

# Transient Non-Conformal Domain Decomposition Using the Laguerre-FDTD Method

Ming Yi, and Madhavan Swaminathan  
School of Electrical and Computer Engineering  
Georgia Institute of Technology  
Atlanta, GA, United States

myi9@gatech.edu, madhavan.swaminathan@ece.gatech.edu

**Abstract**—In this paper, a transient non-conformal domain decomposition scheme is proposed based on the unconditionally stable Laguerre-FDTD method. Field continuity at the non-conformal domain interface is ensured by applying a mortar-element-like method. A time-derivative Lagrange multiplier is introduced at the domain interface which physically represents the interface current excitation. Simulation results have been presented to demonstrate the accuracy and efficiency of the proposed scheme.

**Index Terms**—Domain decomposition, non-conformal, time-domain, Laguerre-FDTD.

## I. INTRODUCTION

The domain decomposition method has been widely used to solve electromagnetic problem for decades. Different from the other numerical methods, the domain decomposition method does not solve the entire computational domain directly, but evaluates each decomposed sub-domain individually by imposing certain interface boundary condition. In recent years, the non-conformal domain decomposition methods, such as mortar-element method and cement method, received wide attention since they largely relaxed mesh generation process by introducing non-conformal domain interface [1]–[3].

However, most of the non-conformal domain decomposition schemes are implemented in the frequency domain based on the finite element method (FEM). In time domain, it is difficult to address the field updating on the non-conformal domain interface for finite-difference time-domain (FDTD) based methods. Sub-gridding and interpolation methods has been applied but only with fixed mesh ratio on the domain interface [4].

In this paper, a transient non-conformal domain decomposition scheme is proposed based on the unconditionally stable Laguerre-FDTD method [5]–[8]. By applying the equivalency between the time-domain FEM (TD-FEM) and the Laguerre-FDTD method, a mortar-element-like method can be used to address the interface non-conformality. A time-derivative Lagrange multiplier term is introduced to couple the adjacent field with physical meaning of boundary excitation.

## II. PROPOSED SCHEME

Assuming an isotropic, non-dispersive, lossless media in two-dimensional Cartesian coordinates, the vector wave equation in time domain can be expressed as

$$\nabla \times \nabla \times \mathbf{E} + \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu \frac{\partial \mathbf{J}}{\partial t}. \quad (1)$$

Multiply (1) by an appropriate testing function  $\mathbf{N}$ , and integrate over domain results in

$$\int_{\Omega} \left[ (\nabla \times \mathbf{N}) \cdot (\nabla \times \mathbf{E}) + \mu\epsilon \mathbf{N} \cdot \frac{\partial^2 \mathbf{E}}{\partial t^2} \right] dS = - \int_{\Omega} \mu \mathbf{N} \cdot \frac{\partial \mathbf{J}}{\partial t} dS. \quad (2)$$

In the computational domain, considering the TE<sub>z</sub> case, the electric field is expanded with vector basis functions

$$\mathbf{E} = \sum_{i=1}^n \mathbf{N}_i E_i \quad (3)$$

where  $n$  is the total edge number,  $E_i$  is the unknown expansion coefficient,  $\mathbf{N}_i$  is the vector basis function. Inserting (3) into (2), with some manipulations, the Laguerre domain coefficient equation for electric field in  $x$ -direction is

$$\begin{aligned} & \left( \frac{1}{\bar{C}_y^E |_{i,j}} + \bar{C}_y^H |_{i,j} + \bar{C}_y^H |_{i,j-1} \right) E_x^q |_{i,j} \\ & + \bar{C}_x^H |_{i,j} E_y^q |_{i+1,j} - \bar{C}_x^H |_{i,j} E_y^q |_{i,j} \\ & - \bar{C}_y^H |_{i,j+1} E_x^q |_{i,j+1} - \bar{C}_x^H |_{i,j-1} E_y^q |_{i+1,j-1} \\ & + \bar{C}_x^H |_{i,j-1} E_y^q |_{i,j-1} - \bar{C}_y^H |_{i,j-1} E_x^q |_{i,j-1} \\ & = -\Delta \bar{y}_j J_x^q |_{i,j} - \frac{4}{\bar{C}_y^E |_{i,j}} \sum_{n=0, q>1}^{q-1} \sum_{m=0}^n E_x^m |_{i,j}. \end{aligned} \quad (4)$$

