligned}$$

Integrating the above inequality yields that $\lim_{t \rightarrow \infty} \int_{t_0}^t \|\dot{\mathbf{q}}_i(\theta)\|^2 d\theta \leq \mathcal{V}_T(t_0) < \infty$. Then, applying Barbalat's Lemma [24] and noting that $\ddot{\mathbf{q}}_i(t)$ is uniformly continuous for $t \geq t_0$, we can conclude that $\dot{\mathbf{q}}_i \rightarrow \mathbf{0}$. Likewise, by taking the time derivative of (5) and (7) we obtain that $\ddot{\mathbf{q}}_i \in \mathcal{L}_\infty$. Then, since $\lim_{t \rightarrow \infty} \int_{t_0}^t \ddot{\mathbf{q}}_i(\theta) d\theta \rightarrow -\dot{\mathbf{q}}_i(t_0) < \infty$, we can apply again Barbalat's Lemma and conclude that $\ddot{\mathbf{q}}_i(t) \rightarrow \mathbf{0}$. The last result along with the fact that $\mathcal{A}(\mathbf{q}_i, \mathbf{q}_j^d, \bar{R}) \equiv 0$ implies that $\partial \mathcal{T}(\mathbf{q}_i, \mathbf{q}_i^d) / \partial \mathbf{q}_i \rightarrow \mathbf{0} \Rightarrow \mathbf{q}_i \rightarrow \mathbf{q}_i^d$ as $t \rightarrow \infty$. ■

VI. SIMULATION EXAMPLE

The results discussed in the previous sections are now illustrated via a numerical example with a team of eight 2-DOF vehicles with dynamics described by (5). We assume that the vehicles have avoidance and detection radii of $r = 2$ [m] and $R = 15$ [m], respectively, and that all vehicles implement the proposed control law (7) and self-triggering rule (9) with control parameters $\nu = 2.0$ [s/m], $k_p = 0.8$ [1/s²], $k_v = 0.5$ [1/s], and $\gamma = 5.5$. The initial and desired configurations are chosen as $\mathbf{q}_1(0) = [-3R, -2R]^T$, $\mathbf{q}_2(0) = [2.5R, -2R]^T$, $\mathbf{q}_3(0) = [3R, 3R]^T$, $\mathbf{q}_4(0) = [-2.5R, 3R]^T$, $\mathbf{q}_5(0) = [0, -2.5R]^T$, $\mathbf{q}_6(0) = [-R, 3R]^T$, $\mathbf{q}_7(0) = [-3R, 0]^T$, $\mathbf{q}_8(0) = [3R, 0.5R]^T$, $\mathbf{q}_1^d = -\mathbf{q}_3^d = [3R, 3R]^T$, $\mathbf{q}_2^d = -\mathbf{q}_4^d = [-3R, 3R]^T$,

$\mathbf{q}_5^d = -\mathbf{q}_6^d = [0, 3R]^T$, and $\mathbf{q}_7^d = -\mathbf{q}_8^d = [3R, 0]^T$. This example simulates an scenario in which all vehicles start opposite to their desired configurations and found each other in risk of a collision when they approximate the center of their workspace.

Fig. 1 illustrates the evolution of the system starting from rest. Note from Fig. 1(a) that the vehicles initially assume linear paths towards their desired configurations until they start sensing the presence of other vehicles at approximately $t = 20$ [s]. At that instant, all vehicles come to a near stop and start to diverge from their linear motion in order to avoid collisions with nearby agents (see Fig. 1(b)-(e)). Once the conflicts have been resolved, the vehicles are able to resume their paths towards their desired trajectories as seen in Fig. 1(f). Fig. 2 plots the minimum distance among all pairs of vehicles as a function of time. Note that the distance between any pair of vehicles is always greater than 3.54 [m] which implies the absence of collisions.

Finally, Fig. 3 illustrates the sequence of observation intervals for two pairs of vehicles. As expected from Corollary 5.1, the observation intervals are positively bounded in both cases, with a minimum of 0.302 [s] for $\tau_{13,k}$ and a minimum of 0.438 [s] for $\tau_{78,k}$.

