

# An Efficient Stochastic Approach to Uncertainty Quantification in 3-D FDTD Magnetized Cold Plasma

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**Abstract** — An efficient stochastic finite-difference time-domain (S-FDTD) method is developed to analyze electromagnetic field variability in three dimensional anisotropic magnetized plasma. The new S-FDTD plasma model provides a full understanding of the true physics due to the associated uncertainties and has broad potential applicability. This new algorithm efficiently calculates in a single simulation not only the mean electromagnetic field values, but also their variance as caused by the variability or uncertainty of the content of the ionosphere. This ability will, for example, provide the capability of determining the confidence level that a communications / remote sensing / radar system will operate as expected under abnormal ionospheric conditions.

## 1 INTRODUCTION

### 1.1 The problem

Communications, surveillance, and navigation capabilities rely heavily on accurate knowledge of electromagnetic (EM) signal propagation through and reflected by the Earth's ionosphere. Satellite communications, over-the-horizon radar, and target direction finding are a few example applications. Poor understanding of either the ionospheric state or the complete signal propagation characteristics through the ionosphere can negatively affect the performance of these applications. For example, inaccurate signal predictions may lead to erroneous target identification and coordinate estimation.

It is crucial for the performance of many these systems to have knowledge of and utilize not just the general (average) structure of the ionosphere, but also its variability (or uncertainty). For example, the variability of the ionosphere strongly effects trans-ionospheric radio propagation. The irregularities in the electron density distribution can cause phase and amplitude scintillation. Therefore, uncertainty analysis becomes an important factor to be considered in ionosphere electrodynamics modeling.

Numerical EM techniques, however, typically use

only average (mean) values of the constitutive parameters of the materials and then solve for expected (mean) electric and magnetic fields. One of the unanswered questions with these simulations is how variation between or uncertainty in the constitutive parameters may impact the electromagnetic fields.

### 1.2 Overview of techniques

The variability of the ionosphere renders many propagation problems too complex to be solved using a deterministic formulation. The structure of the ionosphere can depend not only on the altitude, time of day, and season, but also on the latitude longitude, sun spot cycle, and occurrence of space weather events. A useful approach to such a highly complex problem as EM wave propagation in the ionosphere is to consider it as a random medium problem. For variation analysis of uncertainties effects on the EM field, methods for uncertainty quantification are required. The Monte Carlo method is a well-established and widely-used brute force technique for evaluating random medium problems via multiple realizations. It is significantly more efficient to formulate the problem in such a way that its ensemble averages may be run in a single realization scheme.

The finite-difference time-domain (FDTD) method is a well known technique for modeling electromagnetic (EM) wave propagation and interactions with cold plasmas [1]. We focus on several techniques have been proposed recently to solve uncertainty quantification problems for FDTD model. The article [2] proposes a single-realization scheme to obtain the quantities of interest that are the ensemble average of scattered fields, which makes use of an iterative technique to reformulate multiplicative noise into an additive noise. However, the restriction of this algorithm is that it must meet the condition of weak scattering random medium, where deviation from the mean values of electrical properties is small.

The articles [3, 4] use the generalized polynomial chaos (gPC) method, which is an extension of the homogeneous chaos introduced by Wiener

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(1938). The gPC expands the time-domain electric and magnetic fields in terms of orthogonal polynomial basis functions of the uncertain variables. The infinite sum of polynomial chaos expansion is truncated to a finite number of terms  $P$  of orthogonal basis functions. The number of terms  $P$  is given by:

$$P + 1 = \frac{(n + d)!}{n!d!} \quad (1)$$

where  $d$  is the highest polynomial order in the expansion and  $n$  is number of random variables. It follows that  $P$  grows very quickly with the dimension and the order of the decomposition. In general, the gPC method increases memory consumption by a factor  $P + 1$  and run times proportional to  $(P + 1)^2$ . The gPC techniques typically converge significantly faster than the Monte Carlo method in a number of applications. However, the method has an inherent limitation. It can handle only a limited number of uncertain inputs. For large numbers of random variables, polynomial chaos becomes very computationally expensive and Monte-Carlo methods are typically more feasible.

Another technique, called stochastic FDTD (S-FDTD) developed by Smith and Furse [5], provides a direct estimate of both the mean and variance of the EM fields within a variable ionosphere at every point in space and time. The advantage of this method is that it requires only about twice as much computer simulation time and memory as a traditional FDTD simulation regardless of numbers of random variables. On the other hand, its limitation is that it can only bound the field variances according to a best estimate approximation for the cross correlation coefficients.

In sum, each method has their own strengths and limitations. Given the fact that the ionosphere content can vary even up to 100% or more. In addition, the governing equations are Maxwell's equations coupled to current equations derived from the Lorentz equation of motion. As a result, this increases the complexity of physical model, the governing stochastic equations take complicated forms of large matrix [1]. Therefore, the derivation of explicit equations for the gPC coefficients can be very difficult, or even impossible. In this paper, we extend the Maxwell's equations stochastic FDTD (S-FDTD) methodology of [5] to the fully three-dimensional (3-D) anisotropic magnetized plasma algorithm of [1].

