TOTAL TDEV

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Abstract

We propose an extension of the Time Variance and its square root, the Time Deviation denoted TVAR and TDEV, respectively, to improve the confidence of its estimate, taking advantage of information in time series which remains unused by TDEV. The extensions, called the Total Time Variance and the Total Time Deviation, denoted TotTVAR and TotTDEV respectively, use even reflections of the data at the beginning and at the end to almost triple the data length. Computing TVAR of this extended time series significantly increases the equivalent degrees of freedom used in the variance estimate, while producing little bias in the values for phase-modulation (PM) noise types: white PM, flicker PM, and random-walk PM.

Introduction

The Time Variance, TVAR [1], does not use all available information in a time series, in the sense that the equivalent degrees of freedom (edf) for TVAR values are smaller than necessary. We describe a method here for increasing the edf for TVAR estimates of a given data set, particularly in the presence of phase modulation (PM) noise types: white PM, flicker PM, and random-walk PM. These types of noise are often dominant in clock synchronization and measurement systems. For this reason, TDEV has become accepted as a statistic for specifying standards in telecommunications systems.

TotTDEV Definition

Time-transfer error in time synchronization and measurement systems can be characterized stochastically using

$$TVAR = \frac{\tau^2}{3} MVAR, \tag{1}$$

where MVAR is the modified Allan variance. We identify a new deviation function as TotTDEV, the square root of TotTVAR, defined as the TDEV values of an extension of the original data. Time domain data were extended in the recent definition of a new statistic called Total variance, or Totvar. This was done because the sample Allan variance, or AVAR, can "collapse" at long averaging time due to symmetry in the data [2]. Totvar computes AVAR on an extended data set, similarly to how our proposed TotTVAR is related to TVAR. The improved confidence and other properties of Totvar have recently been investigated in relation to AVAR [3].

The extension designed into TVAR to produce TotTVAR is different from that of Totvar because PM noises are treated as the main focus. After removing a linear fit to the time values, the time difference data set $\{x_n\}$ is extended to a new, longer virtual sequence $\{x_n^*\}$ as follows: if N_x is the number of points in the original data set, for n = 1 to N_x let $x_n^* = x_n$; for j = 1 to $N_x - 1$ let

$$x^*_{1-j} = x_j, \quad x^*_{Nx+j} = x_{N_x+1-j}. \tag{2}$$

TDEV is then applied to $\{x_n^*\}$ to produce TotTDEV. Whereas it is possible to choose different amounts of overlap in computing TDEV [4], we choose the maximal overlap method as is done in the ANSI standard [1].

We investigate properties of TotTDEV including mean, equivalent degrees of freedom, and distribution. The main advantage of TotTDEV is its significantly improved confidence at and near the longest TDEV averaging times of the original data set, for noise types such as white PM, flicker PM, and random-walk PM. For noise types with a stronger low-frequency, phase-modulation component such as flicker or random walk FM, TotTDEV has problems due to the extension procedure. We also note that Total TDEV can produce variance estimates for larger integration times, τ, than TDEV can. We caution against using these as representative of the data, unless the user knows that the stochastic behavior represented in the original data set is representative of a data set as long as the extended one.

Problems in Defining TotTVAR

Comparison of Totvar and TotTVAR

TDEV has found usage as a measure of the time instability in applications involving PM noises in which a constant frequency offset exists between oscillators being compared. The PM noises considered are white PM (WHPM), flicker PM (FLPM), and random walk PM (RWPM).

Totvar is defined as the Allan variance of an extension of the original data set. If we denote a time-difference measurement data set as $\{x_t\}$, and the associated frequency-difference series as $\{y_t\}$, Totvar extends $\{y_t\}$ at both ends to produce a new, longer, virtual sequence $\{y_t^{\#}\}$ by an even reflection at the beginning and end of the sequence. This extension has the effect of applying an odd or mirrored reflection at the adjoins of the phase sequence $\{x_t\}$, as shown in Figure 1a. Figure 1b illustrate the duplication that occurs in the left and right extensions. Our goal in defining TotTVAR was to construct an extension on $\{x_t\}$, apply TDEV to the extended sequence and have results which can be interpreted like TDEV, but with improved confidence for averaging times at and near T/3, the longest averaging time.

