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Direct centroid computation of fuzzy numbers

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***Abstract*—In this paper, we give two direct centroid computation methods for fuzzy numbers: One is to use the membership function and the other is to use the alpha-cuts. Compared with the current centroid computation methods, the new methods are simple both in expression and computation. Weighted samples computation method are also proposed to improve the computational accuracy with numerical integration technique. Three examples illustrate the application of proposed methods.**

I. Introduction

Centroid computation of fuzzy number is very commonly method in fuzzy logic [9]. Cheng [3] used a centroid-based distance method to rank fuzzy numbers. Chu and Tsao [4] utilized the area between the centroid point and the origin to rank fuzzy numbers. Wang et al. [22] found the incorrect centroid formulae in [3, 4], and suggested a pair of centroid formulae for trapezoidal fuzzy numbers. Shieh [15] proposed an extension on the formulae of [22]. Wang and Lee [17] gave a revised ranking method for ranking fuzzy numbers with an area between the centroid and original points. The centroid method is also applied to intuitionistic fuzzy sets [6] and the centroid computation is an important method for the interval type-2 and general type-2 fuzzy set system [8, 14, 16]. Currently, most of the centroid computation are based on the membership function. However, in some cases, such as fuzzy weighted average [20], fuzzy data envelopment analysis [7, 21], fuzzy mathematical programming, goal programming and fuzzy multiple criteria decision making [2, 5], the information of fuzzy numbers are often given in the form of a series of α cut levels. In general type-2 fuzzy set with α-plane, the general centroid are also can only be obtained on the α-cut [11, 13]. Recently, Wang [18, 19] derived two analytical formulas for centroid defuzzification under the assumption that the exact membership functions can be approximated by using piecewise linear functions based on α-level sets, and applied in risk assessment of a software development, a new centroid defuzzification formula based on α-level sets is derived.

Up till now, all the above centroid value computations are usually based the membership function on x-axis, that is we use the membership function to compute the centroid value along x-axis, and use the α-cut value to compute the centroid along α-axis. If only the membership function along x-axis is known, to compute the centroid along α-axis, we have to transform the membership function into α-cuts with inverse, and vice versa. Such centroid computation not only is computational inefficient but also can not deal with complicated cases, such as when the membership function is nonlinear, in which the inverse function is difficult to obtain. In the present paper, based on the centroid computation principle, we give two direct centroid methods for the cases of known membership function and known α-cut values respectively. The inverse function transformations are not needed any longer, and the new centroid computation formulas are simpler both in expression and computation than the current ones. When only the sample point is available, the centroid can also be computed with the numerical integral by adding weights of these samples points to improve the accuracy. Some examples illustrate our proposed approach.

The organization of the rest of the paper is as follows: Section II gives the commonly used centroid computation methods with membership function, and the recent development proposed by Wang [18] in the case of only the alpha cut samples are available. Section III provides some preliminaries of numerical integral that we will use to compute the centroid values directly and more accurately. Section IV proposes two new kinds of methods for the direct centroid computation with the membership function or the alpha cut information respectively. Section V gives some numerical examples to illustrate our proposed approaches. Section VI summarizes the main results and draws conclusions.

II. Developments on the centroid computation of fuzzy numbers

A. The membership function is known

A fuzzy number is a convex fuzzy subset of the real line R and is completely defined by its membership function. Let $\tilde{A}$ be a normal fuzzy number, whose membership function $\mu_{\tilde{A}}(x)$ can generally be defined as

$$\mu_{\tilde{A}}(x)=\begin{cases} f^L_{\tilde{A}}(x), & a\leqslant x\leqslant b,\\ 1, & b\leqslant x\leqslant c,\\ f^R_{\tilde{A}}(x), & c\leqslant x\leqslant d,\\ 0, & \text{otherwise.}\end{cases} \tag{1}$$

$f^L_{\tilde{A}}(x):[a,b]\rightarrow[0,1]$ and $f^R_{\tilde{A}}(x):[c,d]\rightarrow[0,1]$ are two strictly monotonical and continuous mappings from R to the closed interval $[0,1]$. If the membership function $f_{\tilde{A}}(x)$ is piecewise linear, then $\tilde{A}$ is referred to as a trapezoidal fuzzy number and is usually denoted by $\tilde{A} = (a, b, c, d)$. In particular, when $b = c$, the trapezoidal fuzzy number is reduced to a triangular fuzzy number denoted by $\tilde{A} = (a, b, d)$.

