

Moving Ground Target Isolation by a UAV Using Predicted Observations

D. Casbeer, K. Krishnamoorthy, P. Chandler, and M. Pachter

Abstract—An interesting variant of the cops and robbers pursuit-evasion problem is presented. The motivation for the problem comes from the need to monitor a network of roads leading to a protected area. In this paper, an unmanned aerial vehicle (UAV) is searching for a slower moving ground vehicle on a road network that has been out-fitted with unattended ground sensors (UGSs). The UGSs are communication constrained and can only relay observations of the ground vehicle when the UAV visits them. Furthermore, the UAV does not have on-board capability to detect and identify the evader and therefore must investigate the UGSs to learn the evader's location. This scenario leads the UAV's control actions to depend on partial information. A sufficient condition for guaranteed isolation of the intruder is provided along with the corresponding pursuit policy. Since the policy is non-unique, we select a policy that provides the least time to capture among all policies with the guarantee.

I. INTRODUCTION

The UAV-UGS (unmanned aerial vehicle-unattended ground sensor) search and capture problem (U2SC) is a variant of a discrete pursuit-evasion game [1].¹ In discrete pursuit-evasion games, a pursuer and evader takes turns moving from vertex to vertex on a graph. The pursuer wins if it is able to isolate the evader, where isolation means that the pursuer is at the same location as the evader in finite-time, otherwise, the evader wins. Many variations to this problem have been investigated over the years (see [2], [3] for a survey). Typical formulations of discrete pursuit-evasion games deal with evaders that are either invisible or visible at all times (full information). One particular formulation of the problem that is applicable to the U2SC was pursued by [4], where observations of the evader were made by witnesses throughout the graph. The witnesses in [4] are similar in functionality to the UGSs in the U2SC, but they differ in how the observations are reported. In the former, they are reported to the pursuer immediately, whereas the UGS observations are only available when the UAV visits it and are hence delayed.

In the U2SC, the UGS (witnesses) are placed on intersections (nodes) of the road network (graph), and the evader

can win in finite time if it is able to reach a “goal” node before the UAV isolates him. When the evader passes by an UGS, the UGS records the event and time. The UGS have limited communication capability, therefore the UAV/pursuer must fly to an UGS so that the UGS can upload the observations to the UAV. Using these observations, the UAV must decide where to go next, so that it can localize the evader while simultaneously working to isolate the evader (dual-control). The observations are delayed, so the UAV acts on partial information. This makes the problem considered herein harder to solve.

The resulting uncertainty in the evader's location is modeled as the inclusion/exclusion to a prescribed set [5], and we seek a “worst-case” guarantee for isolation [6], [7]. In order to accomplish this task, the UAV must do state-estimation, discussed in a similar setting in [8], [9]. More recently, [10] used the concept of reachability but applied to the full-state problem with bounded ellipsoid regions rather than partial information with discrete sets, and the authors developed an efficient toolbox for implementing their approach. The solution presented in this paper closely follows that of [11] but is different in two important aspects: 1) we deal with discrete-valued state variables and uncertainties and 2) the estimation process must account for delayed observations of the evader's physical state.

More recently, a closely related area of control synthesis has gained attention in the computer science community. The idea is to automatically develop a control policy that satisfies a particular specification, which is typically done using linear temporal logic [12], [13], [14]. In the context of U2SC, the control synthesis problem is to provide a specification that guarantees isolation. The ideas in control synthesis are not directly applicable to the U2SC, because of the delayed and in-complete observation model.

The U2SC was originally posed in [15], where a weaker sufficient condition for evader isolation was given. In this paper, we provide a tighter sufficiency condition for guaranteed isolation. In addition, we also compute the pursuit policy that gives (worst-case) minimum time capture, among all policies that satisfy the sufficient condition. In related work [16], [17], where the road network is restricted to a Manhattan grid, the optimal min-max time to capture (and the corresponding pursuit policy) is provided for both single and two UAVs pursuing an evader. Indeed, we use the Manhattan grid as an example problem in this work to illustrate our solution method. The authors have also investigated the U2SC in a continuous-time setting [18].

The rest of the paper is organized as follows. In Section II,

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¹Discrete pursuit-evasion games are also known as graph searching or “cops and robbers” games.

we give a mathematical formulation of the search problem. The solution method including delayed observations/partial information is presented in Section III. We follow the analysis with an example problem in Section IV and concluding remarks in Section V. We note that the introduction and analysis sections of the paper closely mirror the respective sections in our prior work [15]. This is understandable, given that the current work extends on the prior work and addresses the same problem.

