

# Expansion of MIMO ARX model on Laguerre orthonormal bases

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**Abstract**— In this paper, we propose a new dynamic linear MIMO system representation by using discrete-time MIMO AutoRegressive with eXogenous input (ARX) model. To provide a reduced complexity model, each polynomial function of the MIMO ARX model associated to the inputs and to the outputs is expanded on independent Laguerre orthonormal basis to develop a new black-box linear MIMO ARX-Laguerre model. This reduction is ensured once the poles characterising each Laguerre orthonormal basis are set to their optimal values. Simulation results show the effectiveness of the proposed modeling method.

**Keywords**— MIMO ARX model; MIMO ARX-Laguerre model; Independent Laguerre basis; Pole optimisation

## I. INTRODUCTION

In literature the modeling of multi-input multi-output (MIMO) LTI and stable dynamical systems using orthonormal bases is particularly attractive in parameter number reduction. This approach is developed by many authors [3-5] consists of decomposing the transfer matrix of MIMO LTI stable system on generalized orthonormal bases (GOB). The resulting model, known as MIMO-GOB model, prudes with two main advantages with respect to classical MIMO linear models such as, ARX, ARMAX, .... This first feature is its independence of system delay and sampling interval and the second interests the reduction of parameter number because of the completeness of generalized orthonormal basis in Lebesgue space  $\ell_2[0, \infty[$ . The MIMO-GOB model is handicapped by two essential factors. The first interested in the difficulty of the obtaining of the optima values for the generalized orthonormal bases. The second concerns the restriction of this model to the case of MIMO system decoupled representation. To circumvent these drawbacks, we propose to reduce the parameter complexity of these systems by using the MIMO ARX model projecting its parameter associated with the inputs and the outputs on independent Laguerre orthonormal bases. The resulting model, entitled MIMO-ARX - Laguerre model, could represent coupled and decoupled systems and ensures the parameter number reduction with a recursive and easy representation. Indeed, Bouzrara et al. [1] have proved that the decomposition of a SISO ARX model on Laguerre orthonormal bases ensures the parameter reduction even for a

complex system unlike the classical Laguerre model [1, 2]. This result enables to extend the classical methods of the pole optimization of Laguerre bases and then to obtain a new black-box model with a simple recursive representation and characterizing coupled and decoupled MIMO system. We note that the optimal values of each Laguerre basis are obtained by extending the optimization method proposed by Tanguy et al.

The paper is organized as follow. In section 2, a linear multivariable system is described by using the MIMO ARX model. In section 3 we present the MIMO ARX-Laguerre model by developing the coefficients of the classical MIMO ARX model on independent Laguerre bases. Section 4 is devoted to the simulation results in which we evaluate the performances of the MIMO ARX-Laguerre model compared to those of the classical MIMO ARX one in terms of the parameter complexity reduction and approximation quality. Finally, section 5 presents the conclusions.

## II. THE MIMO ARX MODEL

A multivariable ARX model with  $m$  inputs and  $n$  outputs is given by:

$$A(z^{-1})\underline{y}(k) = B(z^{-1})\underline{u}(k) \quad (1)$$

where  $A(z^{-1})$  and  $B(z^{-1})$  are two matrices of dimension  $m \times m$  and  $m \times n$  respectively whose entries are polynomials in the delay operator  $z^{-1}$ ;  $\underline{y}(k)$  is a  $m$ -dimensional column vector consisting of output signals at time instant  $k$  and  $\underline{u}(k)$  is a  $n$ -dimensional column vector consisting of inputs signals.

The matrix polynomials  $A(z^{-1})$  and  $B(z^{-1})$  are given by:

$$A(z^{-1}) = I + A_1 z^{-1} + \dots + A_a z^{-a} \quad (2)$$

$$B(z^{-1}) = B_1 z^{-1} + \dots + B_b z^{-b} \quad (3)$$

where  $I$  is a square identify matrix of dimension  $m$ .