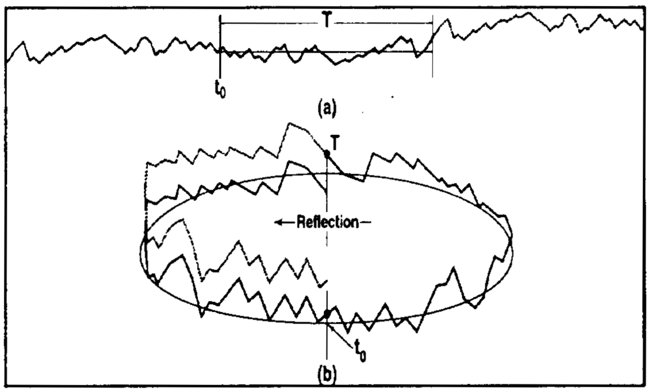

Figure 1 (a) Extension of the original T-length phase data for computation of Total variance. The data are extended to a virtual sequence of 3T-length using an odd mirror-reflection about both ends. (b) Circular representation of the extended data set showing duplication (fold over) of left and right extensions.

If we consider WHPM noise and apply the same mirror-reflection extension to the phase sequence $\{x_t\}$ as done in Totvar, we are likely to create a phase step at the adjoins, as illustrated in Figure 2a. To avoid this, we apply the even extension to sequence $\{x_t\}$ (in the same way that Totvar applies it to $\{y_t\}$) to produce a virtual sequence $\{x_t^{*}\}$ as defined above, and illustrated in Figure 2b..

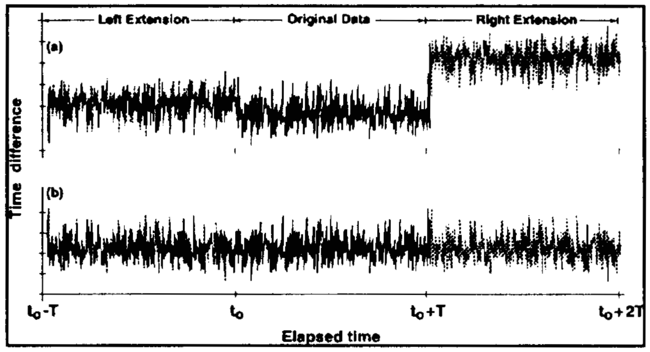

Figure 2 (a) Phase sequence of WHPM noise which is extended using Total variance's odd mirror-reflection at both ends can cause a phase step at each extension interface. (b) Phase sequence of WHPM noise which is extended using Total TDEV's even reflected extension at both ends eliminates the phase steps.

The Effect of an Overall Frequency Difference

A frequency offset between oscillators being compared results in a linear rate offset in phase $\{x_t\}$. When the even extension is applied to data which contains WHPM plus such a linear function, the effect is a periodic term impressed on the virtual sequence with period 2T, as shown in figure 3b (3a shows what would happen with a mirror-reflection extension for comparison). For this reason, an estimate of overall frequency difference must be removed from the $\{x_t\}$ sequence before the virtual sequence $\{x_t^{*}\}$ is constructed. For TotTVAR we remove a linear fit to the time data before extending.

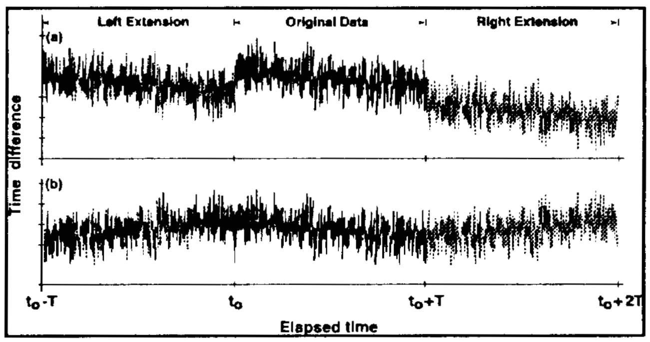

Figure 3 (a) Phase sequence of WHPM noise with a linear phase ramp (which is a frequency offset) is extended using Total variance's odd mirror-reflection. (b) Phase sequence of WHPM noise with a linear phase ramp is extended using Total TDEV's even reflection. An estimate of this linear phase ramp (a frequency offset) must be removed to eliminate the effects of a periodic 2T component impressed on the extended data set.