Since $f^L_{\tilde{A}}(x)$ and $f^R_{\tilde{A}}(x)$ are both strictly monotonical and continuous functions, their inverse functions exist and should also be continuous and strictly monotonical. Let $g^L_{\tilde{A}} : [0, 1] \to [a, b]$ and $g^R_{\tilde{A}} : [0, 1] \to [c, d]$ be the inverse functions of $f^L_{\tilde{A}}(x)$ and $f^R_{\tilde{A}}(x)$, respectively. Then $g^L_{\tilde{A}}(y)$ and $g^L_{\tilde{A}}(y)$ should be integrable on the closed interval $[0, 1]$. In other words, both $\int_0^1 g^L_{\tilde{A}}(y)dy$ and $\int_0^1 g^R_{\tilde{A}}(y)dy$ should exist.

The centroid can be computed as [3, 4, 22]:

$$\begin{aligned}\bar{x}_W(\widetilde{A}) &= \frac{\int_a^d x\mu_{\tilde{A}}(x)dx}{\int_a^d \mu_{\tilde{A}}(x)dx} \\ &= \frac{\int_a^b xf^L_{\tilde{A}}(x)dx + \int_b^c xdx + \int_a^b xf^L_{\tilde{A}}(x)dx}{\int_a^b f^L_{\tilde{A}}(x)dx + \int_b^c dx + \int_a^b f^L_{\tilde{A}}(x)dx} \end{aligned} \tag{2}$$

$$\bar{y}_W(\widetilde{A}) = \frac{\int_0^1 y(g^R_{\tilde{A}}(y) - g^L_{\tilde{A}}(y))dy}{\int_0^1 (g^R_{\tilde{A}}(y) - g^L_{\tilde{A}}(y))dy} \tag{3}$$

B. The alpha cut levels are known

When the fuzzy number is given as α-cuts instead the membership function, Wang [18] derived analytical formulas for the centroid defuzzification. Let $\widetilde{A}$ be a fuzzy set on the universe of discourse X. Then the α-level sets of $\widetilde{A}$ are defined as

$$\begin{aligned}A_\alpha &= \{x \in X | \mu_{\widetilde{A}}(x) \geqslant \alpha\} \\ &= [\min\{x \in X | \mu_{\widetilde{A}}(x) \geqslant \alpha\}, \max\{x \in X | \mu_{\widetilde{A}}(x) \geqslant \alpha\} \\ &= [(x)^L_\alpha, (x)^U_\alpha] = [l(\alpha), r(\alpha)]\end{aligned} \tag{4}$$

According to Zadeh's extension principle, the fuzzy set $\widetilde{A}$ can be equivalently expressed as

$$\widetilde{A} = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha \tag{5}$$

The membership functions can be approximated by using piecewise linear functions based on α-level sets [18]:

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0, & x < l(\alpha_0) \text{ or } x > r(\alpha_n), \\ \alpha_i + \frac{\Delta\alpha_i(x - l(\alpha_i))}{l(\alpha_{i+1}) - l(\alpha_i)}, & l(\alpha_i) \leqslant x \leqslant l(\alpha_{i+1}). \\ 1, & (x)^L_{\alpha_n} \leqslant x \leqslant r(\alpha_n) \\ \alpha_i + \frac{\Delta\alpha_i(r(\alpha_i) - x)}{r(\alpha_i) - r(\alpha_{i+1})}, & l(\alpha_{i+1}) \leqslant x \leqslant r(\alpha_i) \end{cases} \tag{6}$$

where $i = 0, 1, \ldots, n-1$. The defuzzified centroid $\bar{x}(\tilde{A})$ can be computed with (2) by the following equations:

$$\begin{aligned}\int_a^d \mu_{\widetilde{A}}(x)dx = &\frac{1}{2}\Bigg[r(\alpha_n) - l(\alpha_n) \\ &- \sum_{i=1}^{n-1} \alpha_i(r(\alpha_{i+1}) - l(\alpha_{i+1})) \\ &+ \sum_{i=0}^{n-1} \alpha_{i+1}(r(\alpha_i) - l(\alpha_i))\Bigg]\end{aligned} \tag{7}$$