II. PROBLEM SETUP

A. UAV and evader Dynamics

The goal of designing a control policy, which guarantees isolation based on the evader's location, motivates modeling the system dynamics as a combination of both the evader and UAV states. This model is described by the equation:

$$x(k+1) = f(x(k), u(k), w(k)), \quad (1)$$

where, at time k , the (joint) system state is denoted by $x(k)$, the UAV control action is $u(k)$ and the evader's action is $w(k)$. The system state $x(k) = (p_i(k), d_i(k), p_u(k))$ entails the following information: the UAV's current location $p_u(k)$ and the evader's current location $p_i(k)$ and orientation $d_i(k)$. Since the evader and UAV dynamics are de-coupled, (1) can be split into the two equations:

$$p_u(k+1) = f_u(p_u(k), u(k)), \text{ and} \quad (2)$$

$$s_i(k+1) = f_i(s_i(k), w(k)), \quad (3)$$

where $s_i(k) = (p_i(k), d_i(k))$. We assume that $p_u(k) \in P_u(k) \subset \mathcal{P}_u$, $s_i(k) \in S_i(k) \subset \mathcal{S}_i$, $p_i(k) \in P_i(k) \subset \mathcal{P}_i$, $d_i(k) \in D(p_i(k)) \subset \mathcal{D}$, where the sets $\mathcal{P}_u$, $\mathcal{S}_i$, $\mathcal{P}_i$, and $\mathcal{D}$ are all finite and $\mathcal{S}_i = \mathcal{P}_i \times \mathcal{D}$. Furthermore, we assume that an Unattended Ground Sensor (UGS) is located at every possible UAV location, $p \in \mathcal{P}_u$ and also that the intersection $\mathcal{P}_u \cap \mathcal{P}_i$ is non-empty. We define a subset of the set of all common locations, $\mathcal{G} \subset (\mathcal{P}_i \cap \mathcal{P}_u)$ to be the goal locations that the evader is heading towards. The evader's action, $w(k) \in W(s_i(k)) \subset \mathcal{W}$ and the UAV's action, $u(k) \in U(p_u(k)) \subset \mathcal{U}$ where $\mathcal{W}$ and $\mathcal{U}$ are finite sets. We assume that the sets $\mathcal{P}_u$ and $\mathcal{U}$ are consistent with each other, i.e., if $p_u(k) \in \mathcal{P}_u(k)$ and $u(k) \in U(p_u(k))$, then $p_u(k+1) \in \mathcal{P}_u(k+1)$. Similarly, we assume that the sets $\mathcal{P}_i$, $\mathcal{D}$ and $\mathcal{W}$ are also consistent.

We define a policy for the evader, $\pi_i = \{w(0), w(1), \dots\}$ to be a mapping $\pi_i : \mathcal{S}_i \rightarrow \mathcal{W}$ from evader state to action. We assume that, if un-intercepted, the evader will eventually reach a goal location in finite time; regardless of the policy it employs. So, for a given initial evader state $x_i(0)$, and a policy π , let $N_\pi (< \infty)$ be the time taken by the evader (if un-intercepted) to reach a goal location. We define the horizon, $N = \max_{\pi \in \Pi_i} N_\pi$, where Π_i is the set of all possible evader policies. Here, we have tacitly assumed that the evader cannot visit the same location more than once i.e., revisits are not allowed.

The UAV alone cannot isolate the evader because the UAV does not have the onboard capability to detect the

evader; it must rely on the UGS for observations of the evader. When the UAV visits an UGS, the UGS delivers a noise-free observation $z(k) \in \mathcal{Z}$ of the evader, where $\mathcal{Z} = \{G\} \cup \{0, 1, 2, \dots\}$. The observation² $z(k) = G$ indicates that the evader has not visited the UGS, while $z(k) = t$, $t \geq 0$, indicates that the evader was at the current location exactly t time steps ago i.e., $p_i(k-t) = p_u(k)$. The UGS does not provide any information on the orientation $d_i(k-t)$ of the evader, thus the evader (state) information available to the UAV is not only delayed but also incomplete, which complicates the task of isolating the evader. The UAV's mission is accomplished if the current observation $z(k) = 0$ for any k , which implies $p_i(k) = p_u(k)$, however, if the evader were to reach a goal un-intercepted at any time k , i.e., $p_i(k) \in \mathcal{G}$ and $z(j) \neq 0$, $j \leq k$, then the UAV is deemed to have failed its objective. The search commences at time 0, where we assume that the UAV is t_0 steps behind the evader, i.e., $z(0) = t_0 > 0$. In summary, at time k and using all past observations $z(j)$, $j \leq k$, the UAV must decide what action to take, so that it is eventually co-located with the evader. In the following, a method for estimating the location of the evader is described, which will be used in later sections to detail a control policy for guaranteed isolation.