where the superscript  $q$  denotes the Laguerre coefficient of order  $q$ . Note that (4) is derived from TD-FEM with Laguerre discretization. This form is equivalent to the system equation obtained using finite-difference scheme in [5]. Therefore, by maintaining the system equation inside each domain consistent with the standard Laguerre-FDTD method, the coupling between adjacent domains can be addressed using a mortar-element-like method.

For simplicity, considering the computational domain dividing into two sub-domains  $\Omega_1$  and  $\Omega_2$  with non-conformal domain interface as is shown in Fig. 1. Defining the Lagrange multiplier space as

$$\lambda = \sum_{i=1}^n \varphi_i \lambda_i \quad (5)$$

where  $n$  is the total number of expansion terms,  $\lambda_i$  is the unknown expansion coefficient,  $\varphi_i$  is the vector basis function.

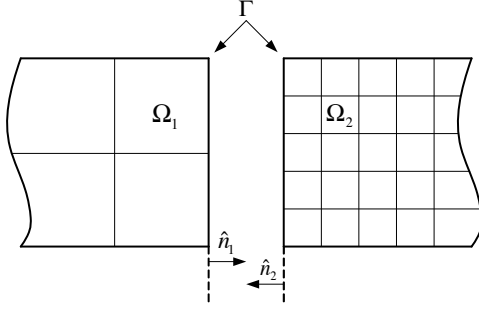


Fig. 1. Two domains with non-conformal domain interface in the domain decomposition scheme.

Different from other mortar-element schemes which introducing the Lagrange multiplier with no physical meanings, a time-derivative Lagrange multiplier term is defined here which is proportional to the boundary current excitation:

$$\frac{\partial \mathbf{J}^{eq}}{\partial t} = \frac{\partial}{\partial t} \sum_{i=1}^n \varphi_i \mathbf{j}_i^{eq} \quad (6)$$

The equations for the sub-domains and the interface are obtained as

$$\int_{\Omega_1} \left[ (\nabla \times \mathbf{N}_1) \cdot (\nabla \times \mathbf{E}_1) + \mu \varepsilon \mathbf{N}_1 \cdot \frac{\partial^2 \mathbf{E}_1}{\partial t^2} \right] dS + \int_{\Gamma} \mathbf{N}_1 \cdot \frac{\partial \mathbf{J}_1^{eq}}{\partial t} dl = - \int_{\Omega_1} \mu \mathbf{N}_1 \cdot \frac{\partial \mathbf{J}_1}{\partial t} dS \quad (7)$$

$$\int_{\Omega_2} \left[ (\nabla \times \mathbf{N}_2) \cdot (\nabla \times \mathbf{E}_2) + \mu \varepsilon \mathbf{N}_2 \cdot \frac{\partial^2 \mathbf{E}_2}{\partial t^2} \right] dS + \int_{\Gamma} \mathbf{N}_2 \cdot \frac{\partial \mathbf{J}_2^{eq}}{\partial t} dl = - \int_{\Omega_2} \mu \mathbf{N}_2 \cdot \frac{\partial \mathbf{J}_2}{\partial t} dS \quad (8)$$

$$\int_{\Gamma} (\mathbf{E}_1 - \mathbf{E}_2) \cdot \boldsymbol{\varphi} dS = 0. \quad (9)$$