$$\begin{aligned}\int_a^d x\mu_{\widetilde{A}}(x)dx = &\frac{1}{6}\Bigg[r^2(\alpha_n) - l^2(\alpha_n) \\ &- \sum_{i=1}^{n-1} \alpha_i(r^2(\alpha_{i+1}) - l^2(\alpha_{i+1}) \\ &+ \sum_{i=0}^{n-1} \alpha_{i+1}(r^2(\alpha_i) - l^2(\alpha_i))\Bigg] \\ &+ \frac{1}{6}\sum_{i=0}^{n-1} \Delta\alpha_i(r(\alpha_i) \cdot r(\alpha_{i+1}) \\ &- l(\alpha_i) \cdot l(\alpha_{i+1}))\end{aligned} \tag{8}$$

When $\Delta\alpha_i = \frac{1}{n}$, $\alpha_i = \frac{1}{n}$, $i = 0, 1, \ldots, n$, the equations are simplified as

$$\begin{aligned}\int_a^d \mu_{\widetilde{A}}(x)dx = &\frac{1}{2n}\Bigg[(r(\alpha_0) - l(\alpha_0)) + (r(\alpha_n) - l(\alpha_n)) \\ &+ 2\sum_{i=1}^{n-1}(r(\alpha_i) - l(\alpha_i))\Bigg]\end{aligned} \tag{9}$$

$$\begin{aligned}\int_a^d x\mu_{\widetilde{A}}(x)dx = &\frac{1}{6n}\Bigg[(r^2(\alpha_0) - l^2(\alpha_0)) + (r^2(\alpha_0) - l^2(\alpha_0)) \\ &+ 2\sum_{i=1}^{n-1} \alpha_i(r^2(\alpha_i) - l^2(\alpha_i))\Bigg] \\ &+ \frac{1}{6n}\sum_{i=0}^{n-1}(r(\alpha_i) \cdot r(\alpha_{i+1}) \\ &- l(\alpha_i) \cdot l(\alpha_{i+1}))\end{aligned} \tag{10}$$

From the above centroid computation methods, we can see that:

- With known membership function $\mu_{\tilde{A}}(x)$, the computation of $\bar{y}(\tilde{A})$ needs to compute the reverse function of $f^L_{\tilde{A}}(x)$ and $f^R_{\tilde{A}}(x)$, which is difficult to manipulate in nonlinear and some other complicated cases.
- When the α-cut value is provided, the computation of $\bar{x}(\tilde{A})$ is some complex because of the membership function is computed with inverse, and it is also an approximate method when the membership function is nonlinear.

III. Some Preliminaries of Numerical Integration

Next, we will give some preliminaries about numerical integral that we will use in our new proposed fuzzy number centroid computation methods.

The goal of numerical integration is to approximate the definite integral of $f(x)$ over the interval $[a,b]$ by evaluating $f(x)$ at a finite number of sample points. All the concepts and conclusions are adopted from [12].

Definition 1: (Quadrature Formula) Suppose that $a = x_0 < x_1 < \cdots < x_m = b$. A formula of the form

$$\begin{aligned} Q(f) &= \sum_{k=0}^{m} w_k f(x_k) \\ &= w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + \\ &\quad \cdots + w_m f(x_m) \end{aligned} \tag{11}$$

with the property that

$$\int_a^b f(x)dx = Q(f) = E(f) \tag{12}$$

is called a numerical integration or quadrature formula. The term $E[f]$ is called the truncation error for integration. The values $\{x_k\}_{k=0}^m$ are called the quadrature nodes and $\{w_k\}_{k=0}^m$ are called the weights.

For all applications, it is necessary to know something about the accuracy of the numerical solution. This leads us to the next definition.

Definition 2: (Degree of Precision) The degree of precision of a quadrature formula is the positive integer n such that $E(P_i) = 0$ for all polynomials $P_i(x)$ of degree $i \leqslant n$, but for which $E(P_{i+1}) \neq 0$ for some polynomial $P_{i+1}(x)$ of degree $n+1$. That is $\int_a^b P_i(x)dx = Q(P_i)$ when degree $i \leqslant n$, and $\int_a^b P_{i+1}(x)dx \neq Q(P_{i+1})$ when degree $i = n+1$.