as well as the matrices

$$A(z^{-1}) = \begin{bmatrix} A_{11}(z^{-1}) & \cdots & A_{1m}(z^{-1}) \\ \vdots & \ddots & \vdots \\ A_{m1}(z^{-1}) & \cdots & A_{mm}(z^{-1}) \end{bmatrix} \quad (4)$$

$$B(z^{-1}) = \begin{bmatrix} B_{11}(z^{-1}) & \cdots & B_{1p}(z^{-1}) \\ \vdots & \ddots & \vdots \\ B_{m1}(z^{-1}) & \cdots & B_{mp}(z^{-1}) \end{bmatrix} \quad (5)$$

where the entries  $A_{ir}(z^{-1})$  and  $B_{it}(z^{-1})$  are polynomials in the delay operator  $z^{-1}$  for  $i = 1, \dots, m$ ;  $r = 1, \dots, m$ ;  $t = 1, \dots, p$ .

$$A_{ir}(z^{-1}) = \delta_{ir} - a_{ii,1}z^{-1} - \cdots - a_{ir,n_{ir}}z^{-n_{ir}} \quad (6)$$

$$B_{it}(z^{-1}) = b_{it,1}z^{-1} - \cdots - b_{it,k_{it}}z^{-k_{it}} \quad (7)$$

where  $\delta_{ir}$  is the Kronecker-delta; it equals 1 when  $i=r$ , otherwise, it is 0. The polynomial  $A_{ir}(z^{-1})$  describes how old values of output number  $r$  affect output number  $i$ .

We consider the parameter matrices  $\theta_1$  and  $\theta_2$  given by:

$$\theta_1 = [A_1 \quad \cdots \quad A_m]; \quad \theta_2 = [B_1 \quad \cdots \quad B_p] \quad (8)$$

and the regression matrices are given by:

$$\begin{cases} \varphi_1(k) = [\underline{y}(k-1) \quad \cdots \quad \underline{y}(k-a)] \\ \varphi_2(k) = [\underline{u}(k-1) \quad \cdots \quad \underline{u}(k-b)] \end{cases} \quad (9)$$

We get the output equation as

$$\underline{y}(k) + \theta_1^T \varphi_1(k) = \theta_2^T \varphi_2(k) \quad (10)$$

We note that parameter of some particular cases of the model (10) can be identified by the hierarchical identification [6-8].

### III. THE MIMO ARX-LAGUERRE MODEL

#### A. Principle

From relation (1), each MISO system for  $i=1, \dots, m$  can be written:

$$y_i(k) = \sum_{r=1}^m \sum_{j=1}^{n_{ir}} a_{ir}(j) y_r(k-j) + \sum_{t=1}^p \sum_{j=1}^{k_{it}} b_{it}(j) u_t(k-j) \quad (11)$$

The coefficients  $a_{ir}(j)$  and  $b_{it}(j)$  can be decomposed on independent Laguerre bases as follow:

For  $i=1, \dots, m$ ;  $r=1, \dots, m$ ;  $t=1, \dots, p$

$$a_{ir}(j) = \sum_{n=0}^{\infty} g_{n,a_{ir}} \ell_n^{a_{ir}}(j) \quad \text{and} \quad b_{it}(j) = \sum_{n=0}^{\infty} g_{n,b_{it}} \ell_n^{b_{it}}(j) \quad (12)$$

where  $\ell_n^{a_{ir}}(j)$  and  $\ell_n^{b_{it}}(j)$  are the orthonormal functions of Laguerre bases,  $g_{n,a_{ir}}$  and  $g_{n,b_{it}}$  are the Fourier coefficients.

By substituting the linear combinations of relation (12) in the MISO ARX model (11), the resulting model, entitled the MIMO ARX - Laguerre which is composed of  $m$  MISO ARX-Laguerre model is written as:

$$y_i(k) = \sum_{r=1}^m \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} g_{n,a_{ir}} \ell_n^{a_{ir}}(j) y_r(k-j) + \sum_{t=1}^p \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} g_{n,b_{it}} \ell_n^{b_{it}}(j) u_t(k-j) \quad i=1, \dots, m \quad (13)$$

$$y_i(k) = \sum_{n=0}^{\infty} g_{n,a_{ir}} \sum_{r=1}^m x_{n_{ir},y}(k) + \sum_{n=0}^{\infty} g_{n,b_{it}} \sum_{t=1}^p x_{n_{it},u}(k) \quad (14)$$