Data Discontinuity at the Extension

It is necessary to formulate the extension so that a particular statistic such as TDEV is expected to have the same value with extended data as without. In simulation trials that appear later, Total TDEV, with its extensions at the beginning and end, can be interpreted like TDEV for the PM noises. However, Total TDEV in the presence of highly dispersive processes such as random-walk FM can lead to large, undesirable transient components in TDEV. A discontinuity may occur at the extension which is not typical of the process as a whole and which affects values corresponding to smaller integration times.

Studies Using Simulation

We used simulation to compare TDEV and TotTDEV. In particular, we generated 10,000 time series each of length 1024 points, simulating the five most common noise types in time and frequency equipment, white PM, flicker PM, random walk PM, flicker FM and random-walk FM. We compared the mean values, the estimated edf, and the distribution functions.

Degrees of freedom in a random variable are defined as follows. If s^2 denotes the usual sample variance of n independent and identically distributed Gaussian measurements (i.e., white noise) with actual variance σ^2, then it is well known that the random variable

$$U = \frac{s^2}{\sigma^2} \cdot \nu \qquad (3)$$

has a chi-square distribution with ν = n-1 degrees of freedom [5]. In the classical situation, the degrees of freedom associated with σ^2 are integer values depending only on the number of measurements. Exact confidence limits on the measurement variance are calculated using percentiles of the appropriate chi-square distribution.

Since the common time and frequency stability measures (AVAR, MVAR, TVAR) are calculated from data arising from non-white noise processes, the confidence limit procedure outlined above is an approximate method [6]. The method is based on approximating the distribution of U in (3) with the chi-square distribution with degrees of freedom

$$\nu = \frac{2(\sigma^2)^2}{Var\ (s^2)}, \qquad (4)$$

where σ^2 represents the appropriate stability measure (e.g., TVAR), s^2 represents its corresponding estimator, and Var(s^2) is the expected variance of the s^2 estimators. The quantity ν here is called the equivalent degrees of freedom, edf, since it need not be integer-valued. Equations for the edf of MVAR and TVAR have been published previously [4], [7], [8], [9]. In our simulation we used the mean of the 10,000 variances as σ^2, and the variance of those variances as Var(s^2). We then applied equation (4) to determine the edf we report here.

The edf found for TDEV and TotTDEV are presented below in Table 1. Note that TotTDEV can estimate deviations at integration times beyond those possible for TDEV. These should usually not be taken as representative of the underlying physical system that the data measure. They are available because we have extended the data to almost three times the original data length. However, this extension may not reflect the performance of the system at times greater than were actually measured.

We also present, in Table 2, the ratios of the rms values of TDEV/TotTDEV taken over the simulation results. For these ratios we use only the values of τ available from TDEV.

Results from Simulation

First we note that the distribution of both the TDEV and TotTDEV results from simulation approximate chi-squared distributions with appropriate degrees of freedom for all noise types. Figure 4 below illustrates the distribution of the TotTDEV values adjusted according to equation (3) from the τ= 128 integration time for flicker PM data. The agreement with the chi-squared distribution with the same edf is good. The only apparent difference is that the simulated data have a higher peak than the theoretical chi-squared distribution. This justifies using the chi-squared distribution to estimate confidence intervals.

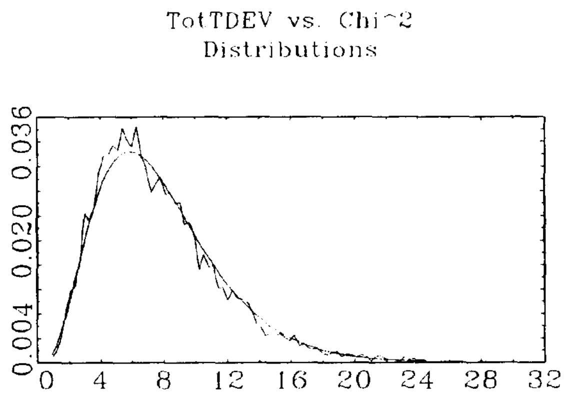

Figure 4 The jagged line is the distribution of the TotTDEV values for τ=128, from simulated flicker PM data, adjusted as in equation (3). The smooth line is the chi-squared distribution with the same number of edf.