When the polynomial $P_m(x)$ of degree m is used to approximate $f(x)$, the integral of $f(x)$ is approximated by the integral of $P_m(x)$, the resulting formula is called a Newton-Cotes quadrature formula. When approximating polynomials of degree $m = 1, 2, 3$, it is called Trapezoidal rule, Simpson's rule and Simpson $3/8$ rule respectively. Because of the non-smooth or oscillatory of the function $f(x)$ in $[a,b]$, we usually split $[a,b]$ with quadrature nodes, and apply the composite Newton-Cotes quadrature formula in a similar way of Riemann Sum. For these three rules, the quadrature nodes $\{x_k\}_{k=0}^m$ are chosen to be equally spaced.

Definition 3: (Riemann Sum) Let $f(x)$ be continuous over the interval $[a,b]$, and let $P: a = x_0 < x_1 < x_2 < \cdots, x_n = b$ be a partition, then the definite integral is given by

$$\int_a^b f(x)dx = \lim \sum_{k=1}^{n} f(t_k)\Delta x_k \tag{13}$$

where $t_k \in [x_{k-1}, x_k]$, $\Delta_k = x_k - x_{k-1}$, and the mesh size of the partition $\max \Delta x_k$ goes to zero in the limit, i.e. $\Delta x_k \to 0$ as $n \to 0$.

We can get the left Left Riemann sum, Right Riemann sum, and Midpoint rule by setting $t_k = x_{k-1}$, $t_k = x_k$ or $t_k = \frac{1}{2}(x_{k-1} + x_k)$ respectively. The average of the left Riemann sum and the right Riemann sum becomes the following composite Trapezoidal rule.

The Trapezoidal rule approximate $f(x)$ using straight lines.

Theorem 1: (Composite Trapezoidal rule) Consider $y = f(x)$ over $[a,b]$. Suppose that the interval $[a,b]$ is subdivided into m subintervals $\{x_{k-1}, x_k\}_{k=1}^m$ of equal width $h = \frac{b-a}{m}$ by using the equally spaced nodes $x_k = x_0 + kh$ for $k = 1, 2, \ldots, m$. The numerical approximation to the integral of $f(x)$ with composite Trapezoidal rule is

$$\begin{aligned} \int_a^b f(x)dx = \frac{h}{2}\left(f(a) + f(b) + 2\sum_{k=1}^{m} f(x_k)\right) \\ + E_T(f,h) \end{aligned} \tag{14}$$

If f is second-order continuous differentiable on $[a,b]$, that is $f(x) \in C^2[a,b]$, the error term $E_T(f,h) = -\frac{(b-a)f''(\xi)}{12}h^2 = O(h^2)$, where $a < \xi < b$. $O(h^2)$ means when the step size is reduced by a factor of $1/2$, the error term $E_T(f,h)$ should be reduced by approximately $(\frac{1}{2})^2 = 0.25$.

The Simpson's rule approximate $f(x)$ using quadratic lines.

Theorem 2: (Composite Simpson's rule) Consider $y = f(x)$ over $[a,b]$. Suppose that the interval $[a,b]$ is subdivided into $2m$ subintervals $\{x_{k-1}, x_k\}_{k=1}^{2m}$ of equal width $h = \frac{b-a}{2m}$ by using the equally spaced nodes $x_k = x_0 + kh$ for $k = 1, 2, \ldots, 2m$. The numerical approximation to the integral of $f(x)$ with composite Simpson's rule is

$$\begin{aligned} \int_a^b f(x)dx = \frac{h}{3}\Bigg((f(a) + f(b) + 2\sum_{k=1}^{m-1} f(x_{2k}) \\ + 4\sum_{k=1}^{m} f(x_{2k-1})\Bigg) + E_S(f,h) \end{aligned} \tag{15}$$

If f is fourth-order continuous differentiable on $[a,b]$, that is $f(x) \in C^4[a,b]$, the error term $E_S(f,h) = -\frac{(b-a)f^{(4)}(\xi)}{180}h^4 = O(h^4)$, where $a < \xi < b$. $O(h^4)$ means when the step size is reduced by a factor of $1/2$, the error term $E_T(f,h)$ should be reduced by approximately $(\frac{1}{2})^4 = 0.0625$.