where  $x_{n_{ir},y}(k)$  and  $x_{n_{it},u}(k)$  are the filtered outputs and the filtered inputs respectively by Laguerre functions such as

$$\begin{cases} x_{n_{ir},y}(k) = \sum_{j=1}^{\infty} \ell_n^{a_{ir}}(j) y_r(k-j) = \ell_n^{a_{ir}}(k) * y_r(k) \\ x_{n_{it},u}(k) = \sum_{j=1}^{\infty} \ell_n^{b_{it}}(j) u_t(k-j) = \ell_n^{b_{it}}(k) * u_t(k) \end{cases} \quad (15)$$

In practice, the infinite series in (13) and (14) can be truncated to a finite orders as follow:

$$y_i(k) = \sum_{n=0}^{Na_{ir}-1} g_{n,a_{ir}} \sum_{r=1}^m x_{n_{ir},y}(k) + \sum_{n=0}^{Nb_{it}-1} g_{n,b_{it}} \sum_{t=1}^p x_{n_{it},u}(k) \quad (16)$$

#### B. Recursive representation of the MIMO ARX-Laguerre model

The orthonormal functions defining both independent Laguerre bases are given by their Z-transform [1]:

$$L_n^{c_{ij}}(z) = \frac{\sqrt{1-\xi_{c_{ij}}^2}}{z - \xi_{c_{ij}}} \left( \frac{1 - \xi_{c_{ij}} z}{z - \xi_{c_{ij}}} \right)^n, \quad c = a, b; \quad n = 0, 1, 2, \dots \quad (17)$$

where  $\xi_{a_{ir}} (|\xi_{a_{ir}}| < 1)$  and  $\xi_{b_{it}} (|\xi_{b_{it}}| < 1)$  are the Laguerre poles.

As mentioned in [1], each polynomial of the ARX model can be decomposed on Laguerre orthonormal basis. We propose to extend this idea to the MIMO ARX model. The same procedure will be applied at each polynomial of the MIMO

ARX model. We proceed by determining the recursive representation given by the polynomial  $A_{ir}(z^{-1})$  and  $B_{it}(z^{-1})$ .

$$\begin{cases} L_0^{a_{ir}}(z) = \frac{\sqrt{1-\xi_{a_{ir}}^2}}{z - \xi_{a_{ir}}} \\ L_n^{a_{ir}}(z) = \frac{1-\xi_{a_{ir}}z}{z - \xi_{a_{ir}}} L_{n-1}(z, \xi_{a_{ir}}), n=1,2,\dots \end{cases} \quad (18)$$

$$\begin{cases} L_0^{b_{it}}(z) = \frac{\sqrt{1-\xi_{b_{it}}^2}}{z - \xi_{b_{it}}} \\ L_n^{b_{it}}(z) = \frac{1-\xi_{b_{it}}z}{z - \xi_{b_{it}}} L_{n-1}(z, \xi_{b_{it}}), n=1,2,\dots \end{cases} \quad (19)$$

The Z-transform of relation (16) gives:

$$Y_i(z) = \sum_{n=0}^{Na_{ir}-1} g_{n,a_{ir}} \sum_{r=1}^m X_{n_{ir},y}(z) + \sum_{n=0}^{Nb_{it}-1} g_{n,b_{it}} \sum_{t=1}^p X_{n_{it},u}(z) \quad (20)$$

where  $X_{n_{ir},y}(z)$ ,  $X_{n_{it},u}(z)$  and  $Y_i(z)$  are the Z-transform of  $x_{n_{ir},y}(k)$ ,  $x_{n_{it},u}(k)$  and  $y(k)$  respectively.