VII. CONCLUSIONS AND FUTURE WORK

In this paper, we have proposed and studied the use of a decentralized, cooperative, self-triggered collision avoidance control law for an arbitrarily large group of vehicles with double-integrator dynamics. We assumed that each vehicle updates its local information and control law continuously but receives or observes the position of nearby vehicles at discrete intervals. Using Lyapunov-based analysis we then proposed a cooperative collision avoidance strategy that increases the vehicles observation intervals while guaranteeing collision avoidance at all times. Being able to maximize the intervals between observations is specially appealing for cyber-physical and networked control systems, where different subsystems have to share a limited amount of computational and communication resources. In addition, the

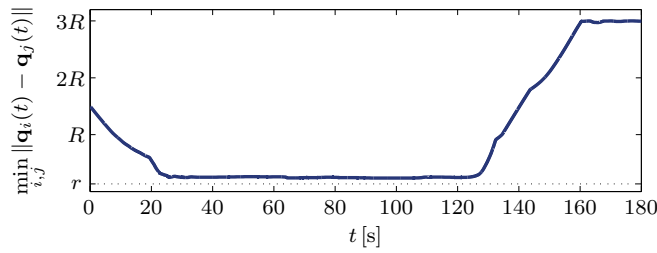


Fig. 2. Minimum distance between any pair of vehicles.

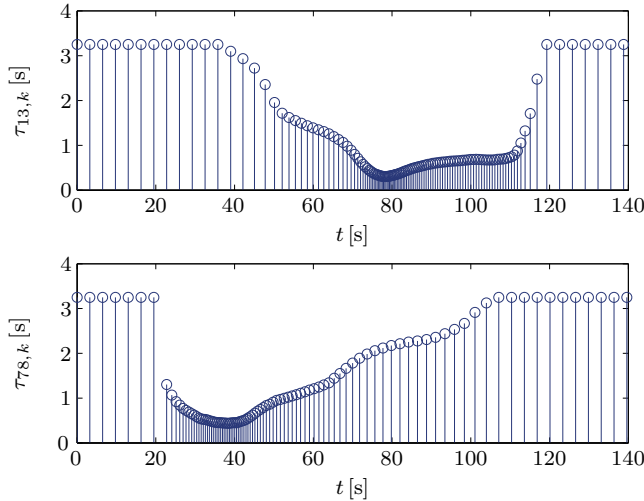


Fig. 3. Sequence of observation intervals, $\tau_{ij,k}$, between the 1st and 3rd vehicle (top) and the 7th and 8th vehicle (bottom).

proposed avoidance control strategy is decentralized in the sense that only position of nearby vehicles are required and remains inactive whenever all others vehicles are safely away. The latter implies that the vehicle can satisfactorily perform other required tasks without the interference of the avoidance control.

Future research directions include the imposition of a known, positive lower bound to the observation intervals. Although we showed that the observation intervals are positively bounded and, therefore, the infamous Zeno effect is avoided; we cannot provide an explicit positive lower bound that could comply with potential predefined constraints imposed by the vehicle's computational and communication resources. In addition, we would like to conduct experiments and test the self-triggered avoidance control in a real multi-vehicle system.

REFERENCES

- [1] G. Cai, J. Dias, , and L. Seneviratne, "A survey of small-scale unmanned aerial vehicles: Recent advances and future development trends," *Un. Syst.*, vol. 2, no. 2, pp. 1–25, Apr. 2014.
- [2] John A. Volpe National Transportation Systems Center, "Unmanned aircraft system (UAS) service demand 2015 - 2035: Literature review and projections of future usage," U.S. Department of Transportation, Tech. Rep. OMB No. 0704-0188, Aug. 2013.