The Simpson's 3/8 rule approximate $f(x)$ using cubic lines.

Theorem 3: (Composite Simpson's 3/8 rule) Consider $y = f(x)$ over $[a,b]$. Suppose that the interval $[a,b]$ is subdivided into $3m$ subintervals $\{x_{k-1}, x_k\}_{k=1}^{3m}$ of equal width $h = \frac{b-a}{3m}$ by using the equally spaced sample points $x_k = x_0 + kh$ for $k = 1, 2, \ldots, 3m$. The numerical approximation to the integral of $f(x)$ with composite Simpson's $3/8$ rule is

$$\begin{aligned} \int_a^b f(x)dx = \frac{3h}{8}\sum_{k=1}^{m}\Bigg(f(x_{3k-3}) + 3f(x_{3k-3}) \\ + 3f(x_{3k-1}) + f(x_{3k})\Bigg) + E_{SC}(f,h) \end{aligned} \tag{16}$$

If f is fourth-order continuous differentiable on $[a,b]$, that is $f(x) \in C^4[a,b]$, the error term $E_{SC}(f,h) = -\frac{(b-a)f^{(4)}(\xi)}{80}h^4 = O(h^4)$, where $a < \xi < b$.

In the following, we will provide alternative methods with the integration principle. Comparing with the current centroid computation methods, our new centroid computation methods are more simple in expression and computation. They can adjust the centroid computation to the different forms of the fuzzy number. The computation is also concise and accurate whatever forms the membership are provided.

IV. Two new kinds of methods for the centroid computation of fuzzy numbers

First, we recall the centroid computation with integral principle. Consider a region in the fuzzy number of area S. We can think of the region as a thin plate with uniform thickness and density. The centroid of the region has coordinates $(\bar{x}, \bar{y})$. We can locate the centroid of the area using integration. The approach is we consider small strips, and then use integration to add them up. The centroid values can be computed as [10]:

$$\bar{x} = \frac{1}{S}\int_S x_e dS, \qquad \bar{y} = \frac{1}{S}\int_S y_e dS \tag{17}$$

where (x_e, y_e) is the coordinates of the centroid of the differential element of area dS.

Next, we will give two methods for the computation centroid of fuzzy number based on the different known conditions on the membership function of fuzzy number.

A. The membership function is known

In such case, the area of fuzzy number $\tilde{A}$ can be seen lies between $y = 0$ and $y = \mu_{\tilde{A}}(x)$ we begin by looking at a thin strip lies between x and $x + \delta x$. The strip may be thought as a rectangle. The width is dx, and so the area is $dS = \mu_{\tilde{A}}(x)dx$. When dx is very small, the centroid of this strip on the x axis is $x_e = x$, the centroid of this trip on the y axis is $y_e = \mu_{\tilde{A}}(x)/2$. With (1), the centroid of the fuzzy number $(\bar{x}(\tilde{A}), \bar{y}(\tilde{A}))$ can be computed as:

$$\bar{x}_{L1}(\tilde{A}) = \frac{\int_a^d x\mu_{\tilde{A}}(x)dx}{\int_a^d \mu_{\tilde{A}}(x)dx} \tag{18}$$

$$\bar{y}_{L1}(\tilde{A}) = \frac{\int_a^d \mu_{\tilde{A}}^2(x)dx}{2\int_a^d \mu_{\tilde{A}}(x)dx} \tag{19}$$

(18) is the same as the current centroid method (2), but (19) is different from the current centroid method (3). In (19), the reverse function of $\mu_{\tilde{A}}(x)$ is not needed, which is simpler than (3) in computation.