From (15) we can write

$$\begin{cases} X_{n_{ir},y}(z) = L_n^{a_{ir}}(z) \cdot Y_r(z) & i,r=1,\dots,m \\ X_{n_{it},u}(z) = L_n^{b_{it}}(z) \cdot U_t(z) & t=1,\dots,p \end{cases} \quad (21)$$

By combining the recurrent form (18) of Laguerre orthonormal functions with relation(21), we can formulate the recurrence relation for the filters  $X_{n_{ir},y}(z)$  and  $X_{n_{it},u}(z)$   $n=1,2,\dots$ :

$$\begin{cases} X_{0_{ir},y}(z) = \frac{\sqrt{1-\xi_{a_{ir}}^2}}{z - \xi_{a_{ir}}} Y_r(z) \\ X_{n_{ir},y}(z) = \frac{1-\xi_{a_{ir}}z}{z - \xi_{a_{ir}}} X_{n_{ir}-1,y}(z, \xi_{a_{ir}}) \end{cases} \quad (22)$$

$$\begin{cases} X_{0_{it},u}(z) = \frac{\sqrt{1-\xi_{b_{it}}^2}}{z - \xi_{b_{it}}} U_t(z) \\ X_{n_{it},u}(z) = \frac{1-\xi_{b_{it}}z}{z - \xi_{b_{it}}} X_{n_{it}-1,u}(z, \xi_{b_{it}}) \end{cases} \quad (23)$$

From(20), (22) and (23), we represent in Figure 1, the filter network of the MIMO ARX - Laguerre model of the first output for two inputs two outputs systems.

From Figure 1, we can obtain the following recursive representation for the MIMO ARX-Laguerre model as a set of

a MISO ARX Laguerre model and each one sub model depends on all input and outputs without omitting coupling between the outputs:

$$\begin{cases} \text{for } r=1,\dots,m; t=1,\dots,p \\ X_y^{ir}(k+1) = A(\xi_{a_{ir}}, Na_{ir}) X_y^{ir}(k) + b(\xi_{a_{ir}}, Na_{ir}) y_r(k) \\ X_u^{it}(k+1) = A(\xi_{b_{it}}, Nb_{it}) X_u^{it}(k) + b(\xi_{b_{it}}, Nb_{it}) u_t(k) \\ y_i(k) = C_i^T X_i(k), \quad i=1,\dots,m \end{cases} \quad (24)$$

where :

$$X_i(k) = \left[ (X_y^{i1}(k))^T, \dots, (X_y^{im}(k))^T, (X_u^{i1}(k))^T, \dots, (X_u^{ip}(k))^T \right]^T \quad (25)$$

$$C_i = [g_{0,a_{i1}}, \dots, g_{Na_{i1}-1,a_{i1}}, \dots, g_{0,a_{im}}, \dots, g_{Na_{im}-1,a_{im}}, \dots, g_{0,b_{i1}}, \dots, g_{Nb_{i1}-1,b_{i1}}, \dots, g_{0,b_{ip}}, \dots, g_{Nb_{ip}-1,b_{ip}}]^T \quad (26)$$

with for  $i,r=1,\dots,m; t=1,\dots,p$ :

$$A(\xi_{a_{ir}}, Na_{ir}) = \begin{bmatrix} \xi_{a_{ir}} & 0 & \dots & 0 \\ 1-\xi_{a_{ir}}^2 & \xi_{a_{ir}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-\xi_{a_{ir}})^{Na_{ir}-2}(1-\xi_{a_{ir}}^2) & (-\xi_{a_{ir}})^{Na_{ir}-3}(1-\xi_{a_{ir}}^2) & \dots & \xi_{a_{ir}} \end{bmatrix} \quad (27)$$

$$A(\xi_{b_{it}}, Nb_{it}) = \begin{bmatrix} \xi_{b_{it}} & 0 & \dots & 0 \\ 1-\xi_{b_{it}}^2 & \xi_{b_{it}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (-\xi_{b_{it}})^{Nb_{it}-2}(1-\xi_{b_{it}}^2) & (-\xi_{b_{it}})^{Nb_{it}-3}(1-\xi_{b_{it}}^2) & \dots & \xi_{b_{it}} \end{bmatrix} \quad (28)$$

$$b(\xi_{a_{ir}}, Na_{ir}) = \sqrt{1-\xi_{a_{ir}}^2} \begin{bmatrix} 1 \\ -\xi_{a_{ir}} \\ (-\xi_{a_{ir}})^2 \\ \vdots \\ (-\xi_{a_{ir}})^{Na_{ir}-1} \end{bmatrix} \quad (29)$$

$$b(\xi_{b_{it}}, Nb_{it}) = \sqrt{1-\xi_{b_{it}}^2} \begin{bmatrix} 1 \\ -\xi_{b_{it}} \\ (-\xi_{b_{it}})^2 \\ \vdots \\ (-\xi_{b_{it}})^{Nb_{it}-1} \end{bmatrix} \quad (30)$$

According to the recursive vector representation (24) each estimated parameter vector  $\hat{C}_i$  can be computed by the standard parameter estimation methods like RLS method.

