The solution to Arrow's difficulty in social choice

Thomas Colignatus

Sheets for the Politicologenetmaal, Leiden, June 8 2018

This concerns *Voting Theory for Democracy*, chapter 9.2, p239-251

https://zenodo.org/record/291985

http://www.thomascool.eu/Papers/VTFD/Index.html

Aggregation of preferences

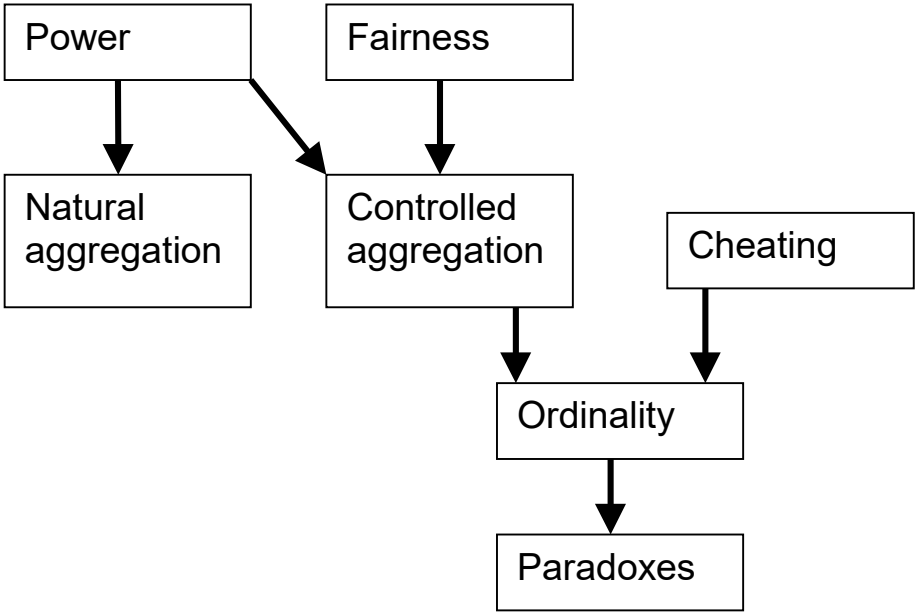
1. Voting rules are always embedded in a social environment.

2. Deductions on those voting rules *by themselves* have only limited validity.

3. Thus be careful with conclusions on the *meaning* and *impact.*

Mathematics:

- (i) adoption of assumptions, (ii) deduction, (iii) the question remains what the assumptions and conclusions really mean
- *Arrow's Impossibility Theorem on Social Choice* and *Sen's Theorem on the Impossibility of a Paretian Liberal* are examples of correct deduction but with incorrect interpretation by these authors



Paradoxes are the price paid for restricting cheating

Paradox = *seeming* contradiction (and no real one)

Arrow's Impossibility Theorem on Social Choice

Kenneth Arrow 1951, 1963 (with a correction by Julian Blau 1957)

The New Palgrave 1988:125

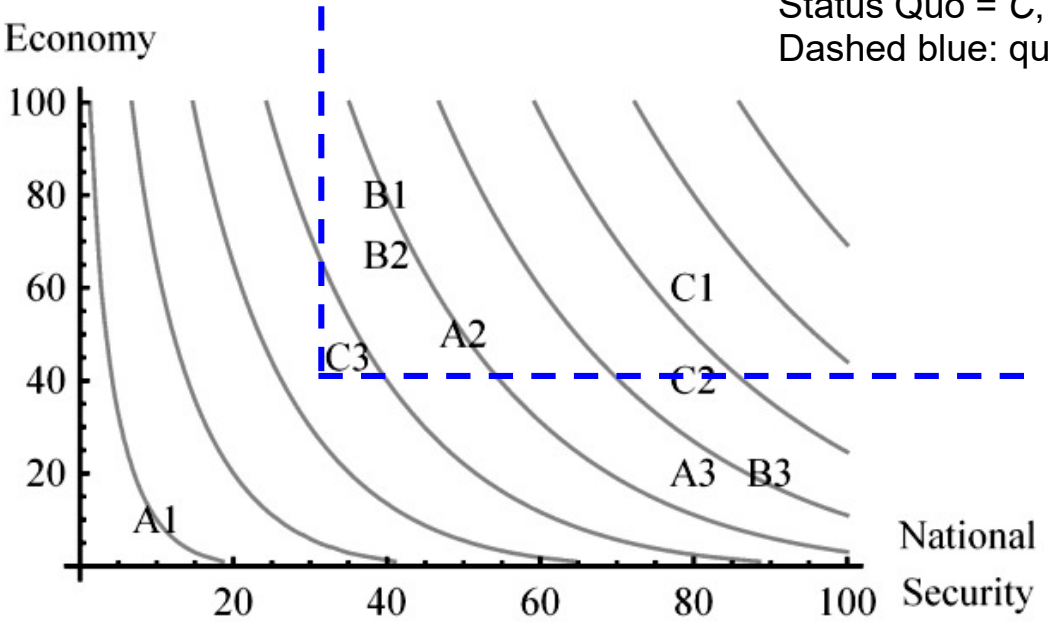
- "(...) conditions to be imposed on constitutions (...)"
- "(...) there is no social choice mechanism which satisfies a number of reasonable conditions"