B. Evader State Estimation

Let $S(j|k)$ be the set of evader states $s(j)$ at time j that are consistent with the measurements obtained through time k , viz., $z(0), z(1), \dots, z(k)$. This section describes the prediction step and measurement update to obtain $S(k+1|k+1)$ from $S(k|k)$. Note that at time k , the UAV knows its current location, $p_u(k)$; it is not aware of the (exact) evader state $s(k)$, but we assume that the dynamics governing the evader's motion (3) are available to the UAV.

Define $\Gamma : 2^{\mathcal{S}} \rightarrow 2^{\mathcal{S}}$ as the mapping that projects a set of evader locations one-step forward in time using the evader dynamics (3), namely

$$\begin{aligned} \Gamma S(k|k) &\triangleq S(k+1|k) \\ &= \{s^+ : s^+ = f_i(s, w), \\ &\quad s \in S(k|k), w \in W(s)\}, \end{aligned} \quad (4)$$

and $S(k+\tau|k) = \Gamma^\tau S(k|k)$. Similarly, the mapping $\Gamma^{-1} : 2^{\mathcal{S}} \rightarrow 2^{\mathcal{S}}$ projects a set of evader locations one-step backward in time is

$$\Gamma^{-1} S \triangleq \{s | f_i(s, w) \in S, \text{ for some } w \in W(s)\}. \quad (5)$$

Note that Γ^{-1} is not the inverse of Γ since $S \subseteq \Gamma^{-1} \Gamma S$ and $\Gamma \Gamma^{-1} S \subseteq S$. We define the state estimate at time k according to:

$$\mathcal{E}_k = \{S(0|k), S(1|k), \dots, S(k|k)\}. \quad (6)$$

Let $\mathcal{M} : \mathcal{E}_{k-1} \times \mathcal{Z} \rightarrow \mathcal{E}_k$ update the evader estimate with a new measurement, where $\mathcal{M}[\mathcal{E}_{k-1}, z(k)] = \mathcal{E}_k$.

²Associated with each observation is the location of the UGS where the observation was made, i.e., $z(k) = \{G, p_u(k)\}$. For brevity, we simply write $z(k) = G$.

In the following, the calculation of the estimate sets $S(j|k)$ is enumerated below for the different possible observations.

Case 1 $z(k) = G$: Recall that this observation implies that the evader has not visited the current UAV/UGS location, $p_u(k)$. With a slight abuse of notation,³ the estimate set $\mathcal{E}_k = \mathcal{M}[\mathcal{E}_{k-1}, G]$ is found by

$$S(0|k) = S(0|k-1) \setminus p_u(k), \text{ and} \quad (7)$$

$$S(j|k) = [\Gamma S(j-1|k)] \setminus p_u(k), \quad j = 1, \dots, k. \quad (8)$$

Case 2 $z(k) = t > 0$: Here the evader was at the current UAV/UGS location $p_u(k)$ at time $k-t$. The evader estimate set (6) is updated in three steps. First,

$$S(k-t|k) = S(k-t|k-1) \cap p_u(k)$$

is then the set of all evader states at time $k-t$ that are consistent with the observation $z(k)$. Second, all past sets $S(j|k)$ for $j = 0, \dots, k-t-1$ also need to be updated to ensure that they are consistent with the evader arriving at $p_u(k)$ at time $k-t$. Working backwards from time $k-t$, this update is written as

$$S(j|k) = S(j|k-1) \cap \Gamma^{-1}S(j+1|k), \quad (9)$$

for $j = k-t-1, \dots, 0$. Lastly, using (4), the updated estimate sets for time $k-t$ to k are given by