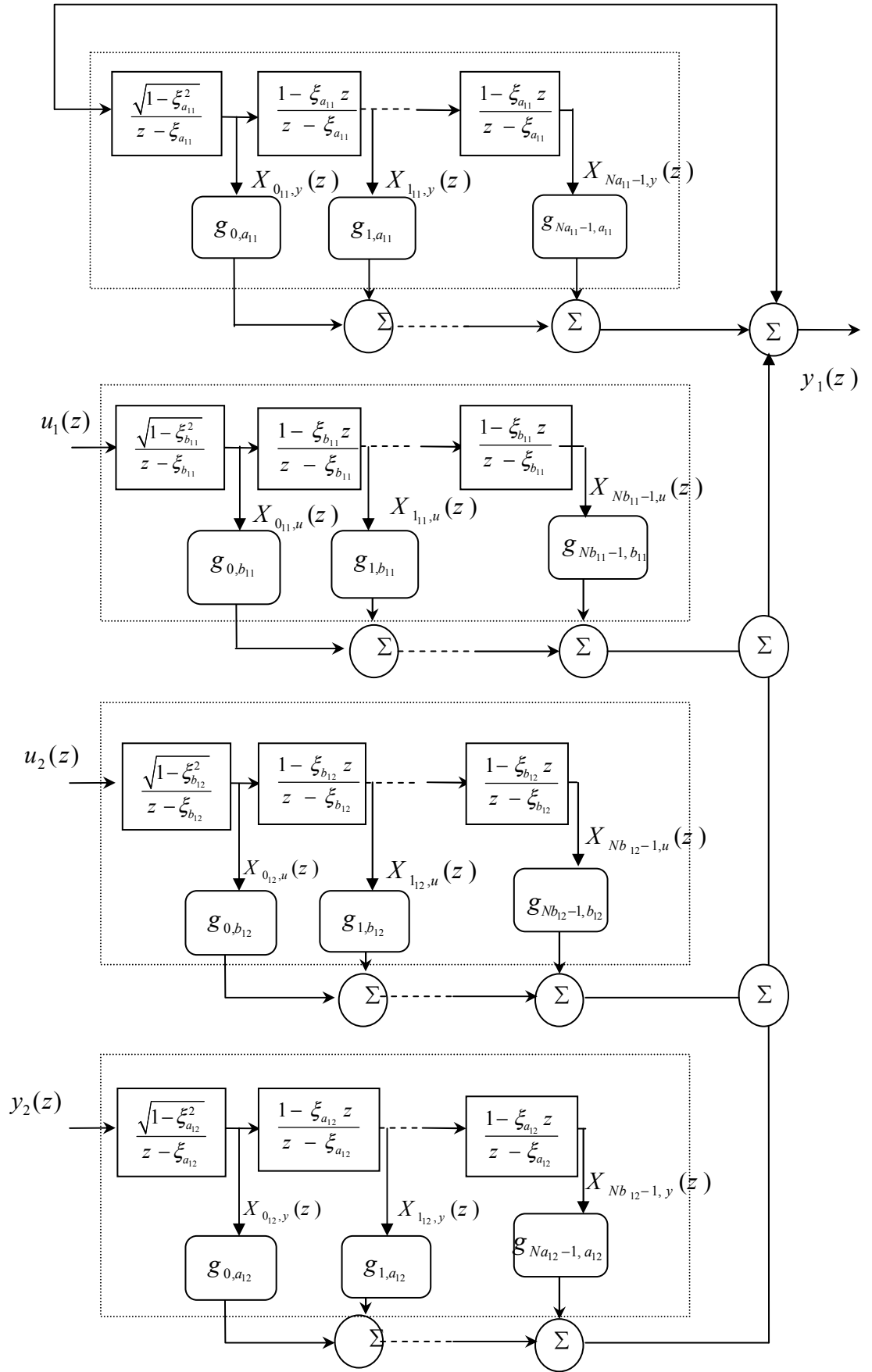


Figure 1. Discrete-time MIMO ARX - Laguerre filter network of the first output  $y_1$

