Link Budget Calculations for Nonlinear Scattering

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Abstract— Interest in harmonic and multitone radar has motivated new design efforts based on the standard approach found in the radar literature, i.e., the classical radar equation. In this paper we show that such approaches can be problematic in the presence of an active or nonlinear target, and indicate how a proper link budget should be constructed based on anticipated target behaviors at high signal levels.

Keywords- intermods; scattering; radar equation; radar cross section.

I. Introduction

Recently, there has been considerable interest in the use of harmonic and multitone radar for various applications, from tracking animals and insects [1] to detecting concealed electronics or the quality of steel welding joints [2]. In designing systems based on these new technologies, it is incumbent on the designer to resist the temptation to use traditional methodologies carelessly. This applies, in particular, to the radar equation, which requires taking into account the way a nonlinear scatterer (target) responds to illumination, which (as we will see below) differs significantly from the response of an ordinary target.

II. Linear Radar

In order to discuss nonlinear effects in radar, we must first specify what we mean by a *linear target*. It is reasonable to identify a target as linear when upon illumination by an electromagnetic (EM) field it radiates a scattered field that is directly proportional to the incident field. For a monostatic radar, this makes it possible to reduce the properties of the scatterer to a single parameter, the (backscattering) *cross section* [3], which is purely a property of the scatterer and independent of any property of the excitation source.

Let us first discuss in detail how the use of this definition leads to the well-established radar equation [4]. Assume such a linear target is illuminated by an antenna with area A a distance R away. Let us define the antenna gain in the usual way:

$$G = \frac{4\pi A\rho}{\lambda^2} \tag{1}$$

where λ is the radar wavelength and ρ is the antenna efficiency, which we assume to be 1. A power P fed to this antenna creates a power *flux* at the target of

$$\wp_{inc} = \frac{PG}{4\pi R^2} \tag{2}$$

and an electric field at the target

$$E_{inc} = \left(Z\wp_{inc}\right)^{1/2} = \frac{(ZPG)^{1/2}}{(4\pi)^{1/2}R} \tag{3}$$

Z is the antenna's impedance. At the target, this field induces a current, which in turn radiates a scattered field back to the antenna. The scattered power flux at the antenna is

$$\wp_{sc} = \frac{\sigma}{4\pi R^2}\wp_{inc} \tag{4}$$

σ is the radar cross section. This leads to a scattered field

$$E_{sc} = \left(Z\wp_{sc}\right)^{1/2} = \left(Z\frac{\sigma}{4\pi R^2}\wp_{inc}\right)^{1/2} = \left(Z\frac{\sigma}{4\pi R^2}\frac{PG}{4\pi R^2}\right)^{1/2} = \frac{(\sigma ZPG)^{1/2}}{4\pi R^2} \tag{4}$$

If the distance R back to the receiver is large, this scattered field looks like a plane wave whose amplitude has fallen off by $1/R^2$. Then considering the receiver gain, we have $\wp_{rec} = G\wp_{sc}$ for the fluxes and the field induced at the receiver is

$$E_{rec} = G^{1/2}\cdot E_{sc} = \frac{(\sigma ZP)^{1/2}G}{4\pi R^2} \tag{5}$$

Then the received power is

$$P_{rec} = Z^{-1}E_{rec}^2 A = \sigma\frac{PGA}{(4\pi)^2R^4} = \sigma\frac{PG^2\lambda^2}{(4\pi)^3R^4} \tag{6}$$

which is proportional to $1/R^4$, the classical result.

III. NONLINEAR RADAR

This derivation fails for a nonlinear scatterer. To illustrate why, let us assume that the nonlinearity of the scatterer is small enough to admit a power-law expansion, a situation that illustrates our argument and is also simple mathematically (although other types of nonlinearities associated with clipping, Schottky barriers, etc., lead to similar results). For the linear case we found that

$$\wp_{sc} = \frac{\sigma}{4\pi R^2}\wp_{inc} \Rightarrow (Z\wp_{sc})^{1/2} = \left(\frac{\sigma}{4\pi R^2}\right)^{1/2}(Z\wp_{inc})^{1/2}$$
$$\Rightarrow E_{sc} = \left(\frac{\sigma}{4\pi R^2}\right)^{1/2} E_{inc} \tag{7}$$