Comparing the results in Table 1 we see a significant increase in edf with TotTDEV over TDEV for all of the noise types except RWFM. Improving confidence at long integration times is particularly valuable. In order to obtain an estimate of TDEV(τ) with a minimum of confidence, one must take data for three times the averaging time, τ. Therefore measuring TDEV at long averaging times with significant confidence may be difficult. The improvement in edf with TotTDEV is greatest at long averaging times.

For example, from Table 1 we find that the edf for τ=256 for flicker PM increase from 2.1 to 3.6 using TotTDEV over TDEV. This is an improvement in edf by more than a factor of three. The corresponding confidence intervals change from 5.0 ns to 2.2 ns. In addition, we see in Table 2 that there are no significant biases for any of the common noise types except RWFM.

Table 1

TDEV | TotTDEV Degrees of Freedom					
Integrated Points	White PM	Flicker PM	Random Walk PM	Flicker FM	Random Walk FM
1	517.2|517.5	609.3|609.7	693.8|694.4	859.8|852.5	1024.9|7.3
2	493.4|494.4	491.3|492.4	514.7|515.5	519.5|515.8	435.4|7.2
4	304.8|306.7	261.8|263.3	250.2|251.8	247.2|245.3	198.6|7.1
8	160.2|162.2	127.0|128.9	121.2|123.0	115.4|114.9	97.2|6.9
16	80.5|83.1	61.2|63.2	59.3|61.3	57.3|57.7	48.1|6.5
32	38.9|41.5	29.6|31.6	29.1|31.2	28.8|29.3	23.0|5.7
64	17.5|20.3	14.2|16.2	13.3|15.4	12.6|13.4	10.3|4.6
128	7.2|9.9	5.7|7.7	5.5|7.3	5.1|6.1	4.2|3.1
256	3.0|5.1	2.1|3.6	1.7|3.0	1.6|2.5	1.3|1.7
512	|1.8	|1.3	|1.0	|1.0	|1.0

Table 2

Ratio of Values: TDEV/TotTDEV					
Integrated Points	White PM	Flicker PM	Random Walk PM	Flicker FM	Random Walk FM
1	1.0	1.0	1.0	1.0	0.7
2	1.0	1.0	1.0	1.0	0.7
4	1.0	1.0	1.0	1.0	0.7
8	1.0	1.0	1.0	1.0	0.7
16	1.0	1.0	1.0	1.0	0.7
32	1.0	1.0	1.0	1.0	0.7
64	1.0	1.0	1.0	0.9	0.8
128	1.0	1.0	1.0	0.9	0.8
256	1.0	1.0	1.1	1.0	1.0

One might wonder whether the improvement in edf is artificial, in that we have extended the measured data in a somewhat artificial way. To check this we extended the data with even reflections to increase the data length far beyond the factor of three in the definition of TotTDEV. We then computed the values of TDEV for these large data extensions, and found the edf using equation (4) as we had done for TotTDEV. We found that the edf do not increase significantly beyond what we already found in TotTDEV. This implies that there is a specific amount of information in a given data set, as reflected in the edf for particular variance estimates, and that TotTDEV makes nearly optimal use of this information.

We illustrate the results of our simulations in Figures 5-9. We find both an increase in confidence from using TotTDEV and a lack of biases from TDEV, for all noise types except RWFM. For RWFM we find that TotTDEV significantly worsens both the confidence and the biases over the use of TDEV. This is due to the phenomenon of discontinuity that we discussed earlier. Because of the change in average frequency from the beginning to the end of a RWFM data set, we introduce a frequency discontinuity at the points of extension when we extend with an even reflection.

Conclusions

1. Total TDEV increases confidence on the deviation estimates, particularly at long averaging times, by using more equivalent degrees of freedom which are available in the data, except in the case of RWFM.