#### IV. SIMULATION RESULTS

As a complete illustration of the proposed model, let us consider the following two-input, two-output ARX model:

$$\begin{pmatrix} A_{11}(z^{-1}) & A_{12}(z^{-1}) \\ A_{21}(z^{-1}) & A_{22}(z^{-1}) \end{pmatrix} \begin{pmatrix} y_1(k) \\ y_2(k) \end{pmatrix} = \begin{pmatrix} B_{11}(z^{-1}) & B_{12}(z^{-1}) \\ B_{21}(z^{-1}) & B_{22}(z^{-1}) \end{pmatrix} \begin{pmatrix} u_1(k) \\ u_2(k) \end{pmatrix} \quad (31)$$

with

$$\begin{aligned} A_{11}(z^{-1}) &= 1 - 1.4067z^{-1} + 0.4348z^{-2} \\ A_{12}(z^{-1}) &= 0.0086z^{-1} - 0.0082z^{-2} \\ A_{21}(z^{-1}) &= -1.5319z^{-1} + 1.4497z^{-2} \\ A_{22}(z^{-1}) &= 1 - 1.5939z^{-1} + 0.6151z^{-2} \\ B_{11}(z^{-1}) &= 0.0056z^{-1} + 0.0894z^{-2} + 0.8134z^{-3} + 2.1714z^{-4} \\ B_{12}(z^{-1}) &= -0.0104z^{-1} + 0.0336z^{-2} + 0.0287z^{-3} + 0.3387z^{-4} \\ B_{21}(z^{-1}) &= -2.2561z^{-1} + 5.859z^{-2} + 4.6272z^{-3} - 0.5204z^{-4} \\ B_{22}(z^{-1}) &= 3.5972z^{-1} + 3.4753z^{-2} + 3.2996z^{-3} + 1.5961z^{-4} \end{aligned}$$

The MIMO ARX model corresponding to (31) contains 24 parameters. The system inputs are a pseudo-random sequence with variable amplitude (Figure 1). The corresponding outputs are given in Figure 2.

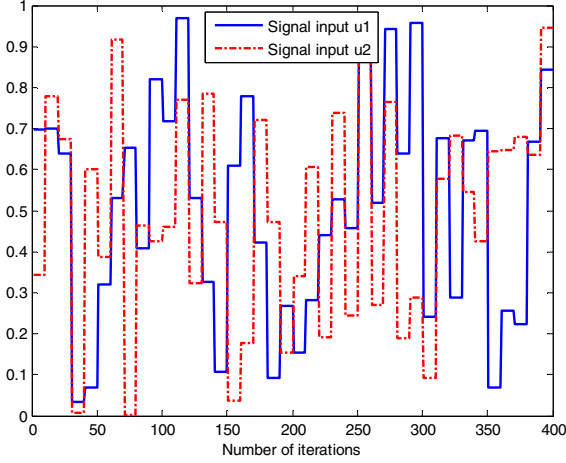


Figure 2. Signal inputs

To validate the proposed model, we fix the truncated orders as follow:

$$Na_{11}=Na_{12}=Na_{21}=Na_{22}=2 \text{ and } Nb_{11}=Nb_{12}=Nb_{21}=Nb_{22}=2.$$

The Laguerre poles are fixed as follow:

$$\begin{aligned} \xi_{a_{11}} &= \xi_{a_{12}} = \xi_{a_{21}} = \xi_{a_{22}} = -0.9 \\ \xi_{b_{11}} &= \xi_{b_{12}} = \xi_{b_{21}} = \xi_{b_{22}} = 0.8 \end{aligned}$$