We generalize this to a nonlinear target by writing

$$E_{sc} = \left(\frac{\sigma}{4\pi R^2}\right)^{1/2}\left[E_{inc} + \alpha E_{inc}^2 + \beta E_{inc}^3\right] \tag{8}$$

where α and β are constants that are specific to the target. Suppose we illuminate the target with two signals E_1 and E_2 close enough in frequency that the antenna can emit and receive both efficiently. Since the incident field E_{inc} contains two fields E_1 and E_2 at two frequencies ω_1 and ω_2, the scattered field E_{sc} will also contain components at these frequencies. For a linear target, there will be no frequencies in the return signal other than these. However, if we view the target as a "device" in the circuit-theoretic sense, we know that the scattered field induced by E_{inc} must also contain additional frequencies based on the quadratic expansion of

$$E_{inc} = E_1\cos(\omega_1 t + \phi_1) + E_2\cos(\omega_2 t + \phi_2) \tag{9}$$

which are

$$0,\ 2\omega_1,\ 2\omega_2,\ \omega_1+\omega_2,\ \text{and}\ \omega_1-\omega_2 \tag{11}$$

If the two initial frequencies are close to one another, i.e., $\omega_1 \approx \omega_2$, then all of these signals are outside the receiver's bandwidth, which may be useful or harmful depending on the application. Likewise, the contributions to E_{sc} from the cubic terms of E_{inc}, which contain the frequencies, are

$$3\omega_1,\ 3\omega_2,\ \omega_1,\ \omega_2,\ 2\omega_1+\omega_2,\ 2\omega_1-\omega_2,\ 2\omega_2+\omega_1,\ \text{and}\ 2\omega_2-\omega_1 \tag{12}$$

There are two things to note about this result:

- The responses at the illuminating radar frequencies ω_1 and ω_2 are no longer linear with amplitude, which makes the nominal cross sections at these frequencies field dependent.
- Unlike the frequencies generated by the squared term in E_{sc}, the fact that $\omega_1 \approx \omega_2$ implies that $2\omega_1 - \omega_2 \approx \omega_1$ and $2\omega_2 - \omega_1 \approx \omega_2$, i.e., these frequencies lie within the antenna bandwidth. In circuit theory these contributions are referred to as intermodulation products.

Let us consider only the in-band signals returning to the antenna. Assuming the antenna gain is the same for both fields E_1 and E_2, the scattered fields at the various frequencies take the form

$$E_{sc}(\omega_1) = \left(\frac{\sigma}{4\pi R^2}\right)^{1/2}\left(E_1 + \beta\left[\frac{3}{4}E_1^3 + \frac{3}{2}E_1^2 E_2\right]\right)$$
$$= \left(\frac{\sigma}{4\pi R^2}\right)^{1/2}\left(\frac{(ZG)^{1/2}}{(4\pi)^{1/2}R}P_1^{1/2} + \frac{3}{4}\beta\frac{(ZG)^{3/2}}{(4\pi)^{3/2}R^3}\left[P_1 + 2P_2\right]P_1^{1/2}\right)$$
$$E_{sc}(\omega_2) = \left(\frac{\sigma}{4\pi R^2}\right)^{1/2}\left(E_2 + \beta\left[\frac{1}{4}E_2^3 + \frac{3}{2}E_1 E_2^2\right]\right)$$
$$= \left(\frac{\sigma}{4\pi R^2}\right)^{1/2}\left(\frac{(ZG)^{1/2}}{(4\pi)^{1/2}R}P_2^{1/2} + \frac{3}{4}\beta\frac{(ZG)^{3/2}}{(4\pi)^{3/2}R^3}\left[P_2 + 2P_1\right]P_2^{1/2}\right) \tag{13}$$

$$E_{sc}(2\omega_1-\omega_2) = \left(\frac{\sigma}{4\pi R^2}\right)^{1/2}\left(\frac{3}{4}\beta E_1^2 E_2\right) = \left(\frac{\sigma}{4\pi R^2}\right)^{1/2}\left(\frac{3}{4}\beta\frac{(ZG)^{3/2}}{(4\pi)^{3/2}R^3}P_1 P_2^{1/2}\right)$$
$$E_{sc}(2\omega_2-\omega_1) = \left(\frac{\sigma}{4\pi R^2}\right)^{1/2}\left(\frac{3}{4}\beta E_1 E_2^2\right) = \left(\frac{\sigma}{4\pi R^2}\right)^{1/2}\left(\frac{3}{4}\beta\frac{(ZG)^{3/2}}{(4\pi)^{3/2}R^3}P_2 P_1^{1/2}\right) \tag{14}$$