B. The alpha cuts are known

In such case, the area of fuzzy number $\tilde{A}$ can be seen lies between $x = l(\alpha)$ and $x = r(\alpha)$. We begin by looking at a thin strip lies between α and $\alpha + d\alpha$. The strip is still thought of as a rectangle. The width is $r(\alpha) - l(\alpha)$, and so the area is $dS = (r(\alpha) - l(\alpha))d\alpha$. When $\delta\alpha$ is very small, the centroid of this strip on the x axis is $x_e = (r(\alpha) + l(\alpha))/2$, the centroid of this trip on the y axis is $y_e = \alpha$, so the centroid of the fuzzy number $(\bar{x}(\tilde{A}), \bar{y}(\tilde{A}))$ is:

$$\bar{x}_{L2}(\tilde{A}) = \frac{\int_0^1 (r(\alpha) + l(\alpha))(r(\alpha) - l(\alpha))d\alpha}{2\int_0^1 (r(\alpha) - l(\alpha)))d\alpha} = \frac{\int_0^1 (r^2(\alpha) - l^2(\alpha))d\alpha}{2\int_0^1 (r(\alpha) - l(\alpha)))d\alpha} \tag{20}$$

$$\bar{y}_{L2}(\tilde{A}) = \frac{\int_0^1 \alpha(r(\alpha) - l(\alpha))d\alpha}{\int_0^1 (r(\alpha) - l(\alpha)))d\alpha} \tag{21}$$

If the α-cuts are given in discrete case, such as in the form of interval number as $[l(\alpha_i), r(\alpha_i)]$, $i = 1, \ldots, m$, then (20) and (21) can be expressed

$$\bar{x}_{L2}(\tilde{A}) = \frac{\sum_{i=1}^m (r^2(\alpha_i) - l^2(\alpha_i))}{\sum_{i=1}^m (r(\alpha_i) - l(\alpha_i))} \tag{22}$$

$$\bar{y}_{L2}(\tilde{A}) = \frac{\sum_{i=1}^m \alpha_i(r(\alpha_i) - l(\alpha_i))}{\sum_{i=1}^m (r(\alpha_i) - l(\alpha_i)))} \tag{23}$$

(22) and (23) only approximate the integral with simple Reimman sum, the accuracies of them are often very low. Numerical integral method can be applied to increase the accuracy significantly.

If composite trapezoidal rule, composite Simpson's rule and composite simpson3/8 rule are applied to (20), (21) respectively, we can get the centroid value with weighted alpha sample points:

$$\bar{x}_{L2}(\tilde{A}) = \frac{\sum_{i=1}^m w_i(r^2(\alpha_i) - l^2(\alpha_i))}{\sum_{i=1}^m w_i(r(\alpha_i) - l(\alpha_i))} \tag{24}$$

$$\bar{y}_{L2}(\tilde{A}) = \frac{\sum_{i=1}^m w_i\alpha_i(r(\alpha_i) - l(\alpha_i))}{\sum_{i=1}^m w_i(r(\alpha_i) - l(\alpha_i)))} \tag{25}$$

The weight values can be assigned from composite trapezoidal rule, composite Simpson's rule and composite simpson3/8 rules respectively, which are given in Table I. In fact, (24) and (25) are the approximate computation of (20) and (21) with numerical integral, where the step size h is omitted, as it is common in both numerator and denominator. The Riemann sum become the special case when all the weights $w_i = 1$.

(21) is the same as the current method (3). Comparing the current method computed with (7) and (8) based on of (2) , (20) is a direct and accurate computation method. It is also simpler than (7), (8) in computation.

The advantages of these fuzzy number centroid computation methods are:

- Whatever the membership function is given in the domain x or the α-cut, the centroid value can be computed directly.
- We do not need to compute the reverse function.
- The centroid values can always be computed in an accurate way, especially in the α-cut case.

TABLE I
WEIGHT ASSIGNMENT METHODS IN (22) AND (23)

Integration Rule	Weight value
Riemann sum	$w_i = 1 (i = 1, 2, \ldots, m)$
Trapezoidal Rule	$w_i = \begin{cases} 1/2 & \text{if } i = 1, m, \\ 1 & \text{if } i \neq 1, m. \end{cases}$
Simpson's Rule	$w_i = \begin{cases} 1/2 & \text{if } i = 1, m \\ 1 & \text{if } i = 1 \mod {}^a(2) \text{ and } i \neq 1, m, \\ 2 & \text{if } i = 0 \mod (2) \text{ and } i \neq m. \end{cases}$
Simpson's $3/8$ Rule	$w_i = \begin{cases} 1/3 & \text{if } i = 1, m \\ 2/3 & \text{if } i = 1 \mod (3) \text{ and } i \neq 1, m, \\ 1 & \text{if } i = 2 \mod (3) \text{ and } i \neq m, \\ 1 & \text{if } i = 0 \mod (3) \text{ and } i \neq m. \end{cases}$

[a] mod is modular arithmetic operator. $i = j \mod (d)$ means $i = nd + j$, where n is an integer.