A proper summary of the reasoning also debunks it

- Arrow's theorem, and the conundrum of its interpretation, can be summarised in the following manner, which also debunks it.
- For 4 axioms we can argue that *each* axiom seems to be reasonable and morally required. Putting them together gives an impossibility. Thus *together* they *cannot* be reasonable and morally required.
- Who wants social choice must drop one of the axioms, otherwise one becomes inconsistent. The real issue is to determine which axiom to drop. We thus must reconsider Arrow's suggestion that *each* axiom would be reasonable and morally required, seen by itself. What *seemed* to be so, suggested by Arrow, in reality isn't so.
- In 1951 Arrow did not adopt all axioms because otherwise he would have adopted an inconsistency. Apparently he liked the conundrum in itself. He didn't say which axiom he would drop to get consistency.

Pareto optimality

Candidates *A, B* and *C*
Voters 1, 2 and 3
Indifference map of 3 shown
Status Quo = *C*, look at 3
Dashed blue: quantity only



Majority is only a tie-breaking rule for Pareto points

Arrow's Impossibility Theorem

WP	weak Pareto principle
U	universal domain for the preferences
D	no dictator
PDM	pairwise decision making (the correct name) = IIA = independence of irrelevant alternatives
a	*WP* & *U* & *D* & *PDM*

AT	Arrow's Theorem, in three forms:
AT	$a \Rightarrow falsum$
AT'	$a \Rightarrow \sim a$
AT"	~a

Key insight for us: voting (summing votes) differs from deciding
Key insight for us: correct math but Arrow confuses voting and deciding

Key: *Put Arrow's interpretation into mathematics too*

Society chooses a constitution:

1. "(...) conditions to be imposed on constitutions (...)"

Define: "Moral desirability" = Ought[*a*] = *Oa* (Deontic logic)

Arrow: *Oa* & ~*a*

2. "(...) there is no social choice mechanism which satisfies a number of reasonable conditions"

Define: "Reasonable" = rational and feasible

- rational = at least consistent
- feasible = in the budget set (money, items, candidates)

1. “(...) conditions to be imposed on constitutions (...)”

Define: “Moral desirability” = Ought[*a*] = *Oa* (Deontic logic)

Arrow: *Oa* & ~*a*

Deontic axiom: (O*p* & (*p* $\Rightarrow$ *q*)) $\Rightarrow$ *Oq*

- Arrow: *Oa*
- Arrow: (*a* $\Rightarrow$ ~*a*)
- Deontic logic: *O*~*a*
- Hence: *Oa* & *O*~*a*

Preference inconsistency !

Hence the axioms *a* are *not* morally desirable: ~*Oa*

2. “(...) there is no social choice mechanism which satisfies a number of reasonable conditions”

Define: “Reasonable” = rational and feasible

- rational = at least consistent
- feasible = in the budget set (money, items, candidates)

But:

- *a* is inconsistent (*a* $\Rightarrow$ *falsum*)
- *a* is infeasible (~*a*)

Hence the axioms *a* are *not* reasonable.

1&2: Arrow’s interpretation evaporates. Only true math for vote counts.

Axiom of pairwise decision making (PDM) (a.k.a. IIA)

- There are arguments to reject PDM
- Ultimately it remains a matter of preference

Crucial distinction:

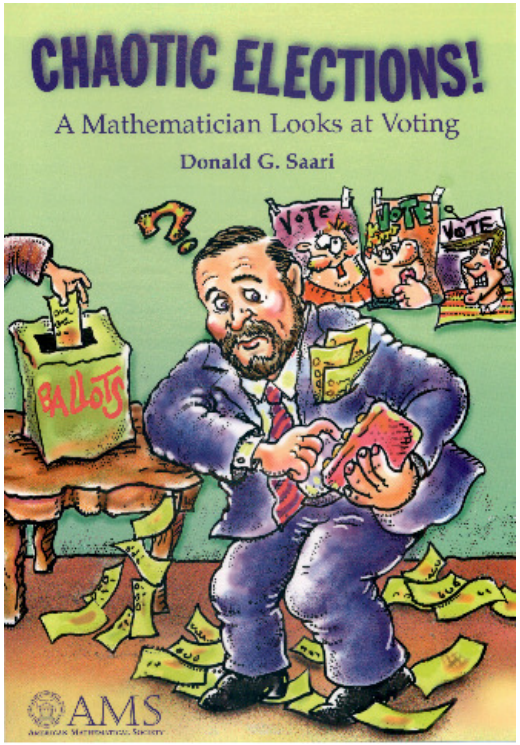
- Voting: summing of votes, voting fields
- Deciding: based upon information of all vote scores, also cycles