$$S(k-t+j|k) = \Gamma_i^j S(k-t|k), \text{ for } j = 1, \dots, t \quad (10)$$

Case 3 $z(k) = 0$: In this case, the evader has been isolated, and there is no need to update the estimate set. Nonetheless, for completeness $\mathcal{M}[\mathcal{E}_{k-1}, z(k)]$ is given by $S(k|k) = \{p_u(k)\}$ and

$$S(j|k) = S(j|k-1) \cap \Gamma^{-1}S(j+1|k),$$

for $j = k-1, \dots, 0$ are the evader's best estimate sets consistent with the evader being at $p_u(k)$ at time k . Note that the estimation update requires all past sets $S(0|k-1), \dots, S(k-1|k-1)$ in order to calculate the current evader estimate set $S(k|k)$. In essence, the "state" under delayed information is the current UAV position $p_u(k)$ and all the evader position estimate sets $S(0|k), \dots, S(k|k)$ and as such grows unbounded!

For practical implementation, only the current evader uncertainty set $S(k-1|k-1)$ is maintained and all past uncertainty sets (except $S(0|0)$) are discarded. In other words, let the practical estimate set be given by $\tilde{\mathcal{E}}_k = \tilde{S}(k|k)$, and let $\tilde{\mathcal{M}}$ denote the relaxed measurement update that estimates only the current evader uncertainty set. Given a detection, or red observation, the *relaxed measurement update* is given by

$$\begin{aligned} \tilde{\mathcal{E}}_k &\triangleq \tilde{\mathcal{M}}[\tilde{\mathcal{E}}_{k-1}, z(k)] = \tilde{S}(k|k) \\ &= \tilde{S}(k|k-1) \cap \left(\bigcup_{d \in \mathcal{D}} \Gamma^{z(k)}(p_u(k), d) \right), \end{aligned} \quad (11)$$

³For simplicity, $S \setminus p_u$ denotes $S \setminus (p_u, d)$, $\forall d$. In other words, $S \setminus p_u$ removes all evader states $s_i = (p_u, d)$ from S . Also, the directional information is implicitly conveyed with a similar abuse of notation for $S \cap p_u$ and $x_i = p_u$.

and for a green observation the current evader estimate set is

$$\tilde{\mathcal{E}}_k = \tilde{\mathcal{M}}[\tilde{S}(k|k-1), z(k)] = \tilde{S}(k|k-1) \cap \tilde{S}_z(k), \quad (12)$$

where $\tilde{S}_z(k)$ is found recursively as $\tilde{S}_z(j) = [\Gamma S_z(j-1)] \setminus p_u(k)$ starting from $\tilde{S}_z(0) = S(0|0) \setminus p_u(k)$. Note that the set of estimated evader locations, $\tilde{\mathcal{E}}_k = \tilde{S}(k|k)$, that is found using the relaxed measurement update $\tilde{\mathcal{M}}$ is not the same as that found in the true measurement update $\mathcal{E}_k = \mathcal{M}$. The loss from this relaxation will be discussed in Section III.

III. PARTIAL INFORMATION SCENARIO

Given the initial UAV position $p_u(0)$ and observation $z(0) = t > 0$, we are interested in guaranteeing isolation given the initial information state $(p_u(0), \mathcal{E}_{0|0})$, which consists of the UAV location and the uncertainty about the evader location. Recall that the initial evader uncertainty set $\mathcal{E}_{0|0}$ is the set of all evader states consistent with the observation $z(0)$. If there was no additional information available to the UAV other than $z(0)$, then the UAV has recourse only to the evader model (dynamics), and the corresponding "prior" uncertainty set at time $k > 0$ is

$$\tilde{\mathcal{E}}_{k|0} = \tilde{S}(k|0) = \Gamma^k \tilde{S}(0|0). \quad (13)$$

Let $S^j(k) \subset \tilde{S}(k|0)$; $j = 1, \dots, n_k$, $k = 1, \dots, N$, to be the collection of subsets, except for the empty set, of the prior estimate set $\tilde{S}(k|0)$, where $n_k = 2^{|\tilde{S}(k|0)|} - 1$. Notice at time 0, $n_0 = 1$ and $S^1(0) = \tilde{S}(0|0)$.