## 2 METHODOLOGY

The magnetized cold plasma governing equations are cast in terms of Maxwell's equations coupled to

current equations derived from the Lorentz equation of motion. The resulting whole governing equation set is given by:

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad (2a)$$

$$\nabla \times \mathbf{H} = -\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_e \quad (2b)$$

$$\frac{\partial \mathbf{J}_e}{\partial t} + v_e \mathbf{J}_e = \epsilon_0 \omega_{Pe}^2 \mathbf{E} + \omega_{Ce} \times \mathbf{J}_e \quad (2c)$$

Here  $v_e$ ,  $J_e$  and  $\omega_{Pe}$  are the collision frequency, the current density and the plasma frequency of electrons, respectively. The plasma frequency is a function of the electron density  $n_e$  given by,

$$\omega_{Pe} = \sqrt{\frac{q_e^2 n_e}{\epsilon_0 m_e}} \quad (3)$$

Ionosphere electron densities vary in a complex manner as a function of location and time. Thus, we consider the electron density as a random variable with its own statistical variation. This variability in the electron density causes variability in the EM fields, which will also be treated as random variables.  $\omega_{Ce}$  is the cyclotron frequency of the electrons given by  $\omega_{Ce} = q_e \mathbf{B} / m_e$ . Here,  $B$  is the applied magnetic field.

For the S-FDTD derivation, there are initially three stochastic equations (2a), (2b) and (2c) that for Cartesian coordinates contain ten random variables for the 3-D case:  $E_x$ ,  $E_y$ ,  $E_z$ ,  $H_x$ ,  $H_y$ ,  $H_z$ ,  $J_{ex}$ ,  $J_{ey}$ ,  $J_{ez}$  and  $\omega_{Pe}$ . By using the delta method [6], Smith and Furse demonstrated that the average (or expected) fields can be found by solving the field equations using the mean or averages of the variables [5]. Thus, the mean EM field and current density values are found by using the mean plasma frequency of  $\omega_{Pe}$ , or equivalently, the mean of electron density  $n_e$ .

In order to derive the standard deviation (or variance) equations, we must take the variance of (2a), (2b) and (2c). This step results in two cases as described below:

**Case 1:** If a function is formed by the sum of multiple variables (equations (2a) and (2b)), its variance is:

$$\sigma^2 \left\{ \sum_{i=1}^n a_i X_i \right\} = \sum_{i=1}^n a_i^2 \sigma^2 \{X_i\} + 2 \sum_{1 \leq i < j \leq n} a_i a_j \rho_{X_i, X_j} \sigma \{X_i\} \sigma \{X_j\} \quad (4)$$

Here,  $\rho_{X_i, X_j}$  is the correlation coefficient ( $-1 \leq \rho_{X_i, X_j} \leq 1$ ). The closer this coefficient is to zero, the more independent the terms are from each

other. If the correlation coefficients  $\rho_{X_i, X_j}$  ( $1 \leq i < j \leq n$ ) = 1, we obtain:

$$\sigma \left\{ \sum_{i=1}^n a_i X_i \right\} = \sum_{i=1}^n a_i \sigma \{X_i\} \quad (5)$$

**Case 2:** If a function is formed by the product of multiple variables (equation (2c)), its variance is solved by using the delta method [6]:

$$\sigma^2 \{f(X_1, \dots, X_m) g(X_{m+1}, \dots, X_{m+n})\} \quad (6)$$

$$= \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} \frac{\partial(fg)}{\partial X_i} \frac{\partial(fg)}{\partial X_j} \bigg|_{\mu_{X_1}, \dots, \mu_{X_{m+n}}} \text{Cov}(X_i, X_j)$$

Equations (5) and (6) will be used in the derivation of the standard deviation equations. When standard deviation equations are derived, covariances are needed of the  $E$ ,  $H$  fields and current density  $J_e$  in both time and space. The equations also relate the electric field to the plasma frequency of the ionosphere, resulting in additional covariance terms between the electric field and the plasma frequency. As for the Maxwell's equations S-FDTD methodology of [5], for the 3-D S-FDTD magnetized cold plasma algorithm, the magnetic fields, electric fields and current densities are highly correlated to each other. As such, the correlation coefficients of the  $E$ ,  $H$  fields and current density  $J_e$  may be approximated as 1. However, it is challenging to decide which method should be used to evaluate the remaining cross correlation coefficients between the electric field and the plasma frequency. There are many factors in choosing the best  $\rho_{\omega, E}$  values, such as the field component orientation, the cell's location relative to the source, the type of source wave, and the direction of the background magnetic field. The approximation of these correlation coefficients will control the accuracy of the algorithm.

### 3 CONCLUSION

An initial magnetized ionosphere plasma S-FDTD model was created [7, 8] in which we successfully extended the Maxwell's equations stochastic FDTD (S-FDTD) methodology developed by Smith and Furse and applied to biomedical applications [5] to the fully three-dimensional (3-D) anisotropic magnetized plasma of [1]. When applied to the ionosphere, this model uses as input not only average electron (or ion) densities, but also their variance due to uncertainties or due to disturbances such as space weather events. Although it still remains a challenge to develop the best methodology for determining these cross correlation values, this

method provides a good starting point to develop a stochastic optimization FDTD-based algorithm that is well-suited for large uncertainty quantification of the ionosphere, so that the variability of the EM wave propagation is well under control and understood.

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