Then the fields arriving with the intermod frequencies are

$$E_{rec}(2\omega_1-\omega_2) = G^{1/2}\cdot E_{sc}(2\omega_1-\omega_2) = \frac{3}{4}\sigma^{1/2}\beta\frac{Z^{3/2}G^2}{(4\pi)^2 R^4}P_1 P_2^{1/2}$$
$$E_{rec}(2\omega_2-\omega_1) = G^{1/2}\cdot E_{sc}(2\omega_2-\omega_1) = \frac{3}{4}\beta\sigma^{1/2}\frac{Z^{3/2}G^2}{(4\pi)^2 R^4}P_2 P_1^{1/2} \tag{15}$$

and the powers received at these frequencies are

$$P_{rec}(2\omega_1-\omega_2) = Z^{-1}E_{sc}(2\omega_1-\omega_2)^2 A = \frac{9}{16}\sigma\beta^2\frac{Z^2G^4A}{(4\pi)^4R^8}P_1^2P_2 = \frac{9}{16}\sigma\beta^2\frac{Z^2G^5\lambda^2}{(4\pi)^5R^8}P_1^2P_2$$
$$P_{rec}(2\omega_2-\omega_1) = Z^{-1}E_{sc}(2\omega_2-\omega_1)^2 A = \frac{9}{16}\sigma\beta^2\frac{Z^2G^4A}{(4\pi)^4R^8}P_2^2P_1 = \frac{9}{16}\sigma\beta^2\frac{Z^2G^5\lambda^2}{(4\pi)^5R^8}P_2^2P_1 \tag{16}$$

Looking past the complexity of these expressions, let us focus on the R–dependence of the intermodulation terms [5]. (1) reveals that $P_{rec}(2\omega_1-\omega_2)$ and $P_{rec}(2\omega_2-\omega_1)$ both fall off as $1/R^8$ rather than the traditional $1/R^4$. Note that this drastic attenuation of receiver power at the new frequencies will be somewhat mitigated by the increased antenna gain coefficient G^5 [6].

At first glance, it appears that the far-field expressions we have derived for the gain imply that the intermods are undetectable, since there is no power at infinity from a field that falls off faster than $1/R$. To counter this assertion, we consider the bistatic case. Let the transmit antenna have a gain G_t and be a distance R_t from the target, while the receive antenna has a gain G_r and is at a distance R_r from the target. Assume that both R_t and R_r are large compared to wavelength and target dimensions. Then the same derivation as above leads to the following expression for the intermod fields at the scatterer:

$$E_{sc}(2\omega_1-\omega_2)=\left(\frac{\sigma}{4\pi R_r^2}\right)^{1/2}\left(\frac{3}{4}\beta E_1^2E_2\right)=\left(\frac{\sigma}{4\pi R_r^2}\right)^{1/2}\left(\frac{3}{4}\beta\frac{(Z_tG_t)^{3/2}}{(4\pi)^{3/2}R_t^3}P_1P_2^{1/2}\right)$$
$$E_{sc}(2\omega_2-\omega_1)=\left(\frac{\sigma}{4\pi R_r^2}\right)^{1/2}\left(\frac{3}{4}\beta E_1E_2^2\right)=\left(\frac{\sigma}{4\pi R_r^2}\right)^{1/2}\left(\frac{3}{4}\beta\frac{(Z_tG_t)^{3/2}}{(4\pi)^{3/2}R_t^3}P_2P_1^{1/2}\right) \quad (17)$$