The question we answer in this section is: given the current evader state estimate set, what are the favorable UAV locations from which isolation can be guaranteed? The favorable UAV locations are defined recursively starting with the base case, for the recursion, at time N . Let $S^\ell(N) \subset \tilde{S}(N|0) = \tilde{\mathcal{E}}_{N|0}$, $\ell \in \mathcal{L}$ be those sets which contain only one goal location. Note that the $S^\ell(N)$ could contain multiple evader states, $x_i(N)$ corresponding to a single goal location because of the different possible orientations. With a slight abuse of notation, we denote this goal location as $p_\ell \in \mathcal{G}$, and $\mathcal{L} \subset \{1, \dots, n_N\}$ is the subset of indices that satisfy this property. Clearly, isolation can only be guaranteed for those sets, because N is the terminal time and so, the UAV needs to precisely know the evader location (if not the orientation) and, furthermore, be at that location. So, the set of favorable UAV locations corresponding to $S^\ell(N)$ is given by,

$$\tilde{P}_u^\ell(N) = \{p_\ell\}, \quad \ell \in \mathcal{L}. \quad (14)$$

For all other subsets $S^j(N)$, $j \notin \mathcal{L}$, the corresponding set of favorable UAV locations $\tilde{P}_u^j(N)$ is empty.

Using the sets $\tilde{P}_u^\ell(N)$ at the time horizon N as the base case for a backward recursion, the set of favorable UAV locations for times $k = (N-1), \dots, 0$ are computed as follows.

- 1) If $S^j(k)$ contains states with multiple (different) goal locations, then $\tilde{P}_u^j(k) = \emptyset$.

2) If $(p_i, d_i) \notin S^j(k)$, $\forall p_i \in \mathcal{G}$, then

$$\begin{aligned} \tilde{P}_u^j(k) = & \left\{ p_u : \exists u \in U(p_u) \right. \\ & \left. \text{s.t. } p^+ = f_u(p_u, u) \in \cap_{z \in \mathcal{Z}(k+1, p^+)} \tilde{P}_u^{\ell_j(z)}(k+1) \right\} \end{aligned} \quad (15)$$

where $\mathcal{Z}(k+1, p^+)$ is the set of possible observations made at location p^+ at time $k+1$ and $\tilde{P}_u^{\ell_j(z)}(k+1)$ is the set of favorable locations corresponding to the estimate set,

$$S^{\ell_j(z)}(k+1) \triangleq \tilde{\mathcal{M}}[\Gamma_i(S^j(k) \setminus p_u), z]. \quad (16)$$

3) If for some $p_i \in \mathcal{G}$, $(p_i, d) \in S^j(k)$ for any $d \in D(p_i)$, and $(p_j, d_j) \notin S^j(k)$, $\forall p_i \in \mathcal{G}$, $j \neq i$, then

$$\begin{aligned} \tilde{P}_u^j(k) = & \{p_i\} \cap \left\{ p_u : \exists u \in U(p_u) \right. \\ & \left. \text{s.t. } p^+ = f_u(p_u, u) \in \cap_{z \in \mathcal{Z}(k+1, p^+)} \tilde{P}_u^{\ell_j(z)}(k+1) \right\} \end{aligned} \quad (17)$$

Lemma 1: A sufficient condition for guaranteed isolation is that the initial UAV position satisfy: $p_u(0) \in \tilde{P}_u^1(0)$.

Proof: The result directly follows from the construction of the sets $\tilde{P}_u^j(k)$ in (15) and (17) with the initialization (14). ■

The definition in 1) follows from the fact that if the intruder could be at multiple goal locations then there does not exist a UAV location that guarantees isolation, *e.g.*, if the UAV were at one goal location, then the intruder could escape via the other goal location. In 2), there are no goal locations in the evader estimate set $S^j(k)$. Hence, the UAV can isolate the intruder by being co-located with the intruder at the current time or moving to a favorable location in the next time step, regardless of the evader's action. For this to occur, the UAV must move to a location $p(k+1)$, where, for any observation $z(k+1)$, the resulting uncertainty set $\tilde{\mathcal{M}}[\Gamma S^j(k), z(k+1)]$ has $p(k+1)$ included in the set of favorable locations that guarantees isolation at time $k+1$. In 3) there is a goal location included in $S^j(k)$. The UAV must be at the goal location to prevent the evader from escaping via the goal location. In addition, as with 2), the UAV must also move to a favorable location with its next move, in case the intruder is not at the goal location.