Note that at the domain interface, there is no source excitation, therefore, the right-hand side of (7) and (8) equals zero. Also, we have

$$\mathbf{J}_1^{eq} = -\mathbf{J}_2^{eq} = \mathbf{J}^{eq}. \quad (10)$$

Expanding the electric field using (3) and discretizing in Laguerre domain yields

$$\left( \frac{s^2}{4} \mathbf{T}_1 + \mathbf{S}_1 \right) \mathbf{E}_1^q + \frac{s}{2} \mathbf{B}_1^T \mathbf{j}^{eq,q} = -\mathbf{T}_1 s^2 \sum_{n=1, q>0}^{q-1} \sum_{m=0}^n \mathbf{E}_1^m - s \mathbf{B}_1^T \sum_{n=1, q>0}^{q-1} \mathbf{j}^{eq,n} - \mathbf{f}_1^q \quad (11)$$

$$\left( \frac{s^2}{4} \mathbf{T}_2 + \mathbf{S}_2 \right) \mathbf{E}_2^q - \frac{s}{2} \mathbf{B}_2^T \mathbf{j}^{eq,q} = -\mathbf{T}_2 s^2 \sum_{n=1, q>0}^{q-1} \sum_{m=0}^n \mathbf{E}_2^m + s \mathbf{B}_2^T \sum_{n=1, q>0}^{q-1} \mathbf{j}^{eq,n} - \mathbf{f}_2^q \quad (12)$$

$$\mathbf{B}_1 \mathbf{E}_1^q - \mathbf{B}_2 \mathbf{E}_2^q = \mathbf{0} \quad (13)$$

where the coupling matrix  $\mathbf{B}_i$  is expressed as

$$\mathbf{B}_i = \int_{\Gamma} \mathbf{N}_i \cdot \boldsymbol{\varphi}_i dS. \quad (14)$$

To eliminate the redundant unknown vector  $\mathbf{j}^{eq,q}$ , relationship between unknown vector  $\boldsymbol{\lambda}^q$  and  $\mathbf{j}^{eq,q}$  are established by projection matrices related to the interface meshing as:

$$\frac{s}{2} \mathbf{B}_1^T \mathbf{j}^{eq,q} = s \mathbf{P}_1 \mathbf{B}_1^T \boldsymbol{\lambda}^q = \mathbf{C}_1^T \boldsymbol{\lambda}^q \quad (15)$$

$$\frac{s}{2} \mathbf{B}_2^T \mathbf{j}^{eq,q} = -s \mathbf{P}_2 \mathbf{B}_2^T \boldsymbol{\lambda}^q = -\mathbf{C}_2^T \boldsymbol{\lambda}^q \quad (16)$$

The resulting linear system for two domains can therefore be obtained as

$$\begin{bmatrix} \mathbf{K}_1 & \mathbf{0} & \mathbf{C}_1^T \\ \mathbf{0} & \mathbf{K}_2 & \mathbf{C}_2^T \\ \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{E}_1^q \\ \mathbf{E}_2^q \\ \boldsymbol{\lambda}^q \end{bmatrix} = \begin{bmatrix} \mathbf{h}_1^q \\ \mathbf{h}_2^q \\ \mathbf{0} \end{bmatrix}. \quad (17)$$

Note that  $\mathbf{K}_1$ ,  $\mathbf{K}_2$  matrices are equivalent to the left side of (4) which is corresponded to the field components inside each domain. Matrix  $\mathbf{h}_1^q$  and  $\mathbf{h}_2^q$  represent the combined excitation inside each domain and the time derivative equivalent current. The coupling of two domains are represented by  $\mathbf{B}_1$  and  $\mathbf{B}_2$  matrices.