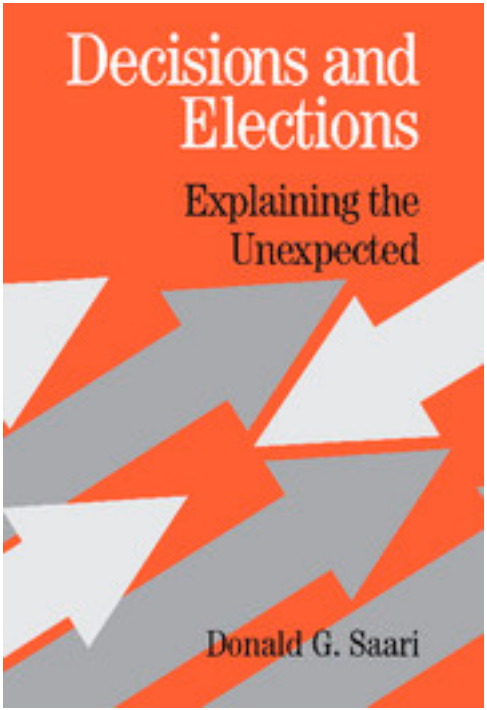
Summing votes does not *by itself* give a decision.

Confusing voting and deciding can cause a contradiction $A > B > C > A$, while in reality there is indecision (deadlock, stalemate, indifference $\cong$).

Arrow's confusion abuses vote cycles for impossibility of social choice.

Donald Saari – 2001





Consistency requires dropping one axiom: likely PDM

1. The conditions *WP* & *U* & *D* are crucial for a Social Welfare Function.

2. Given that PDM forms a confusion, it is logical to drop it. We need *all* information to decide also about a single pair.

3. Saari suggests to adopt *symmetry* as a condition, which gives the Borda method. His geometry is brilliant. A test is to change the electorate. However, *the electorate tends to be given.* Math astray.

4. My suggestion is to require some *dynamic stability.* This gives the *Borda Fixed Point* method. Saari's geometry cannot model this.

VTFD provides *proof* that Arrow's interpretation collapses. VTFD also debunks Sen's mis-interpretation. Saari's *suggestion* on symmetry derives from mathematical esthetics. Saari cannot debunk Sen.

Borda Fixed Point

$X = \{x_1, \ldots, x_n\}$ = budget

A Social Decision Function (SDF) selects a winner:

W = SDF[X] = winner

W^{alt} = SDF[$X \setminus \{W\}$] = alternative = runner up = represents 'the rest'

Fixed point if *pairwise:* W = SDF[W, W^{alt}]

If this is not the case, or tie,
then start a next loop with W^{alt}.