2. There is little indication of bias in Total TDEV over TDEV for any of the five common noise types, except for RWFM. Total TDEV should not be used if the data are consistent with a RWFM model.

3. The distribution of TDEV and Total TDEV estimates from our simulations are consistent with a chi-squared distribution.

4. Total TDEV extends phase sequences $\{x_t\}$ in the same way that the Total variance extends frequency sequences $\{y_t\}$.

5. An estimate of frequency difference is removed from the phase sequence $\{x_t\}$ in the definition of Total TDEV to remove a linear ramp in phase.

6. Total TDEV is designed to characterize principally the three common PM noises (WHPM, FLPM, and RWPM), over integration times available to TDEV.

7. Values of Total TDEV are available for integration times beyond those available for TDEV. These should usually not be taken as characterizing the underlying physical system.

References

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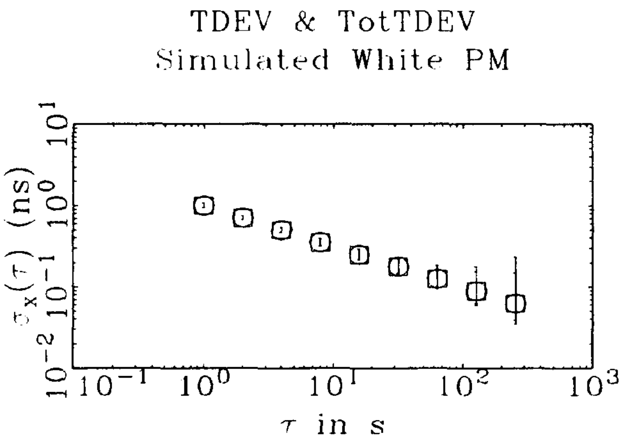

Figure 5 RMS values over 10,000 simulated white PM series of TDEV (circles) and TotTDEV (squares). The smaller confidence intervals are for TotTDEV.

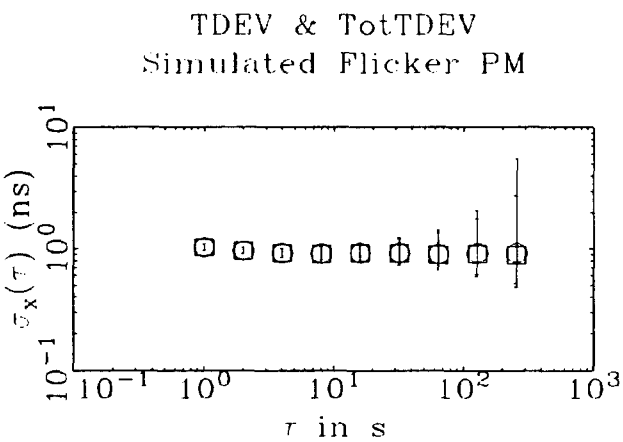

Figure 6 RMS values over 10,000 simulated flicker PM series of TDEV (circles) and TotTDEV (squares). The smaller confidence intervals are for TotTDEV.

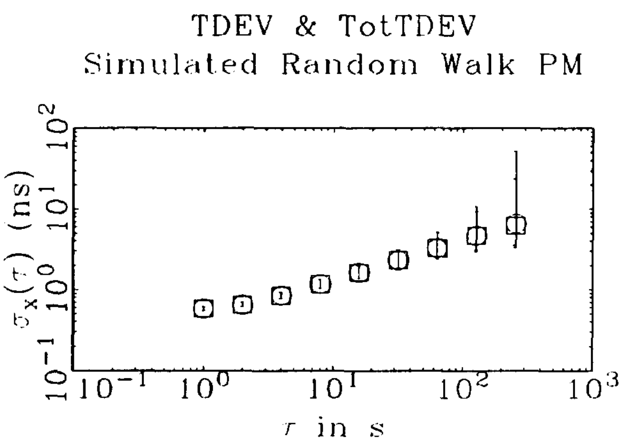

Figure 7 RMS values over 10,000 simulated random walk PM series of TDEV (circles) and TotTDEV (squares). The smaller confidence intervals are for TotTDEV.