By eliminating the field components  $\mathbf{E}_1^q$  and  $\mathbf{E}_2^q$  in (17), the interface equation can be derived as

$$(\mathbf{B}_1 \mathbf{K}_1^{-1} \mathbf{C}_1^T + \mathbf{B}_2 \mathbf{K}_2^{-1} \mathbf{C}_2^T) \boldsymbol{\lambda}^q = \mathbf{B}_1 \mathbf{K}_1^{-1} \mathbf{h}_1^q + \mathbf{B}_2 \mathbf{K}_2^{-1} \mathbf{h}_2^q. \quad (18)$$

After the interface problem is solved, the coefficient of the electric field inside each domain can be evaluated individually using

$$\mathbf{K}_1 \mathbf{E}_1^q = \mathbf{h}_1^q - \mathbf{C}_1^T \boldsymbol{\lambda}^q \quad (19)$$

$$\mathbf{K}_2 \mathbf{E}_2^q = \mathbf{h}_2^q - \mathbf{C}_2^T \boldsymbol{\lambda}^q. \quad (20)$$

Clearly, by introducing the time-derivative Lagrange multiplier, the original problem is reduced to sub-problems with fewer degrees of freedom. The individual sub-problems can be solved in a parallel manner which makes the solution faster especially for large scale problems.

### III. NUMERICAL RESULTS

To validate the proposed non-conformal domain decomposition method, a 2-D parallel plate waveguide shown in Fig. 2 is simulated. The length and width of the structure are  $l = 40$  mm and  $w = 20$  mm, respectively. The waveguide is divided into two domains with line current excitation located inside domain one and observation probe located in the center of domain two. Fig. 3 shows the time-domain waveform at the probe using domain decomposition with different interface grid ratio. For comparison, the conventional FDTD simulation is performed with single domain using the same meshing strategy as in domain one in the decomposed scheme. It can be observed that results of the conventional FDTD method and the decomposed scheme with different grid ratio correlate well.

The scattering from a PEC cylinder with square cross-section is investigated. Plane wave is excited at the left of the

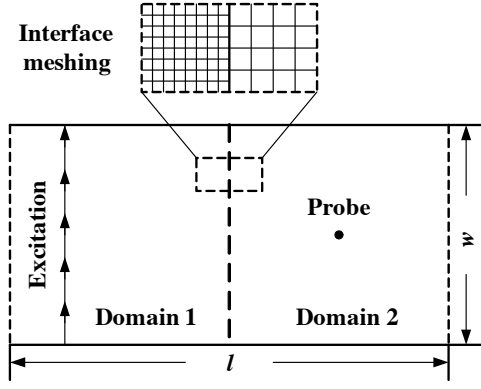


Fig. 2. 2-D parallel plate waveguide separated into two domains with non-conformal domain interface.

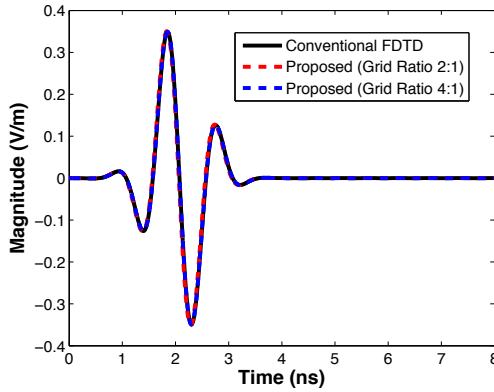


Fig. 3. Comparison of the time-domain waveform of the 2-D parallel plate waveguide.

object shown in Fig. 4. The wavelength corresponding to the upper frequency bound is chosen as  $\lambda = 1 \times 10^{-2}$  m, and the side length of the square is  $a = 2 \times 10^{-2}$  m. The calculation domain is decomposed into two sub-domains with interface grid ratio of 4:1. For comparison, the scattering problem is also simulated with conventional FDTD method with single domain. Fig. 5 shows the radar cross section (RCS) result. The CPU time of the conventional FDTD method and the domain decomposition method are 50 s and 21 s, respectively. The domain decomposition scheme is accurate with improved computational efficiency.