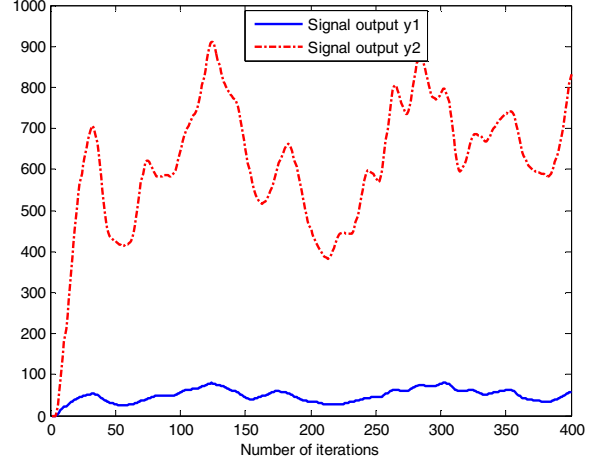


Figure 2. Signal outputs

Then the parameter vectors  $c_1$  and  $c_2$  can be estimated by using the Least square method.

$$\begin{aligned} C_1 &= [2.9931 \quad -0.0420 \quad -0.0529 \quad 0.0408 \\ &\quad 1.0895 \quad 0.0047 \quad 0.2233 \quad -0.0004] \\ C_2 &= [-0.0001 \quad 105.0428 \quad 0.0000 \quad 178.8367 \\ &\quad 0.0001 \quad 14.6207 \quad 9.9119 \quad -0.0000] \end{aligned}$$

In Figure 3, we plot the evolution of the first output  $y_1$  by using the ARX model and the ARX-Laguerre model. We note that with a parameter number equal to 8, the Normalized Mean square Error output NMSE=0.0036%.

In Figure 4, we plot the evolution of the second output  $y_2$  of the ARX model and of the ARX-Laguerre model. With the 8 parameters, the NMSE is equal 0.0033%.

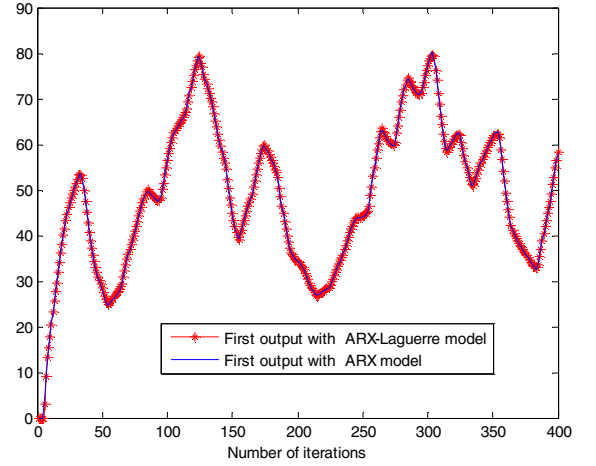


Figure 3. Validation of the proposed model on the first output

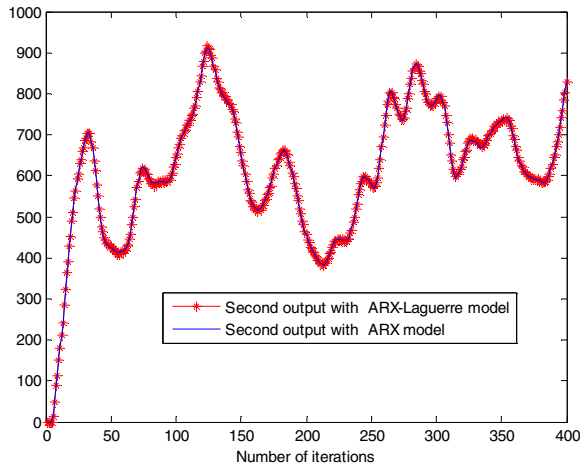


Figure 4. Validation of the proposed model on the second output

The performances in term of parameter number reduction and NMSE illustrate the robustness of the proposed model.

## V. CONCLUSION

In this paper, a new approach for the input-output modelling of linear multivariable systems has been introduced which overcomes the parameter complexity of the MIMO ARX model. This reduction has been insured by developing the MIMO ARX model parameters on independent Laguerre bases.

The proposed model is entitled the MIMO ARX-Laguerre presented as a simple recurrent representation. The proposed

procedure of identifying the MIMO ARX-Laguerre model is tested on simulations and the performances in terms of parameter complexity reduction and approximation quality are highly appreciated.

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