Lemma 1 is however not a necessary condition for guaranteed isolation. This is perhaps not apparent from the recursions (15) and (17) established earlier, but it has to do with the dual control nature of the problem. At each time step, the above recursion looks at all possible future observations, and guarantees isolation by hedging against this uncertainty. However, for a particular evader path, only a subset of all possible observations will occur. Thus, for a particular evader trajectory, the algorithm is conservative in the sense that it is hedging against all uncertainty, rather than the uncertainty at hand. To ensure necessity, the algorithm would need to look at not just the current estimated set of evader locations, but rather all possible uncertainty sets from the initial time. This would likely lead to intractability, since one would have to consider every possible observation

(and the resulting new information thereof) at time $k+1$. Since the observations are delayed, the evader estimate sets $S^j(k)$ will have to be replaced by an indexed set of sets: $(S^j(0), \dots, S^j(k))$.

A. Control Policies

1) *Pursuit Policy with isolation guarantee:* If the sufficient condition in Lemma 1 is satisfied, a UAV control policy that leads to isolation is given by: $\tilde{\pi} = (\tilde{u}(0), \dots, \tilde{u}(T-1))$, where, $\tilde{u}(k) \in U(p_u(k))$ satisfies the following:

$$p^+ = f_u(p_u, \tilde{u}(k)) \in \cap_{z \in \mathcal{Z}(k+1, p^+)} \tilde{P}_u^{\ell, z}(k+1), \quad (18)$$

where the index ℓ is determined by $S^\ell(k+1) = \Gamma S(k|k)$, where $S(k|k)$ is the current best estimate set defined earlier (see Sec. II for details). Note that by definition, the best estimate set $S(k|k)$ is necessarily a subset of the prior estimate set $S(k|0)$ and so, $\Gamma S(k|k) \subset S(k+1|0)$. Here, $T(\leq N)$ is the time at which isolation occurs, *i.e.*, $z(T) = 0$ and the search is concluded. To summarize, the UAV does the following sequence of estimation and control steps:

- 1) Use current measurement $z(k)$ and past estimate sets $S(j|k-1)$; $j = 0, \dots, (k-1)$, to generate the current best estimate set $S(k|k)$.
- 2) Use $S_i(k|k)$ to compute the optimal action $\tilde{u}(k)$ via (18).

The UAV repeats the above sequence for $k = 0, \dots, T-1$, in the prescribed order, until the measurement, $z(T) = 0$, indicating isolation, is received (this is guaranteed to happen if the sufficient condition is met).

2) *Worst Case Minimum-time Guaranteed Capture:* It is clear that the pursuit policy $\tilde{\pi}$ as defined above is non-unique in that multiple control options, $u(k) \in U(p_u(k))$ will likely satisfy the condition (18). So, a judicious choice would be to pick a policy that also achieves isolation in minimum time under worst-case uncertainty. Given the current position of the pursuer, p and the evader uncertainty set $S^j(k)$, let the capture time be defined according to:

$$\begin{aligned} \mathcal{T}(p, S^j(k), u) = & \begin{cases} 0, & \text{if } p \in \tilde{P}_u^j(k) \text{ \& } z(k) = 0 \\ \infty, & \text{if } \tilde{P}_u^j(k) = \emptyset \\ 1 + \max_z \mathcal{T}(f_u(p, u), S^{\ell_j(z)}(k+1), \tilde{u}), & \text{otherwise} \end{cases} \end{aligned} \quad (19)$$

where $u \in U(p)$, $z \in \mathcal{Z}(k+1, f_u(p, u))$, and the index $\ell_j(z)$ is found from $S^{\ell_j(z)} = \tilde{\mathcal{M}}[\Gamma(S^j \setminus p), z]$. From a game theoretic view, the maximum would be over the action space of the evader, which includes all actions taken by the evader up to time $k+1$. By writing the Bellman Equation as in (19), the measurement z , effectively, parameterizes the action space of the evader. Notice that the capture time (19) can be computed recursively along with the guaranteed capture sets (14)-(17). The worst case minimum-time control is given by:

$$u^*(k) = \underset{\tilde{u}(k)}{\operatorname{argmin}} \mathcal{T}(p, S^j(k), u), \quad k = 0, \dots, (T-1), \quad (20)$$

where, $\tilde{u}(k)$ satisfies (18) and the terminal condition, $\mathcal{T}(p(T), S^j(T), u) = 0$, $p(T) \in \tilde{P}_u^j(T)$, $z(T) = 0$.