Then the fields detected by the receiver are of the form

$$E_{rec}(2\omega_1-\omega_2)=G_r^{1/2}\cdot E_{sc}(2\omega_1-\omega_2)=\frac{3}{4}\sigma^{1/2}\beta\frac{Z_t^{3/2}G_r^{1/2}G_t^{3/2}}{(4\pi)^2R_rR_t^3}P_1P_2^{1/2}$$
$$E_{rec}(2\omega_2-\omega_1)=G_r^{1/2}\cdot E_{sc}(2\omega_2-\omega_1)=\frac{3}{4}\sigma^{1/2}\beta\frac{Z_t^{3/2}G_r^{1/2}G_t^{3/2}}{(4\pi)^2R_rR_t^3}P_2^{1/2}P_1 \quad (18)$$

These fields now have a factor of $\frac{G_t^{3/2}G_r^{1/2}}{R_t^3R_r}$, so that the intermod powers received at the antenna go as $\frac{G_t^3G_r^2}{R_t^6R_r^2}$. Clearly the initial trip from transmitter to target satisfies the far-field condition by construction, while the $1/R_r^2$ behavior arising from the trip from target to receive antenna shows *prima facie* that it is also satisfied on this leg of the trip as well. There will, in fact, be near-field contributions to the radiation from the scatterer, but they contain higher powers of $1/R_r^2$ and hence are even more attenuated by distance. Note that in the bistatic case the coefficient β may depend on the angle between the receiver and transmitter boresights.

IV. POWER BUDGETS

To see how these results affect the properties of the radar, we consider an example from Hovhaness's book [4] of an S-band radar with the following parameters:

Antenna parameters:

$$\begin{cases}\text{Diameter: 50 in } =1.27\text{ m} \Rightarrow \text{ area } A=1.25\text{ m}^2\\ \text{Power: } P_t=10^3\text{ W}\\ \text{Frequency: 3 GHz} \Rightarrow \text{ wavelength } =.1\text{ m}\\ \text{Loss factor: 10}\end{cases}$$

Receiver parameters:

$$\begin{cases}\text{Noise figure: } F = 2.66\\ \text{Noise temperature: } T = 300\text{ K}\\ \text{Noise bandwidth: } B_n = 1000\text{ Hz}\end{cases}$$

Target parameter: radar cross section = 90.9 m^2

Using the linear radar equation, we can derive the following expression for the signal-to-noise ratio (SNR) of the radar in dB versus distance:

$$\left(\frac{\text{S}}{\text{N}}\right)_R=10\log\left\{\frac{P_{rec}}{(k\cdot FT\cdot B_n)L}\right\}=10\log\left\{\frac{P_tG^2\lambda^2\sigma}{(4\pi)^3R^4(F\cdot k_BT\cdot B_n)L}\right\} \quad (19)$$

where k_B is Boltzmann's constant. The maximum range, defined by the distance at which $\left(\frac{\text{S}}{\text{N}}\right)_{R_0}=0\text{ dB}$, is easily found to be

$$R_0=\left[\frac{PG^2\lambda^2\sigma}{(4\pi)^3(k\cdot FT\cdot B_n)L}\right]^{1/4}\Rightarrow\left(\frac{\text{S}}{\text{N}}\right)_R=40\log\left(\frac{R_0}{R}\right)^4 \quad (20)$$

Using this expression, we easily find that $R_0=173$ nautical miles, while the SNR of the radar at distances of 20, 40, and 100 nautical miles are

$$\left(\frac{\text{S}}{\text{N}}\right)=\begin{cases}\text{37.5 dB at 20 nautical miles}\\ \text{25.4 dB at 40 nautical miles}\\ \text{9.5 dB at 1000 nautical miles}\end{cases} \quad (21)$$

Let us now consider the same expression for a two-tone radar receiving the intermod $2\omega_1-\omega_2$ with the same parameters. The corresponding expressions are