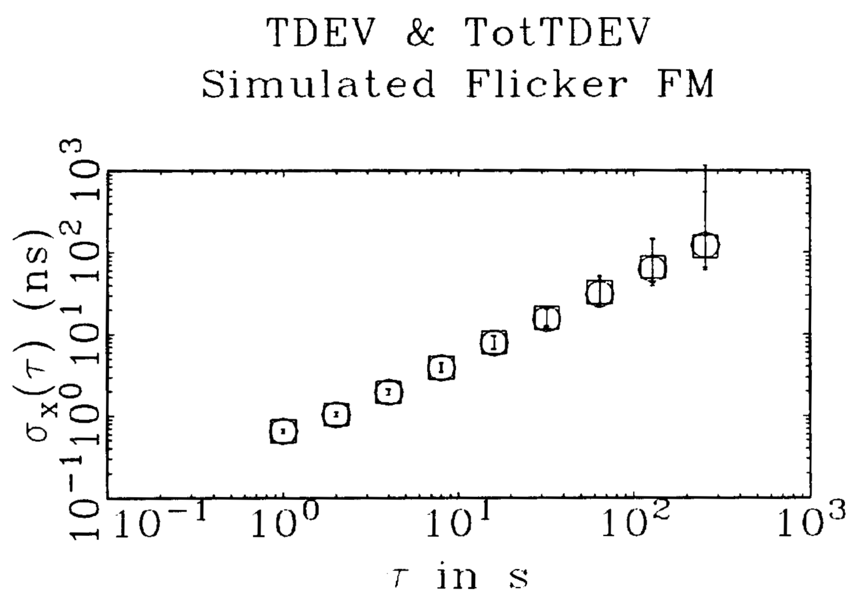

Figure 8 RMS values over 10,000 simulated Flicker FM series of TDEV (circles) and TotTDEV (squares). The smaller confidence intervals are for TotTDEV.

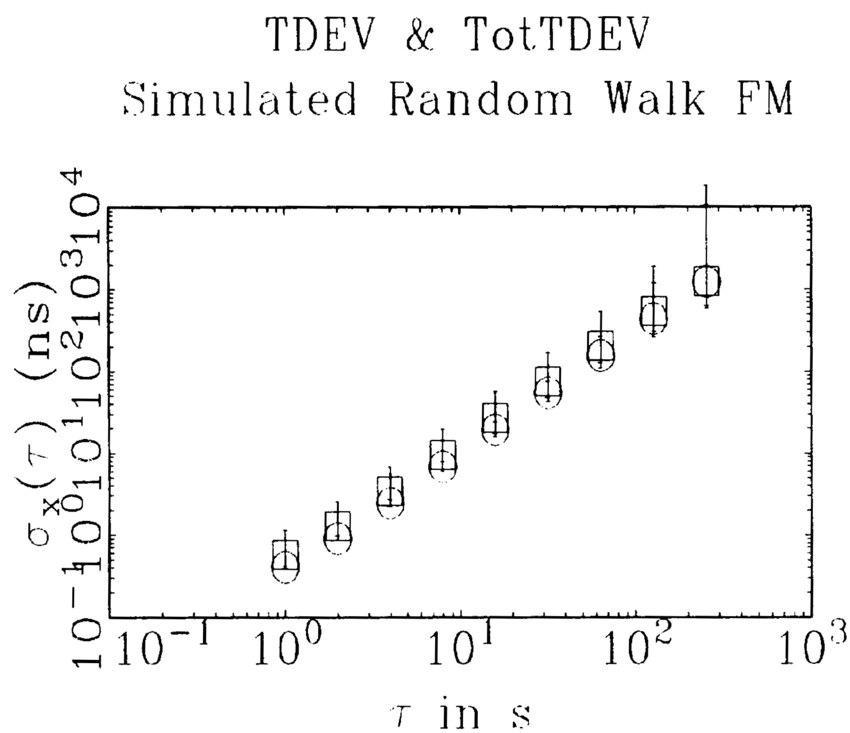

Figure 9 RMS values over 10,000 simulated random walk FM series of TDEV (circles) and TotTDEV (squares). The TotTDEV values are biased, and confidence is degraded.