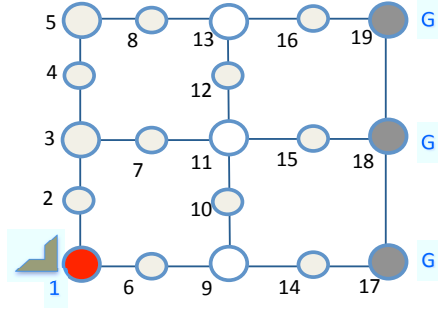


Fig. 1. 3×3 Grid

IV. MANHATTAN GRID EXAMPLE

To illustrate the solution method, we consider a simple road network (see Fig. 1) *i.e.*, a rectangular grid instrumented with UGSs at all the intersections. The set of all evader locations, $S_i = \{1, \dots, 19\}$. The goal locations that the evader is heading toward are $\mathcal{G} = \{17, 18, 19\}$, marked ‘G’ in the figure. The evader is allowed to move east, south or north, provided a path exists in that direction. We denote these actions by E , S and N . However, he is not allowed to make a U-turn *i.e.*, if he had moved south in the previous time step, he cannot choose north (or vice-versa) in the current time step. The UAV, in addition to being able to move north, south and east, can also wait at a particular location, indicated by the action W . Also the UAV is twice as fast as the evader, who has unit speed. The UGSs are placed at those locations that the UAV can occupy, *i.e.*, $\mathcal{P}_u = \{1, 3, 5, 9, 11, 13\} \cup \mathcal{G}$. Note that the same example was considered in [15], where it was shown that the weaker sufficient condition was satisfied for the initial observation, $z(0) = 1$ at UGS location 1. We also showed that the condition failed for the case $z(0) = 2$, even though isolation can be achieved (see Sec V-A in [15] for details). Here, we show that with the tighter sufficiency condition developed earlier is met and we are able to provide a pursuit policy that guarantees capture (in minimum time amongst all policies with the guarantee). So, let the UAV start at node 1 and observe $z(0) = 2$, *i.e.*, the evader was at location 1 two time steps ago. The longest possible path that the evader can take towards a goal location is of length 10 (counting the number of time steps from time 0) and hence, we set $N = 10$.

The prior estimate sets for the evader (13) can immediately be written down (see Table I), where the subscript at each time k indicates the current orientation $d_i(k)$ of the evader. From (14), (15) and (17), we can recursively compute the set of favorable UAV locations shown in Table II. Note that the table is incomplete, in that we have not enumerated (for brevity) all possible subsets of $S(k|k)$ at each time k . To illustrate the logic, we look at time 6, where the set of favorable UAV locations corresponding to the estimate set $\{17_E, 19_E\}$ is empty. This is because both 17 and 19 are goal locations and if the UAV is uncertain as to which of the two locations is being occupied by the evader, isolation

TABLE I
PRIOR EVADER STATE ESTIMATE SET

time (k)	$\mathcal{S}(k 0)$
0	$\{3_N, 9_E\}$
1	$\{4_N, 7_E, 10_N, 14_E\}$
2	$\{5_N, 11_E, 11_N, 17_E\}$
3	$\{8_E, 12_N, 15_E, 10_S\}$
4	$\{13_E, 13_N, 18_E, 9_S\}$
5	$\{12_S, 16_E, 14_E\}$
6	$\{11_S, 19_E, 17_E\}$
7	$\{10_S, 15_E\}$
8	$\{9_S, 18_E\}$
9	$\{14_E\}$
10	$\{17_E\}$

clearly cannot be guaranteed. As a direct consequence of this and the (backward) recursive construction of the sets, the set of favorable UAV locations corresponding to the estimate set $\{14_E, 16_E\}$ in the previous time step 5 is empty and so on, until at time 2, we see that the set corresponding to $\{11_E\}$ is the singleton $\{11\}$. This is interesting in that it says the following: though the UAV precisely knows the evader state to be $\{11_E\}$ at time 2, the only location from which isolation can be guaranteed is 11 itself *i.e.*, the evader has been isolated then and there, else the guarantee is lost! This follows from the observation that if the UAV were not at 11, then the evader estimate set at time 3 (given no new information) is $\Gamma\{11_E\} = \{10_S, 15_E, 12_N\}$ for which the corresponding set of favorable UAV locations is empty.