Borda Fixed Point: do the above with SDF = Borda

Comparison: chess & world champion in chess

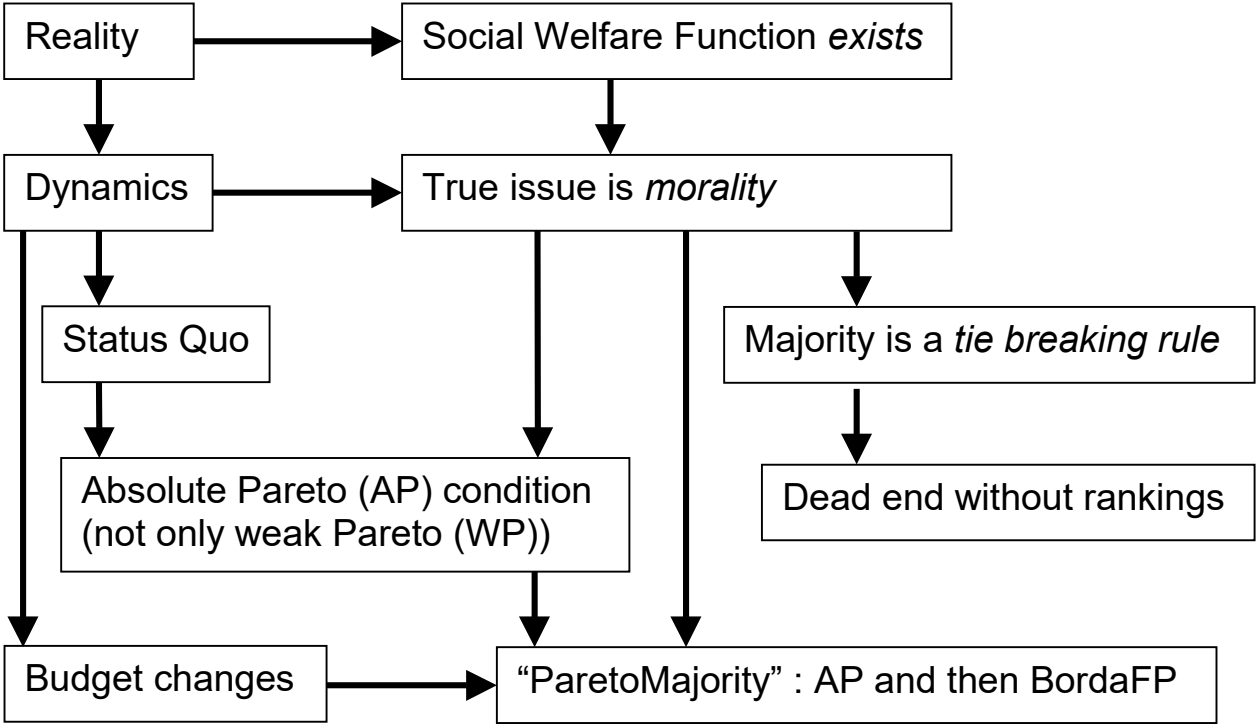
- In a normal tournament: Borda. A winner of a game gets 1, a loser 0, and a tie gives each ½. The winner of the tournament is the one with the highest score (possible ties). Given that white has the advantage (white can select an opening strategy that white has specialised in) it is common that each pair meets twice: a *double round-robin* (ribbon).
- For the world championship: a challenger is selected, and the ruling world champion must defeat the challenger. The 2018 challenger is the winner of a *double round-robin* “Candidates Tournament” with eight players, held in Berlin March 2018. Paradoxes can happen.
- This means that *not all pairwise matches are treated the same*. This looks more like *Borda Fixed Point* rather than *Borda.*
- Borda FP is not designed as such, but works as a compromise of Borda and Condorcet (winner must beat all). Paradoxes can happen.

Key insight: conclusions are conditional to the budget

W.r.t. the axiom of a *universal domain*: this means that the preferences may consider *any budget*. In practice there will be different budgets at different occasions. Thus a key insight is that *conclusions on social choice are always conditional to the budget*. Paradoxes (or seeming contradictions) arise when we presume that the budget wouldn't matter.

- Proper orderings of options are standardised by elimination
  W = SDF[X] = winner
  W' = SDF[$X \setminus \{W\}$]
  W'' = SDF[$X \setminus \{W, W'\}$]
  …
- Proper orderings are less sensitive to changes in budget X
- Borda or Plurality orderings differ from this - and are more sensitive to the budget or budget changes

Overall method is baptised “ParetoMajority”



Conclusions 1

(1) *Voting Theory for Democracy* (VTFD) provides a view that is alternative to the common literature, reorganises and complements it:
- social choice is rational by definition: dynamic reality
- the true issue is moral
- Arrow's theorem is rather irrelevant and mainly confusing
- Arrow's math ≠ verbal explanations
- deontic logic shows Arrow's verbal explanation to be incorrect
- Arrow's mainly confuses voting (summing votes) and deciding

(2) Key issues in voting are:
- sensitivity to the flux of individual preferences
- minimise surprises when the budget changes
- BordaFP is designed to have less surprises than Borda. But there can be surprises, especially when a new item is a fixed point too.

Conclusions 2

In my experience, Social Choice Theorists don't really understand Arrow's Theorem and Sen's Theorem, and these issues around those.

You have to study *Voting Theory for Democracy* (VTFD) if you want to understand this topic in Social Choice. It is the only book in the world that I know about that has this analysis that I consider to be the proper analysis. Please observe that I am a modest person: but I must also inform you about what VTFD does. The use of *Mathematica* is crucial.

These sheets give an overview, the book is precise.

Beware of mathematicians who do not look at Social Choice but who want to impose their own conventions merely to do math.

I protest against the censorship of science and abuse of power since 1990 by the directorate of the Dutch Central Planning Bureau (CPB).

PM. History of this analysis

1990: *Why a social welfare (meta) function does exist: The Arrow Impossibility Theorem for Social Choice resolved, a better analysis suggested*, own & CPB internal note 90-III-37 (Largely included in VTFD)

1992: *On the resonableness and the moral desirability of the conditions of Arrow's Theorem*, p273-285, in *DRGTPE: Background papers*

1997: *The solution to Arrow's difficulty in Social Welfare*, re-edit of 1992, https://econpapers.repec.org/paper/wpawuwpgt/9707001.htm

2001: *Voting Theory for Democracy*, 1st edition. New: Borda Fixed Point

2013: *A short response to a 'Review', with a comment on Arrow's Impossibility Theorem*, Voting Matters, Issue 30, April 2013

2014: *Voting Theory for Democracy*, 4th edition