V. Numerical Examples

Example 1: This example is adopted from [18]. Given the trapezoidal fuzzy number $\tilde{A} = (3; 5; 7; 10)$, we can easily get that the membership function is

$$\mu_{\tilde{A}}(x) = \begin{cases} (x-3)/2 & x \in [3,5]; \\ 1 & x \in [5,7]; \\ (10-x)/3 & x \in [7,10]; \\ 0 & \text{otherwise.} \end{cases} \tag{26}$$

With (18) and (19), we can get $\bar{x}_{L1}(\tilde{A}) = 170/27$ and $\bar{y}_{L1}(\tilde{A}) = 11/27$.

From (26), the left and right α-level set function can be obtained:

$$l(\alpha) = 3 + 2\alpha; \quad r(\alpha) = 10 - 3\alpha \tag{27}$$

With (20) and (21), we can get $\bar{x}_{L2}(\tilde{A}) = 170/27$ and $\bar{y}_{L2}(\tilde{A}) = 11/27$.

It can be verified that $\bar{x}_{L1}(\tilde{A}) = \bar{x}_{L2}(\tilde{A})$, $\bar{y}_{L1}(\tilde{A}) = \bar{y}_{L2}(\tilde{A})$, that is both of the two centroid computation methods, (18) and (19), (20) and (21) produce the same results.

Although (2) and (3) produce the same results, (2) and (3) use both of the expressions of membership function (26) and (27) at the same time. But our method only use one of them. It is obvious that our method are more simple and direct than the current centroid computation methods in Section II.

Example 2: Here is a fuzzy number with nonlinear membership function [1]. Fuzzy number $\tilde{A} = (1, 28, 29, 30)_2$, or, in the parametric form of $\tilde{A}$ with $l(\alpha) = 1 + 27\sqrt{\alpha}$, $r(\alpha) = 30 - \sqrt{\alpha}$, $\alpha \in [0, 1]$. With (20), the centroid value for x is $\bar{x}(\tilde{A}) = 22.21$. But with (7) and (8), the centroid value can only be approximated with piecewise linear function.

In the case of α-cuts are not given as function formula, but as some samples, the centroid can be computed with the numerical integration methods in section 3. Different numerical integration rules can be adopted, such as trapezoidal rule, Simpson's rule, Simpson's 3/8 rule etc. The degree precision of these numerical integration rules are $1, 2, 3$ respectively. In general, the more nonlinear the α-cuts values are, the higher precision order numerical integration rule should be adopted.

Example 3: Following Example 2, the fuzzy number is given by its α-cut levels in Table II.

As the α-cuts are nonlinear, the centroid value with (22) and the weight assignment of Simpson's rule in Table I is $\bar{x}_{L2}(\tilde{A}) = 22.16$. The centroid value with (9), (10) and (2) is $\bar{x}(\tilde{A})_W = 21.97$. We have known the exact centroid value is $\bar{x}(\tilde{A}) = 22.21$. The absolute errors of them are $|\bar{x}_{L2}(\tilde{A}) - \bar{x}(\tilde{A})| = 0.05$, but $|\bar{x}_W(\tilde{A}) - \bar{x}(\tilde{A})| = 0.19$. Our new method is much more accurate than the current method.

VI. Conclusions

The paper proposes two kinds method for the centroid computation of fuzzy numbers. The centroid can be computed either with the ordinary membership function or the α-cuts. The new methods are more simple, computational efficient and accurate than the current fuzzy number centroid computation methods. Four sample weighted methods are also proposed to improve the computational accuracy. Some examples illustrate our proposed approaches.

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TABLE II
THE α-CUTS OF $\tilde{A} = (1, 28, 29, 30)_2$

α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$l(\alpha)$	1	9.54	13.07	15.79	18.08	20.09	21.91	23.59	25.15	26.61	28
$r(\alpha)$	30	29.68	29.55	29.45	29.37	29.29	29.23	29.16	29.11	29.05	29

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