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)_R = 10\log\left\{\frac{P_{rec}(2\omega_1-\omega_2)}{(k\cdot FT\cdot B_n)\,L}\right\} = 10\log\left\{\frac{9}{16}\sigma\beta^2\frac{Z^2G^5\lambda^2}{(4\pi)^5R^8k\cdot FT\cdot B_n\cdot L}P_1^2P_2\right\}$$
$$= 10\log\left\{\frac{9}{16}\sigma\beta^2\frac{Z^2G^5m_1^2m_2\lambda^2}{(4\pi)^5R^8(k\cdot FT\cdot B_n)\cdot L}P_T^3\right\}$$
$$R_0 = \left\{\frac{9}{16}\sigma\beta^2\frac{Z^2G^5\lambda^2m_1^2m_2}{(4\pi)^5k\cdot FT\cdot B_n\cdot L}P_T^3\right\}^{1/8} \Rightarrow \left(\frac{\mathrm{S}}{\mathrm{N}}\right)_R = 80\log\left(\frac{R_0}{R}\right) \tag{22}$$

where P_T is the total power transmitted, $P_{1,2} = Z^{-1}E_{1,2}^2$ are transmitted powers at the two frequencies, and we define the power fractions $m_{1,2} = \frac{P_{t1,2}}{P_T}$. For this radar, we find that $R_0 = 1.45$ nautical miles, i.e., the range is drastically reduced, while the SNR of the radar at distances of 20, 40, and 100 nautical miles are now:

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right) = \begin{cases} -91.2 \text{ dB at 20 nautical miles} \\ -115.3 \text{ dB at 40 nautical miles} \\ -147.2 \text{ dB at 100 nautical miles} \end{cases} \tag{23}$$

Note that R_0 goes as $P_T^{3/8}$, so that recovering the original range of the radar (173 miles) requires that the power be increased to 345 MW [7].

V. MULTIPATH EFFECTS

In situations where the radar and the target are both relatively close to the ground, multipath effects can further exacerbate this range-dependent attenuation. Although in rare cases multipath may actually enhance a target return, the most likely effect at low grazing angles is destructive interference caused by the ground reflection. Vertical polarization can provide some advantages due to pseudo-Brewster angle effects, but the Brewster angle ranges from about 13° (moist soils) to about 30° (arid soils) while the grazing angle from a 2-m-high sensor to a point 100 m away is 1.1°. At these shallow angles the behavior of both horizontally and vertically polarized waves is essentially the same, with an inversion of sign upon reflection, and with very little loss in amplitude.

In general, the effect of multipath cancellation can add another $1/R^2$ to the power dependence each way. To demonstrate where this dependence comes from, consider a radar illuminating a target near the ground as shown in Fig. 5.

An antenna at a height h off the ground illuminates a point on a target a horizontal distance R, where the point is a height s off the ground. The signal from the antenna is taken to be the sum of a direct propagation from antenna to target and an indirect propagation involving a bounce off the ground at a horizontal distance αR from the target, with $\alpha<1$. Assuming that the reflection causes a complete phase reversal with no loss of amplitude (again, this is a reasonable assumption for shallow grazing angles, with horizontal polarization), the signal at the target is of the form

$$E = E_0(R)\left(\exp\left[i\frac{\omega}{c}D\right] - \exp\left[i\frac{\omega}{c}(D_1+D_2)\right]\right) = E_0(R)\exp\left[i\frac{\omega}{c}D\right]\left(1-\exp\left[i\frac{\omega}{c}(D_1+D_2-D)\right]\right) \tag{24}$$

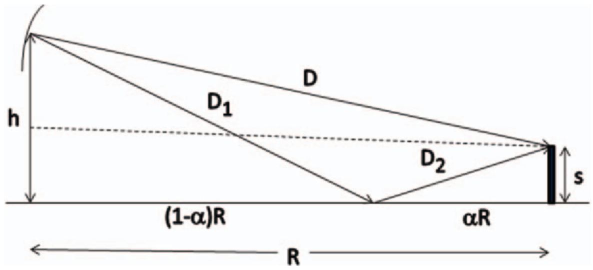


Figure 1. Multipath geometry.

where $E_0(R) \propto \frac{1}{R}$ as before, and

$$\begin{aligned} D_1 &= \sqrt{h^2 + (1-\alpha)^2R^2} \\ D_2 &= \sqrt{s^2 + \alpha^2R^2} \\ D &= \sqrt{(h-s)^2 + R^2} \end{aligned} \tag{25}$$