#### IV. CONCLUSION

A transient non-conformal domain decomposition method has been proposed. By applying the mortar-element-like method including the time-derivative Lagrange multiplier in the Laguerre domain, the method is able to tackle with non-conformal domain interface with field continuity. The numerical results show that the proposed method has good accuracy with improved computational efficiency compared to standard non-decomposed schemes.

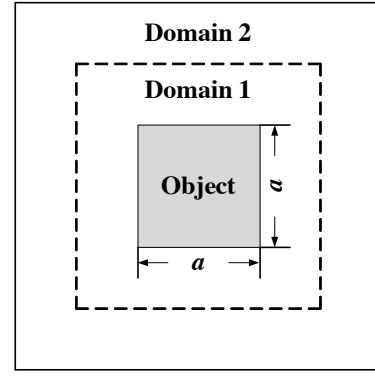


Fig. 4. PEC cylinder structure divided into two domains with non-conformal domain interface.

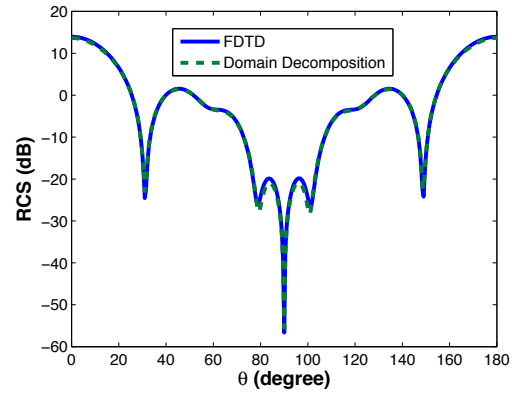


Fig. 5. Comparison of the RCS of the PEC cylinder structure.

#### REFERENCES

- [1] F. Rapetti, "The mortar edge element method on non-matching grids for eddy current calculations in moving structures," *Inter. J. Numerical Modeling: Electronic Networks, Devices and Fields*, vol. 14, no. 6, pp. 457-477, Nov. 2001.
- [2] M. Vouvakis, Z. Cendes, and J.-F. Lee, "A FEM domain decomposition method for photonic and electromagnetic band gap structures," *IEEE Trans. Antennas Propag.*, vol. 54, no. 2, pp. 721-733, Feb. 2006.
- [3] M.-F. Xue, and J.-M. Jin, "Nonconformal FETI-DP methods for large-scale electromagnetic simulation," *IEEE Trans. Antennas Propag.*, vol. 60, no. 9, pp. 4291-4305, Sept. 2012.
- [4] N. Venkatarayalu, R. Lee, Y. Gan, and L. Li, "A stable FDTD subgridding method based on finite element formulation with hanging variables," *IEEE Trans. Antennas Propag.*, vol. 55, no. 3, pp. 907-915, Mar. 2007.
- [5] Y.-S. Chung, T. Sarkar, B. Jung, and M. Salazar-Palma, "An unconditionally stable scheme for the finite-difference time-domain method," *IEEE Trans. Microw. Theory Tech.*, vol. 51, no. 3, pp. 697-704, Mar. 2003.
- [6] M. Yi, M. Ha, Z. Qian, A. Aydin, and M. Swaminathan, "Skin-effect-incorporated transient simulation using the Laguerre-FDTD scheme," *IEEE Trans. Microw. Theory Tech.*, vol. 61, no. 12, pp. 4029-4039, Dec. 2013.
- [7] M. Yi, M. Swaminathan, Z. Qian, and A. Aydin, "Skin effect modeling of interconnects using the Laguerre-FDTD scheme," in *Proc. IEEE Electron. Performance Electron. Packag. Systems Conf.*, Oct. 2012, pp. 236-239.
- [8] M. Yi, M. Swaminathan, M. Ha, Z. Qian, and A. Aydin, "Memory efficient Laguerre-FDTD scheme for dispersive media," in *Proc. IEEE Electron. Performance Electron. Packag. Systems Conf.*, Oct. 2013, pp. 51-54.