Since the initial UAV position, $p_u(0) = 1 \in \tilde{\mathcal{P}}_u^1(0)$, where $\tilde{\mathcal{P}}_u^1(0)$ is the set of favorable UAV locations corresponding to the initial estimate set $S(0|0) = \{3_N, 9_E\}$, isolation can indeed be guaranteed (as per Lemma 1). Furthermore, the

TABLE II
SET OF FAVORABLE UAV LOCATIONS

time (k)	$S(k k)$	$\tilde{\mathcal{P}}_u(k)$
10	$\{17_E\}$	$\{17\}$
9	$\{14_E\}$	$\{9, 17, 18\}$
8	$\{9_S, 18_E\}$ $\{9_S\}$ $\{18_E\}$	$\{18\}$ $\{1, 9, 11, 17, 18, 19\}$ $\{18\}$
7	$\{10_S, 15_E\}$ $\{10_S\}$ $\{15_E\}$	$\{11, 17, 18, 19\}$ $\mathcal{X}_u \setminus \{5\}$ $\{11, 17, 18, 19\}$
6	$\{17_E, 19_E\}$ $\vdots$	ϕ $\vdots$
$\vdots$	$\vdots$	$\vdots$
2	$\{11_E\}$ $\{5_N, 11_N, 17_E\}$ $\{11_E, 17_E\}$ $\vdots$	$\{11\}$ $\{17\}$ ϕ $\vdots$
1	$\{10_N, 14_E\}$ $\{4_N, 7_E\}$ $\{4_N, 7_E, 10_N, 14_E\}$ $\vdots$	$\{9, 17, 18\}$ $\{3, 9, 11, 13\}$ ϕ $\vdots$
0	$\{3_N, 9_E\}$ $\{3_N\}$ $\{9_E\}$	$\{1, 9, 11\}$ $\{1, 3, 5, 9, 11, 13\}$ $\{1, 9, 11, 17, 18, 19\}$

time optimal UAV policy that enables isolation, computed

according to (20), is shown in Table III. For instance, at time 0, the control action $u^*(0)$ satisfies:

$$f_u(1, u^*(0)) \in \cap_{z \in \mathcal{Z}(k+1, p+)} \tilde{P}_u^{\ell, z}(1).$$

If the UAV were to move to location $p^+ = 9$, then there are two possible observations $z(1) = 1$ or $z(1) = G$, corresponding to the two uncertainty sets $\{10_N, 14_E\}$ and $\{4_N, 7_E\}$, respectively. The favorable UAV sets for these two evader estimate sets are $\tilde{P}_u^{\ell, z}(1) = \{9, 17, 18\}$ and $\{3, 9, 11, 13\}$, which both include $p^+ = 9$, thus the optimal action is to move east, $f_u(1, E) = 9$. In Table III, we only show the state evolution corresponding to worst case evader actions. In this case, isolation occurs at time 6 at node 19. For all other paths, the search will terminate early.

Although we have tightened the sufficient condition for guaranteed capture, it is still *not* necessary. As pointed out earlier, computing the necessary condition and the set of UAV locations thereof would lead to intractability. But we are encouraged by the fact that the condition provided herein does faithfully recover the optimal min-max (capture time) pursuit policy for the example problem above. Indeed, we have verified this empirically for bigger grids as well by comparing our policy (for 3 rows and number of columns, 3 to 6) with the optimal min-max pursuit policy developed in [16].

TABLE III
STATE EVOLUTION UNDER THE OPTIMAL POLICY

time (k)	$z(k)$	$S(k k)$	$p_u(k)$	$u^*(k)$
0	1	$\{3_N, 9_E\}$	1	E
1	1	$\{10_N, 14_E\}$	9	E
2	'G'	$\{11_N\}$	17	N
3	'G'	$\{12_N, 15_E\}$	18	W
4	'G'	$\{13_N\}$	18	N
5	'G'	$\{16_E\}$	19	W
6	0	$\{19_E\}$	19	-

V. CONCLUSIONS

We have improved upon a sufficient condition, established in prior work, for guaranteed isolation of the evader on a graph, pursued by a UAV under delayed information. In addition to guaranteed isolation, we also provide a pursuit policy that is time optimal, under worst-case evader actions. The policy is promising in that it recovers the optimal min-max control policy for the special case of a 3-row Manhattan grid. The method is also relatively easy to implement on other general graphs.

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