Now, the quantity in brackets will vanish when $D_1 + D_2 - D = m\lambda$. Note that the triangle inequality insures that $D_1 + D_2 > D$, so m = 0 cannot be satisfied. At distances $R >> h, s$ we find that

$$D_1 + D_2 - D \approx \frac{1}{2R}\left[h^2\left(\frac{\alpha}{1-\alpha}\right) + s^2\left(\frac{1-\alpha}{\alpha}\right) - 2hs\right] \tag{26}$$

and so

$$\begin{aligned} E &\approx E_0(R)\exp\left[i\frac{\omega}{c}D\right]\left(1-\exp\left[i\frac{\omega}{c}(D_1+D_2-D)\right]\right) \\ &\propto \frac{1}{R}\exp\left[i\frac{\omega}{c}D\right] \end{aligned} \tag{27}$$

Clearly this change in amplitude at the target corresponds to a $1/R^4$ change in the power from antenna to target, and a $1/R^8$ change for the round trip, even for a linear target. In Fig. 6 we plot the amplitude, in dB, versus the range to the target for a radar located 2 m off the ground illuminating a target 0.25 m off the ground. The 6-dB peaks in the signals for the 2000 and 3000 MHz curves is due to constructive interference, but further out in range all of the curves approach a 20-dB/decade falloff in amplitude, thus demonstrating the $1/R^2$ rule. Consider the region between 15 and 20 m in the example of Fig. 5. Here the illumination near 1000 MHz is definitely influenced by an additional $1/R^2$ term and hence a $1/R^8$ falloff is expected. Combining this with the effect of the nonlinear target gives a total falloff of $1/R^{16}$. This would mean a change of range from 50 to 100 m would result in a loss of signal of 2^{16} or 48 dB per octave or 160/dB per decade.

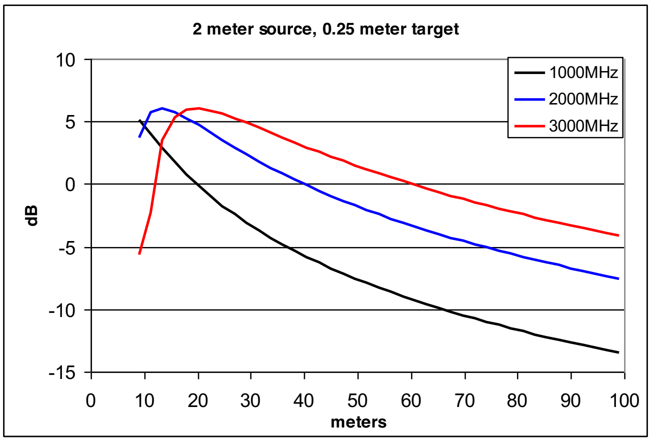
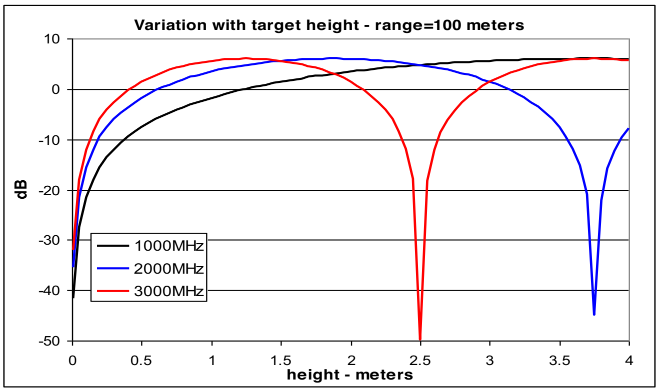


Figure 2. Range dependence of signal in a multipath environment.

Multipath is of particular importance when the target height above ground can vary quickly, as is the case when insects are tagged with nonlinear transponders [1,2]. In flight the insect can appear as a free-space target, but as it approaches or crawls along the ground, its effective cross section is seriously reduced, as can be seen in Fig. 7. Here we examine the effects of varying the target height at a fixed range of 100 m from the radar. Note that there are situations where additional nulls can occur (e.g., 3000 MHz at 2.5 m height, 2000 MHz at 3.75 m height) and this happens more frequently at smaller wavelengths and may cause a fast variation in target signature as the target height varies.



Change in target strength versus height of target for a target range of 100 meters

VI. CONCLUSION

Our conclusions here should not be construed as advocating any modification of the radar equation itself. Rather, we are simply asserting that in nonlinear radar the RCS of a target is no longer independent of distance from the source, and has to be redefined for each order of nonlinearity in the target response.

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