- [3] J. K. Kuchar and L. C. Yang, "A review of conflict detection and resolution modeling methods," *IEEE Trans. Intell. Transp. Syst.*, vol. 1, no. 4, pp. 179–189, 2000.
- [4] D. M. Stipanović, "A survey and some new results in avoidance control," in *Proc. Int. Workshop Dyn. Control*, Tossa de Mar, Spain, 2009, pp. 166–173.
- [5] E. Lalish and K. A. Morgansen, "Distributed reactive collision avoidance," *Autonomous Robots*, vol. 32, no. 3, pp. 207–226, 2012.
- [6] E. J. Rodríguez, C. Tang, , M. W. Spong, and D. M. Stipanović, "Trajectory tracking with collision avoidance for nonholonomic vehicles with acceleration constraints and limited sensing," *Int. J. Robot. Res.*, vol. 33, no. 12, pp. 1569–1592, Oct. 2014.
- [7] V. Venugopal and S. Kannan, "Accelerating real-time LiDAR data processing using GPUs," in *Proc. IEEE MWSCAS*, Columbus, OH, Aug. 2013, pp. 1168–1171.
- [8] M. C. F. Donkers, P. Tabuada, and W. P. M. H. Heemels, "Minimum attention control for linear systems," *Discrete Event Dyn. Syst.*, vol. 24, no. 2, pp. 199–218, June 2014.
- [9] C. Cassandras and W. Li, "Sensor networks and cooperative control," *Eur. J. Control*, vol. 11, no. 4-5, pp. 436–463, 2005.
- [10] A. A. Melikyan, "On minimal observations in a game of encounter," *J. Appl. Math. Mech.*, vol. 37, no. 3, pp. 407–414, 1973, in Russian.
- [11] F. L. Chernousko and A. A. Melikyan, "Some differential games with incomplete information," in *Optimization Techniques IFIP Technical Conference Novosibirsk*, ser. Lecture Notes in Control and Information Sciences. Springer Berlin - Heidelberg, 1975, pp. 445–450.
- [12] G. J. Olsder and O. Pourtallier, "Optimal selection of observation times in a costly information game," in *New Trends in Dynamic Games and Applications*, ser. Annals of the International Society of Dynamic Games, G. J. Olsder, Ed. Birkhäuser Boston, 1995, pp. 227–246.
- [13] D. Neveu, J. P. Pignon, A. Raimondo, J. M. Nicolas, and O. Pourtallier, "Pursuit games with costly information: Application to the ASW helicopter versus submarine game," in *New Trends in Dynamic Games and Applications*, ser. Annals of the International Society of Dynamic Games, G. J. Olsder, Ed. Birkhäuser Boston, 1995, pp. 247–258.
- [14] E. J. Rodríguez-Seda and D. M. Stipanović, "Guaranteed collision avoidance with discrete observations and limited actuation," in *Advances in Intelligent Vehicles*, ser. Intelligent Systems, Y. Chen and L. Li, Eds. Academic Press, 2013, pp. 89–110.
- [15] D. M. Stipanović, A. Melikyan, and N. Hovakimyan, "Guaranteed strategies for nonlinear multi-player pursuit-evasion games," *Int. Game Theory Rev.*, vol. 12, no. 1, pp. 1–17, 2010.
- [16] M. Velasco, P. Marti, and J. Fuentes, "The self triggered task model for real-time control systems," in *Proc. IEEE Real-Time Syst. Symp.*, 2003, p. 6770.
- [17] G. Leitmann and J. Skowronski, "Avoidance control," *J. Optim. Theory Appl.*, vol. 23, no. 4, pp. 581–591, Dec. 1977.
- [18] D. M. Stipanović, P. F. Hokayem, M. W. Spong, and D. Šiljak, "Cooperative avoidance control for multiagent systems," *J. Dyn. Syst. Meas. Control*, vol. 129, pp. 699–707, Sept. 2007.
- [19] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Trans. Automat. Control*, vol. 57, no. 5, pp. 1291–1297, May 2012.
- [20] C. Nowzari and J. Cortés, "Self-triggered coordination of robotic networks for optimal deployment," *Automatica*, vol. 48, no. 6, pp. 1077–1087, June 2012.
- [21] A. Anta and P. Tabuada, "To sample or not to sample: Self-triggered control for nonlinear systems," *IEEE Trans. Automat. Control*, vol. 55, no. 9, pp. 2030–2042, Sept. 2010.
- [22] M. Mazo, A. Anta, and P. Tabuada, "An iss self-triggered implementation of linear controllers," *Automatica*, vol. 46, no. 8, pp. 1310–1314, Aug. 2010.
- [23] W. P. M. H. Heemels, K. H. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *Proc. IEEE Conf. Decision Control*, Maui, HI, Dec. 2012, pp. 3270–3285.
- [24] H. K. Khalil, *Nonlinear Systems*. New Jersey: Prentice Hall, 2002.
- [25] E. J. Rodríguez-Seda and M. W. Spong, "Guaranteed safe motion of multiple lagrangian systems with limited actuation," in *Proc. IEEE Conf. Decision Control*, Maui, HI, Dec. 2012, pp. 2773–2780.